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Numerical Solution of Ill-posed Cauchy Problems for Parabolic Equations

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To Atra, Mohsen and my parents

Abstract

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Ill-posed mathematical problem occur in many interesting scientific and engineering applications. The solution of such a problem, if it exists, may not depend continuously on the observed data. For computing a stable approximate solution it is necessary to apply a regularization method. The purpose of this thesis is to investigate regularization approaches and develop numerical methods for solving certain ill-posed problems for parabolic partial differential equations. In thermal engineering applications one wants to determine the surface temperature of a body when the surface itself is inaccessible to measurements. This problem can be modelled by a sideways heat equation. The mathematical and numerical properties of the sideways heat equation with constant convection and diffusion coefficients is first studied. The problem is reformulated as a Volterra integral equation of the first kind with smooth kernel. The influence of the coefficients on the degree of ill-posedness are also studied. The rate of decay of the singular values of the Volterra integral operator determines the degree of ill-posedness. It is shown that the sign of the coefficient in the convection term influences the rate of decay of the singular values.

Further a sideways heat equation in cylindrical geometry is studied. The equation is a mathematical model of the temperature changes inside a thermocouple, which is used to approximate the gas temperature in a combustion chamber. The heat transfer coefficient at the surface of thermocouple is also unknown. This coefficient is approximated via a calibration experiment. Then the gas temperature in the combustion chamber is computed using the convection boundary condition. In both steps the surface temperature and heat flux are approximated using Tikhonov regularization and the method of lines.

Many existing methods for solving sideways parabolic equations are inadequate for solving multi-dimensional problems with variable coefficients. A new iterative regularization technique for solving a two-dimensional sideways parabolic equation with variable coefficients is proposed. A preconditioned Generalized Minimum Residuals Method (GMRS) is used to regularize the problem. The preconditioner is based on a semi-analytic solution formula for the corresponding problem with constant coefficients. Regularization is used in the preconditioner as well as truncating the GMRES algorithm. The computed examples indicate that the proposed PGMRES method is well suited for this problem.

In this thesis also a numerical method is presented for the solution of a Cauchy problem for a parabolic equation in multi-dimensional space, where the domain is cylindrical in one spatial direction. The formal solution is written as a hyperbolic cosine function in terms of a parabolic unbounded operator. The ill-posedness is dealt with by truncating the large eigenvalues of the operator. The approximate solution is computed by projecting onto a smaller subspace generated by the Arnoldi algorithm applied on the inverse of the operator. A well-posed parabolic problem is solved in each iteration step. Further the hyperbolic cosine is evaluated explicitly only for a small triangular matrix. Numerical examples are given to illustrate the performance of the method.

Populärvetenskaplig sammanfattning

Inom många vetenskapliga och tekniska tillämpningar är det nödvändigt att uppskatta vissa parametrar som kännetecknar ett fysiskt system, men som är oåtkomliga för direkta mätningar. Detta problemområde, så kallade inversa problem, har blivit ett mycket aktivt och väletablerat tvärvetenskapligt forskningsområde under de senaste tre decennierna. Det täcker bl a problem från teknik, industri, medicin, liksom livs- och geovetenskaper.

Ett exempel på ett inverst problem inom värmeteknik är att man vill bestämma yttemperaturen hos ett objekt när ytan i sig är oåtkomlig för mätning. Det kan också vara så att lokalisering av en mätenhet på ytan skulle störa mätningarna så att en felaktig temperatur registreras. I sådana fall är man begränsad till interna mätningar. Detta problem kan modelleras med en värmeledningsekvation, som löses i sidled, dvs i rumsled, till skillnad från standardproblemet som löses framåt i tiden. Inversa problem är ofta illa-ställda. Lösningen till ett illa-ställt matematiskt problem, om den existerar, beror inte kontinuerligt på observerade data. Detta innebär t.ex. att små fel i uppmätta temperaturer kan leda till mycket stora fel i den beräknade lösningen. I praktiken uppstår fel i uppmätta data och det är därför nödvändigt att tillämpa en regulariseringsmetod, som gör lösningen mindre känslig för störningar i uppmätta data.

