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# On the connectedness of the branch locus of the moduli space of Riemann surfaces

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**Abstract.** The moduli space  $\mathcal{M}_g$  of compact Riemann surfaces of genus  $g$  has structure of orbifold and the set of singular points for such orbifold is the *branch locus*  $\mathcal{B}_g$ . In this article we present some results related with the topology of  $\mathcal{B}_g$ , the connectedness of  $\mathcal{B}_g$  for  $g \leq 8$ , the existence of isolated equisymmetric strata of a given dimension in  $\mathcal{B}_g$  and finally the connectedness of the branch locus of the moduli space of Riemann surfaces considered as Klein surfaces.

**Keywords:** Riemann surface, moduli space, automorphism.

**2000 Mathematics Subject Classification numbers:** 32G15 and 14H15.

## 1 Introduction

The moduli space  $\mathcal{M}_g$  of compact Riemann surfaces of genus  $g$  has structure of complex orbifold since it is the quotient of the Teichmüller space by the discontinuous action of the mapping class group. The set of singular points for the orbifold  $\mathcal{M}_g$  is denominated the *branch locus*  $\mathcal{B}_g$ . In this article we present some new contributions in the understanding of the topology of the branch locus. More precisely we shall study the connectedness of  $\mathcal{B}_g$  for low  $g$ , the existence of isolated equisymmetric strata of a given dimension in  $\mathcal{B}_g$  and finally the connectedness of the branch locus of the moduli space of Riemann surfaces considered as Klein surfaces.

In order to study Riemann surfaces our main tool will be the uniformization by *Fuchsian groups*.

Given a Riemann surface  $X$  of genus  $g > 1$ , we consider the universal covering  $\mathcal{H} \xrightarrow{\pi_1(X)} X$ , where  $\mathcal{H}$  is the complex upperplane. Hence there is a representation  $r : \pi_1(X) \rightarrow \text{Isom}^+(\mathcal{H}) = PSL(2, \mathbb{R})$  such that  $X = \mathcal{H}/r(\pi_1(X))$  and  $r(\pi_1(X))$  is a discrete subgroup of  $PSL(2, \mathbb{R})$  (i.e. a Fuchsian group).

If there is  $g \in PSL(2, \mathbb{R})$ , such that  $r_1(\pi_1(X)) = g(r_2(\pi_1(X)))g^{-1}$ , clearly the Fuchsian groups  $r_1(\pi_1(X))$  and  $r_2(\pi_1(X))$  uniformize the same Riemann surface. The space:

$$\{r : \pi_1(X) \rightarrow PSL(2, \mathbb{R}) : \mathcal{H}/r(\pi_1(X)) \text{ is a genus } g \text{ surface}\} / \text{conjugation in } PSL(2, \mathbb{R})$$

is the Teichmüller space  $\mathbf{T}_g$ . The Teichmüller space  $\mathbf{T}_g$  has complex structure of dimension  $3g - 3$  and is simply connected.

The group  $\text{Aut}^+(\pi_1(X))/\text{Inn}(\pi_1(X)) = \text{Mod}_g$  is the modular group or mapping class group and acts by composition on  $\mathbf{T}_g$ . Now we define the *moduli space* by  $\mathcal{M}_g = \mathbf{T}_g/\text{Mod}_g$ .

The projection  $\mathbf{T}_g \rightarrow \mathcal{M}_g = \mathbf{T}_g/\text{Mod}_g$  is a regular branched covering with *branch locus*  $\mathcal{B}_g$ , in other words  $\mathcal{M}_g$  is an orbifold with singular locus  $\mathcal{B}_g$ . The branch locus  $\mathcal{B}_g$  consists of the Riemann surfaces with symmetry, i. e. Riemann surfaces with non-trivial automorphism group (up for genus  $g = 2$ , where  $\mathcal{B}_2$  consists in the surfaces with automorphisms different from the hyperelliptic involution and the identity). Our goal is the study the topology of  $\mathcal{B}_g$ .

As an example, let us describe  $\mathcal{B}_1$ . Each elliptic surface is uniformized by a lattice  $\{z_1, z_2\}$  with  $\text{Im}(z_1/z_2) < 0\}$ , that can be normalized and then parametrized by a complex number, the

modulus of the base  $\{z_1, z_2\}$ :  $z_1/z_2 = \tau \in \{z \in \mathbb{C} : \text{Im}z > 0\}$ . Then the Teichmüller space is:  $\mathbf{T}_1 = \mathcal{H} = \{\tau \in \mathbb{C} : \text{Im}\tau > 0\}$ .