Syftet med denna avhandling är att undersöka regulariseringsstrategier och utveckla numeriska metoder för att lösa vissa illa-ställda problem för paraboliska partiella differential ekvationer. Vi studerar först påverkan av koefficienterna i värmeledningsekvationen i sidled på graden av illa-ställdhet. Vi löser också en värmeledningsekvation i sidled, där ekvationen är en matematisk modell av temperaturförändringar inuti ett termoelement, som används för att approximera gastemperaturen i en förbränningskammare.

Många metoder för att lösa paraboliska ekvationer i sidled är otillräckliga för att lösa multi-dimensionella problem med variabla koefficienter. En ny iterativ regulariseringsteknik för att lösa en två-dimensionell parabolisk ekvation i sidled med variabla koefficienter föreslås. I en iterativ metod startar man med en begynnelseapproximation, som sedan gradvis förbättras. Eftersom iterationerna för denna typ av problem först förbättrar approximationen men senare försämrar den, kan man regularisera genom att stoppa iterationer vid lämplig tidpunkt.

I denna avhandling presenteras också en numerisk metod för lösning av ett parabolisk inverst problem i ett flerdimensionellt rum, där området är cylindriskt i en rumsriktning. Den approximativa lösningen beräknas genom att projicera på ett underrum genererat av en Krylovsrumsmetod.

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Linköping, February 25, 2010
Zohreh Ranjbar

Contents

1

Introduction and Overview

In many scientific and engineering applications there is a need to estimate some parameters characterizing a physical system, which is inaccessible to direct measurements.This problem area, so called *inverse problems*, has become a very active, interdisciplinary and well-established research area over the past three decades. It covers problems from engineering, industry, medicine, as well as life and earth sciences. Some examples of inverse problems are computerized tomography [50], inverse scattering problems [24], inverse problems in geophysics [25]. One particular area of inverse problems is in *heat conduction* [1], where the goal is to reconstruct the values of the parameters characterizing the system from a noisy measured temperature. A classic problem of inverse heat conduction is the *backward problem*, where the initial conditions (temperature) are to be found from later measurements. Numerical solutions of this problem have been considered in [17, 61, 60, 40, 42]. Another interesting problem is Sideways Heat Equation (SHE) problem, where the unknown temperature at the boundary of the object is to be approximated based on measurements of the accessible temperature of the object. The *heat equation* is a prototype parabolic partial differential equation (PDE), used to model heat conduction in a given region over time. The aim of this thesis is to investigate the numerical solution of certain sideways parabolic PDE problems.

1 Inverse Ill-Posed Problems and Ill-Conditioning

Inverse problems are often *ill-posed*, as opposed to the well-posed problems introduced by Hadamard [31]. Consider the linear operator equation

$$Kf = g, \tag{1.1}$$

where $K : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$.

Definition 1.1. The problem of solving (1.1) is called *well-posed* if:

1. Existence: a solution exists for any data g in the data space.
2. Uniqueness: the solution f is unique in the solution space.
3. Stability: the solution f depends continuously on the data g.

Conditions 1 and 2 are equivalent to saying that the operator K has a well defined inverse K^{-1} and that the range of K is all of the data space. The condition 3 is equivalent to K^{-1} being continuous or bounded. For inverse problems this stability condition is most often violated. To stabilize the solution, one needs to incorporate additional a priori information into the problem, so called *regularization*.

In many linear ill-posed problems that arise in science and engineering, instead of the exact right hand side g, only a perturbation of it, $g_\delta = g + \delta$, is available. When the inverse operator K^{-1} is unbounded, it means that if the solution f exists then the solution $f_\delta = K^{-1} g_\delta$ is usually a poor approximation of f, even for very small values of δ.

While inverse problems are often formulated in infinite dimensional spaces, limitations to a finite number of measurements, and the practical consideration of recovering only a finite number of unknown parameters leads to stating the problem in discrete form. Following [35], a problem is called *discrete ill-posed* when it is highly sensitive to high-frequency perturbations and the solution operator has very large norm.

2 Cauchy Problems for Parabolic Equations

A parabolic Cauchy problem (initial-boundary value problem) is a PDE that satisfies certain conditions which are given on the boundary of the domain. As a simple example of a one-dimensional ill-posed Cauchy problem for the heat equation we have,

$$\begin{array}{ll} u_t = u_{xx}, & 0 < x < 1, \quad 0 \le t \le 1, \\ u(x,0) = 0, & 0 \le x \le 1, \\ u(0,t) = g(t), & 0 \le t \le 1, \\ u_x(0,t) = h(t), & 0 \le t \le 1. \end{array} \tag{1.2}$$

From the data at the left boundary we want to determine the solution at the right boundary $u(1,t) = f(t)$, see Figure 1.1. This is a sideways heat equation in a bounded domain. From now on we consider all the involved functions to be in L^2.

2.1 Sideways Heat Equations

The sideways heat equation (SHE) (1.2) is an ill-posed Cauchy problem for a parabolic equation. It is sometimes referred as the inverse heat conduction problem [1, 49]. We label it the SHE in order to distinguish it from other inverse problems for a parabolic equation [12]. In this definition the initial temperature distribution is considered known. The linearity of the differential equation implies that initial data can easily be taken equal to zero. In Paper II we consider a SHE as a simple model of the heat transfer inside a thermocouple of a suction pyrometer. The suction pyrometer [8], is used for measuring and controlling the gas temperature in a combustion chamber. The numerical experiments in this paper, with synthetic data, indicate that it is possible to organize the measurements

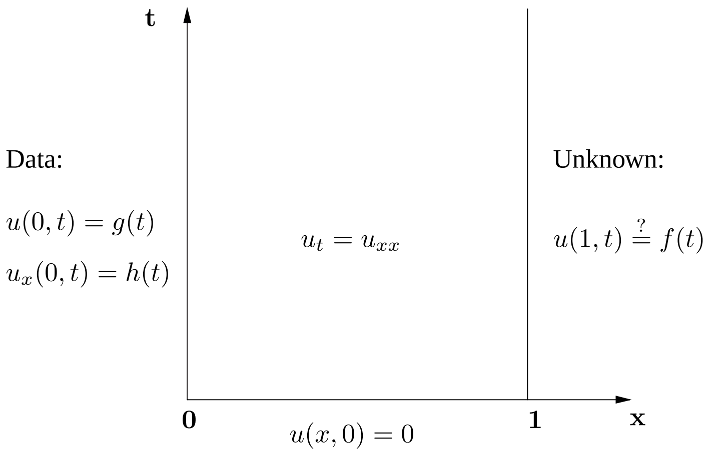

Figure 1.1: *Determination of function f from the interior measurements g and h.*

with the suction pyrometer in such a way that the gas temperature can be computed with sufficient accuracy. For more industrial examples modelled by SHE see [1, 5, 16, 62, 63].

Two types pf problems can be formulated:

- *The quarter plane problem* is a problem in a semi-infinite region, $x \in [0, \infty)$, where one wants to compute $f(t)$ at $x = 0$ from the known data function $g(t)$ (Dirichlet boundary condition) at $x = l$ for $0 < l < \infty$, with the additional condition that the solution is bounded, as x tends to infinity.
- *The bounded interval problem*: the requirement of boundedness of the solution in quarter plane is replaced by the measurement of the heat flux at $x = l$ (Neumann boundary condition). The equation problem (1.2) is a bounded interval problem. All problems studied in this thesis are considered in a bounded region in space.

Both problems can be reformulated as a Volterra integral equation of the first kind [11, 12],

$$g(t) = (Kf)(t) = \int_0^t k(t - \tau) f(\tau)\, d\tau, \qquad 0 \le t \le 1, \tag{1.3}$$

with different kernel functions k. However the kernel functions have similar mathematical properties, e.g. the smoothing property [12, 18].

Recall that the SHE is not well-posed [1] and the solution does not depend continuously on the data. Further the measured data are usually a corrupted version of the exact quantities since we are never able to measure exactly in practice. But in both the quarter and bounded interval problems one can obtain stability of logarithmic convexity type by applying an a priori bound on the solution and bounded measurement error, i.e.,

$$\|f\|_{L^2} \le M \qquad \text{and} \qquad \|g - g_\delta\|_{L^2} < \epsilon,$$

see [12, 26, 45, 46, 52]. Stability estimate for more general parabolic equation can be found in [41].