The modular group  $\text{Mod}_1$  is  $PSL(2, \mathbb{Z})$  and the orbifold  $\mathcal{M}_1$  is the Riemann sphere with a cusp and two conic points, one of isotropy group of order 2 and other one with isotropy group of order 3:  $\widehat{\mathbb{C}}_{2,3,\infty}$ . Then  $\mathcal{B}_1 = \{[i], [e^{2\pi i/3}]\}$ .

It is known that  $\mathcal{B}_2$  is not connected, more concretely R. Kulkarni (see [?]) shows that the curve  $w^2 = z^5 - 1$  is isolated in  $\mathcal{B}_2$ , i. e. this single surface is a connected component of  $\mathcal{B}_2$ . It is easy to show that  $\mathcal{B}_2$  has exactly two connected components (see Section 3). In Section 3 we present the following results: the branch locus  $\mathcal{B}_i$ ,  $i = 3, 4, 7$  are connected,  $\mathcal{B}_5, \mathcal{B}_6$  are connected up an isolated point,  $\mathcal{B}_8$  is connected up two isolated points.

There is a natural stratification of  $\mathcal{B}_g$  by equisymmetric strata:  $\mathcal{M}_g = \bigcup \overline{\mathcal{M}}^{G,a}$ , see Section 2. In such stratification the isolated points of  $\mathcal{B}_g$  are the isolated strata of smaller dimension. After the results on  $\mathcal{B}_g$ , with  $g \leq 8$ , it is natural to ask if the only obstruction to the connectedness of the branch locus is the existence of isolated points. In Section 4 we give a negative answer to the above question studying the possible isolated one-dimension equisymmetric strata. We obtain that such strata exist in  $\mathcal{B}_{p-1}$ , with  $p$  a prime  $\geq 11$ . Furthermore we show that for  $g$  large enough we can find isolated equisymmetric strata of  $\mathcal{B}_g$  of dimension as large as we want.

In [?] we give a geometrical interpretation of the existence of isolated points in  $\mathcal{B}_g$  using the moduli space of Riemann surfaces considered as Klein surfaces,  $\mathcal{M}_g^K$ , i.e. the space of classes of Riemann surfaces of a given genus considering in the same class the surfaces that are conformal or anti-conformally equivalent. In such a context  $\mathcal{M}_g^K$  has also an orbifold structure with branch loci  $\mathcal{B}_g^K$  and there is a two fold covering  $c : \mathcal{M}_g \rightarrow \mathcal{M}_g^K$ . The isolated points in  $\mathcal{B}_g$  are the preimage by  $c$  of some intersection of strata of  $\mathcal{B}_g^K$  (see [?]). In Section 5 we announce that  $\mathcal{B}_g^K$  is connected for every  $g$ .

We just sketch the proof of some results, complete proofs will be published elsewhere.

## 2 Symmetric Riemann surfaces

Let  $X$  be a Riemann surface and assume that  $\text{Aut}(X) \neq \{1\}$ . Hence  $X/\text{Aut}(X)$  is an orbifold and there is a Fuchsian group  $\Gamma \leq PSL(2, \mathbb{R})$ , such that:

$$\mathcal{H} \rightarrow X = \mathcal{H}/\pi_1(X) \rightarrow X/\text{Aut}(X) = \mathcal{H}/\Gamma$$

The algebraic structure of  $\Gamma$  is given by the signature  $s(\Gamma) = (h; m_1, \dots, m_r)$ , where  $h$  is the genus of  $\mathcal{H}/\Gamma$  and  $m_1, \dots, m_r$  are de orders of the conic points of the orbifold  $\mathcal{H}/\Gamma$ .

If  $\mathcal{G}$  is an abstract group isomorphic to the Fuchsian groups of signature  $s = (h; m_1, \dots, m_r)$ , the Teichmüller **space** of Fuchsian groups of signature  $s$  is:

$$\{r : \mathcal{G} \rightarrow PSL(2, \mathbb{R}), \text{ such that } s(r(\mathcal{G})) = s\}/\text{conjugation in } PSL(2, \mathbb{R}) = \mathbf{T}_s.$$

The Teichmüller space  $\mathbf{T}_s$  is a complex ball of dimension  $2g - 3 + r$ .