In the Volterra integral equation (1.3) we have a *causality principle* [19] in the sense that what happens at the "solution" boundary at time t_0 can only affect the data $g(t)$ for $t \geq t_0$. Therefore the solution $f(t)$ can not be reconstructed for t close to the final time 1. This is another reason of ill-posedness of the SHE beside the smoothness of the Volterra operator. Note that the coefficients of SHE play a role for the smoothness of the Volterra operator (ill-posedness) which is analyzed and illustrated in Paper I.

A common family of methods for solving the SHE transforms the problem into an integral equation of first kind. The weak point of these methods is that often the kernel in the corresponding integral equation is not known explicitly, e.g., when the differential equations has variable coefficients.

2.2 Ill-Posedness and Singular Values

Singular values and singular functions of the operator give information about the nature of ill-posedness, for more details see [24, chapter 2]. If in (1.3) the kernel function $k \in L_2([0,1] \times [0,1])$ then the operator K is compact and the inverse operator K^{-1} exists and is unbounded [12]. For any compact operator we have the following definition of a singular system.

Definition 2.1. The set $(\sigma_n, u_n, v_n)_{n=1}^{\infty}$ is the *singular system* of the compact operator $K : \mathcal{H}_1 \to \mathcal{H}_2$, if

$$Kv_n = \sigma_n u_n, \qquad K^* u_n = \sigma_n v_n,$$

where the singular values are ordered, $\sigma_1 \geq \sigma_2 \geq \cdots > 0$, and the sets of singular functions $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ constitute complete orthonormal systems.

The operator $K^* : \mathcal{H}_2 \to \mathcal{H}_1$ is the adjoint of K. For compact operators the singular values have only one accumulation point 0, which means $\lim_{n\to\infty} \sigma_n = 0$. The discrete analogue of the above singular system is a singular value decomposition (SVD) of a rectangular matrix [28, p.70], and is considered later in this chapter. The singular value expansion (SVE) of the operator is

$$Kf = \sum_{n=1}^{\infty} \sigma_n (f, v_n) u_n, \qquad K^* g = \sum_{n=1}^{\infty} \sigma_n (g, u_n) v_n.$$

Formally the solution of $Kf = g$ can be written

$$f = K^{\dagger} g = \sum_{n=1}^{\infty} \frac{(g, u_n)}{\sigma_n} v_n, \tag{1.4}$$

The operator $K^{\dagger}$ is called the *Moore-Penrose pseudo-inverse* [24]. The following condition for the existence of a solution is called the *Picard criterion*,

$$\|f\|_{L^2}^2 = \sum_{n=1}^{\infty} \frac{|(g, u_n)|^2}{\sigma_n^2} < \infty. \tag{1.5}$$

This implies that (g, u_n) must decay more rapidly as $n \to \infty$ than σ_n, which means that the data function g that satisfies the Picard condition must be very smooth. In practice, due to the measurement error we have inexact data $g_\delta = g + \delta$. Since the measurement errors are usually stochastic we can not assume that the function δ satisfies the Picard condition. Therefore in the case of measurement errors (even very small) any naive solution via the infinite sum (1.4) usually diverges or has a very large norm. The analogue of the condition (1.5), which should be satisfied by the solution of a finite dimensional linear system is called the *discrete Picard condition*, introduced in [34].

The Volterra operator (1.3) in discrete form is a triangular Toeplitz matrix which is extremely ill-conditioned. Even for noise free measured data g, applying the least squares solution of solving this ill-conditioned system always exhibits very large oscillations due to round-off error in the computer arithmetic.

Therefore there is a need to regularize the problem. There are various approaches to regularizing and stabilizing the ill-posedness of parabolic Cauchy problems, see e.g., [24, 40, 3] and the references therein. In the following we will give a short presentation of some regularization methods for discrete ill-posed problems.

3 Direct Regularization Methods

Direct regularization method can be based on some decomposition in numerical linear algebra like, e.g. Singular Value Decomposition (SVD) or orthogonal transformation. These methods require a direct access to all elements of matrix K when solving the linear system $Kf = g$. Some methods based on *Fourier transform* where the ill-posedness is dealt with by cutting off the high frequencies are proposed in [2, 22, 53, 59]. In this thesis the following direct methods are employed: Tikhonov regularization in discrete form (Papers I-III), truncated SVD (Papers III, IV), a method of lines combined with the approximation of the time derivative by a bounded operator (Paper I-III). These methods will be discussed shortly in following subsections.

Other examples of regularization methods for solving ill-posed Cauchy parabolic problems are the method of *Beck* [1], *mollification* [48] and *variational methods* [44, 38].

3.1 Truncated Singular Value Decomposition

The association of small singular values with ill-posedness has been discussed in the previous section. Therefore an obvious idea to regularize the problem is to truncate the small singular values of the operator i.e. truncate the series (1.4). Suppose that the operator K in (1.1) is discretized to a $n \times n$ matrix with the following singular value decomposition

$$K = \sum_{i=1}^{n} \sigma_i u_i v_i^T,$$

where $\sigma_1 > \sigma_2 > \cdots > \sigma_n \geq 0$ are the singular values and u_k and v_k are the left and right singular vectors, respectively. The solution to $Kf = g_\delta$ can be written

$$f = \sum_{i=1}^{n} \frac{u_i^T g_\delta}{\sigma_i} v_i.$$

Recall that small perturbations in the coefficients $u_i^T g_\delta$ are magnified extremely by a factor σ_i^{-1} for i large enough, because the most of singular values $\{\sigma_i\}$ are very small.

The truncated singular value decomposition (TSVD) method only utilizes the $N \leq n$ largest singular values for the solution and damps the effects caused by division by the small singular values, i.e. the TSVD solution is

$$f_N = \sum_{i=1}^{N} \frac{u_i^T g_\delta}{\sigma_i} v_i.$$

The number of terms N is the regularization parameter which should be properly selected to get a good accuracy of the approximated solution.

3.2 Tikhonov Regularization

Tikhonov regularization is one of the most commonly used methods for regularization of ill-posed problems. The goal is to minimize $\|g - Kf\|^2$ subject to a constraint on the size or smoothness of the f. The regularized solution solves the problem,

$$\min_f \|g - Kf\|^2 + \lambda \|Lf\|^2,$$

where $\lambda > 0$ is the regularization parameter and the matrix L is either the identity matrix or a discrete differentiation operator. The parameter λ controls the smoothness of the solution. In other words it is a trade-off between fitting the data and reducing a norm of the solution. Both Tikhonov and TSVD are expensive to implement for large-scale applications. For more details about these two methods we refer to [24, 35].

3.3 Method of Lines

The idea of the method of lines is to rewrite the original PDE (1.2) as a system of Ordinary Differential Equation (ODE). The source of ill-posedness is that the time derivative $\partial u/\partial t$ is unbounded, see [19, 21, 55]. Therefore by replacing the time derivative by a bounded approximation one can regularize the problem. Afterwards the resulting initial-value problem can be solved numerically essentially as an ODE in the space variable. This is a reason to treat the space and time discretization separately. There are some methods to approximate the time derivative, like difference quotient [20], wavelet-Galerkin [55, 56], mollification approximation [39, 48], spectral in terms of Fast Fourier Transform(FFT) [2], spline approximation [4]. We have used the last two methods in Papers I and II.

4 Iterative Regularization Methods

Iterative regularization methods are based on iteration schemes that access the matrix K only via matrix-vector multiplications. This property is suitable for sparse matrices and also when solving problems, where the matrix is not explicitly available. Many iterative methods have a self regularizing property in that early termination of the iterative process has a regularizing effect. The approximate solution initially tends to the true solution and then it diverges to some other undesired vector in later stages of iteration. This is called the *semi-convergence* phenomenon of iterative methods applied to ill-posed problems [24, chapter 6]. In other words, the iteration number can be considered as a regularization parameter. This is due to the fact that in its first few steps the iterative method approximates mainly solution components associated with the largest singular values. Then gradually the smaller singular values start to influence the solution and finally after many steps, those singular values that are close to zero are approximated, which makes the solution explode. Therefore iterative methods may give useful results if terminated early.

4.1 Classical Stationary Methods

One of the first iterative regularization methods is the Landweber iteration, which is a variant of the steepest-descent method for least square problems, see [24, chapter 6]. Such methods are known as stationary methods. The Landweber iteration method has very slow convergence compared to Krylov subspace methods. The Landweber method is most used in signal and image processing, see e.g., [6, 54] and references therein. The iterative method proposed in [15] also can be viewed as a Landweber type method.

4.2 Krylov Subspace Methods

Some of the best known Krylov subspace methods are the conjugate gradient (CG), Arnoldi and generalized minimum residual (GMRES) methods, see [58]. The CG algorithm is used for solving sparse symmetric positive definite (SPD) linear systems [30]. In [42, 57] a CG type method based on minimizing a certain functional is proposed to solve a backward heat conduction problem. For a description about the regularization property of CG we refer to [24, chapter 7].

The Arnoldi method is used to solve both large and sparse linear systems and eigenvalue problems by projecting the problem onto a Krylov subspace of smaller dimension $m < n$. Given $K \in \mathbb{R}^{n\times n}$ and $g \in \mathbb{R}^n$ the *Krylov subspace* of dimension m is defined by

$$\mathcal{K}_m(K, g) = \text{span}\{g, Kg, K^2 g, \dots, K^{m-1}g\}.$$

Let $v_1 = g/\|g\|$ then the orthonormal basis $V_m = [v_1, \dots, v_m]$ of Krylov subspace $\mathcal{K}_m(K, g)$ can be obtained one vector at a time by computing Kv_j and orthonormalizing this vector against all previous v_i's by modified Gram-Schmidt procedure. This gives a relation of the form

$$KV_m = V_m H_m + h_{m+1,m} v_{m+1} e_m^T, \tag{1.6}$$

where H_m is a $m \times m$ Hessenberg matrix containing the orthonormalization coefficients. Further by regularizing the Hessenberg matrix we can find a good approximation of the original problem recovered from the solution of projected problem [7, 23, 27, 51].

The GMRES method is based on the Arnoldi recursion and can be used for solving linear systems of equations (1.1) with a arbitrary (non-symmetric) square matrix K. At step n we approximate the exact solution $f = K^{-1}g$ by a vector $f_m \in \mathcal{K}_m(K, r_0)$ such that residual

$$\|r_m\|_2 = \|g - Kf_m\|_2,$$

is minimized, see Paper III and the references therein. The regularizing properties of the GMRES method have recently been studied in several papers [10, 14, 13, 37, 43, 9].

Usually preconditioning (for well-posed problems) is implemented to speed up the convergence process by clustering the singular values of the iteration matrix. In our work in Paper III preconditioning (for an ill-posed problem) is meant to reduce the number of iterations required to reconstruct the information from the large singular values, may improves the quality of the computed solution. But the small singular values still remain and should be suppressed. This means that the preconditioner should only precondition the well-posed part of operator (cluster the larger singular values of the operator) and it must not precondition the ill-posed part of the operator. In some papers the preconditioner of ill-posed problems acts upon the ill-posed part of the problem with the identity operator [33, 32]; in Paper III we simply truncate the ill-posed part. The problem in Paper III is a two-dimensional sideways parabolic equation with variable coefficients. As a preconditioner we take the solution of an ill-posed Cauchy problem for an equation corresponding to original ill-posed problem with constant coefficients. Thus we regularize the problem by regularizing the preconditioner as well as truncating the GMRES iteration.

5 Choice of Regularization Parameter

The regularization parameter in any regularization scheme is a quantity that is used to control the degree of regularization of the solution. Techniques for choosing the regularization parameter can be considered in two categories: those which are based on knowledge of error norms in the data e.g., the Morozov discrepancy principle [47], and those that use just the information in the data, e.g., the L-curve criterion [36] and the method of generalized cross validation [29]. The discrepancy principle is most widely used method for selecting the level of regularization. Basically the discrepancy principle states that the optimal solution is that in which the discrepancy between the true and predicted data is compatible with the estimated data error. However when the estimate of the noise level is not available, then the discrepancy principle cannot be used. For such cases heuristic methods are needed. These parameter choice approaches depend on both the regularization method and the inverse problem in a complex way, i.e., there is not one best approach, see [24] for more details. In this thesis we have used the discrepancy principle and the L-curve method to choose the parameters.

2

Summary of Papers

Paper I: Analysis of an Ill-Posed Cauchy Problem for a Convection-Diffusion Equation

In this paper we investigate mathematical and numerical properties of an ill-posed problem for a convection-diffusion equation. We study the influence of the coefficients on the degree of ill-posedness. The problem is reformulated as a Volterra integral equation of the first kind with a smooth kernel. Then we compute numerically the singular values of the Volterra operator. The rate of decay of the singular values of the integral operator determines the degree of ill-posedness. In this paper it is shown that the sign of the coefficient in the convection term influences the rate of decay of the singular values. The problem is also analyzed by taking the Fourier transform, giving a linear ordinary differential equation. The eigenvalues of the matrix of the differential equation is then computed. Numerical experiments confirm our results.

Paper II: A Sideways Heat Equation Applied to the Measurement of the Gas Temperature in a Combustion Chamber

We propose an approach to approximate the gas temperature in a combustion chamber, when a shielded aspirated thermocouple has been used to calibrate the temperature sensor. The gas temperature and the heat transfer coefficient at the surface of thermocouple are unknown. The mathematical model of the temperature changes inside the thermocouple is described as a sideways heat equation problem. Since this is an inverse problem which is severely ill-posed, regularization methods are accomplished to determine the solution. The gas temperature and the heat transfer coefficient at the surface of thermocouple are unknown. First the coefficient is approximated via a calibration experiment. Since the thermocouple is made of two different materials, magnesium oxide and steel, the coefficients in the inverse problem are functions of radial distance. Therefore the problem in the space domain is divided into two parts. In the first part, where the material is magnesium oxide, the problem is more ill-posed and the inverse problem is reformulated as a Volterra integral equation. In order to rule out unphysical solutions, we impose a monotonicity constraint on the regularized solution when applying Tikhonov regularization in the calibration experiment. Using the approximate heat transfer coefficient and finding the temperature and heat flux at the surface of thermocouple we compute an approximation of the gas temperature with a relatively high accuracy.

Paper III: A Preconditioned GMRES Method for Solving a Sideways Parabolic Equation in Two Space Dimensions

We present a new iterative regularization technique for solving a two-dimensional sideways parabolic equation with variable coefficients using a preconditioned Generalized Minimum Residuals Method (GMRES). Large-scale ill-posed problems are often solved using iterative methods, in particular Krylov methods, e.g. GMRES, where the number of iterations serves as a regularization parameter. But for this type of problem we can not make such an iteration converge to a reasonable solution at all. Therefore we use preconditioned GMRES, with a preconditioner based on a semi-analytic solution formula for the corresponding problem with constant coefficients. Since the preconditioning operator is singular and a pseudo-inverse is used. Regularization is used in the preconditioner as well as truncating the GMRES algorithm. An important feature of the preconditioner is that it is based on a truncated expansion in terms of trigonometric functions. Therefore it can be implemented using fast discrete trigonometric transforms, in combination with the solution of a relatively small number of simple 1D sideways parabolic problems. The Discrepancy principle is used for determining when to terminate the iteration process [10]. Regularizing the preconditioner stabilizes the convergence behavior of the PGMRES iteration makes which in its turn the quality of the solution less sensitive to choice of regularization parameter (number of PGMRES steps). The computed examples indicate that the proposed PGMRES method is well suited for the solution of 2D sideways parabolic problems with variable coefficients.

Paper IV: Numerical Solution of a Cauchy Problem for a Parabolic Equation in Two or more Space Dimensions by the Arnoldi Method

In this paper we consider the numerical solution of a Cauchy problem for a parabolic equation in multi-dimensional space, where the domain is cylindrical in one spatial direction. It is desired to find the lower boundary values from the Cauchy data on the upper boundary. This problem is severely ill-posed. The formal solution is written as a hyperbolic cosine function in terms of a multi-dimensional parabolic (unbounded) operator. The approximate solution is computed by projecting onto a smaller subspace generated by the Arnoldi algorithm applied on the inverse of the operator. A well-posed parabolic problem is solved in each iteration step. Further the hyperbolic cosine is evaluated explicitly only for a small triangular matrix. Numerical examples are given to illustrate the performance of the method.

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