If  $X/\text{Aut}(X) = \mathcal{H}/\Gamma$  and genus( $X$ ) =  $g$ , there is a natural inclusion  $i : \mathbf{T}_s \subset \mathbf{T}_g$  :

$$r : \mathcal{G} \rightarrow PSL(2, \mathbb{R}), \pi_1(X) \subset \mathcal{G}, r' = r|_{\pi_1(X)} : \pi_1(X) \rightarrow PSL(2, \mathbb{R}).$$

If we have  $\pi_1(X) \triangleleft \mathcal{G}$  then there is a topological action of a finite group  $G = \mathcal{G}/\pi_1(X)$  on surfaces of genus  $g$ . The inclusion  $a : \pi_1(X) \rightarrow \mathcal{G}$  produces  $i_a(\mathbf{T}_s) \subset \mathbf{T}_g$  and the image of  $i_a(\mathbf{T}_s)$  by  $\mathbf{T}_g \rightarrow \mathcal{M}_g$  produces  $\overline{\mathcal{M}}^{G,a}$ , where  $\mathcal{M}_g^{G,a}$  is the set of Riemann surfaces with automorphisms group containing a subgroup acting in a topologically equivalent way to the action of  $G$  on  $X$  given by the inclusion  $a$ .

Furthermore  $\mathcal{M}_g = \bigcup \overline{\mathcal{M}}^{G,a}$  and  $\mathcal{B}_g = \bigcup_{G \neq \{1\}} \overline{\mathcal{M}}^{G,a}$ , such covers are called the equisymmetric stratifications [?].

Since all non-trivial group  $G$  contains subgroups of prime order, we have the following remark that will be very useful in the sequel:

**Remark 1**

$$\mathcal{B}_g \subset \bigcup_{p \text{ prime}} \overline{\mathcal{M}}^{C_p,a}$$

where  $\overline{\mathcal{M}}^{C_p,a}$  is the set of Riemann surfaces of genus  $g$  with an automorphism group containing  $C_p$ , the cyclic group of  $p$  elements, acting on surfaces of genus  $g$  in a fixed way given by  $a$ .

### 3 The connectedness of $\mathcal{B}_g$ for $g \leq 8$

For  $g = 2$ , we have that:  $\mathcal{B}_2 = \{\text{surfaces with } \text{Aut}(X) \supseteq \{id, \text{hyperelliptic involution}\}\}$ .

**Theorem 2**  $\mathcal{B}_2$  has two connected components.

Proof. It is easy to show that the prime order groups that can act on a surface  $X$  of genus  $g$  are:  $C_2$ ,  $C_3$  and  $C_5$ . There are two possible actions of groups of order two: the action topologically equivalent to the hyperelliptic involution and the action with exactly two fixed points (giving as orbit space a surface of genus 1). For  $C_3$  and  $C_5$  there is only a topological type of actions. Hence we have:

$$\mathcal{B}_2 \subset \overline{\mathcal{M}}^{C_2} \cup \overline{\mathcal{M}}^{C_3} \cup \overline{\mathcal{M}}^{C_5}$$

In order to finish the proof of the theorem it is sufficient to show the existence of a surface in  $\overline{\mathcal{M}}^{C_2} \cap \overline{\mathcal{M}}^{C_3}$ . For that we observe that we can construct a finite group of homeomorphisms acting on a topological surface of genus 2 having a homeomorphism of order three and an homeomorphism of order two with exactly two fixed points. For that consider the unit sphere  $S^2$  in the space  $\mathbb{R}^3$  and in  $S^2$  consider the graph  $\delta$  consisting in three geodesic arcs from the north to the south poles and making on the poles three angles  $\frac{2\pi}{3}$ . Let  $U_\varepsilon(\delta)$  be the set of points at distance  $\leq \varepsilon$  and  $X = \partial U_\varepsilon(\delta)$ . On  $X$  acts as group of homeomorphisms the restriction of a group of rotations of  $\mathbb{R}^3$  isomorphic to  $D_3$ , in such a group there are homeomorphisms of order three and homeomorphisms of order two with two fixed points. Finally,  $\overline{\mathcal{M}}^{C_5}$  consists exactly in one surface: the isolated Kulkarni curve, with automorphism group  $\mathbb{Z}_{10}$ .  $\square$

**Theorem 3**  $\mathcal{B}_3$  is connected.

Proof. The prime orders of cyclic actions on surfaces of genus  $g$  are 2, 3 and 7. There are several actions of  $C_2$  and  $C_3$ , each topological action is determined for the genus  $h$  of the orbit surface of the action. For  $C_2$  there are three actions where  $h = 0, 1, 2$  and for genus 3 two actions with  $h = 0, 1$ . For order 7 there are two different actions the two producing as quotient a sphere. Hence:

$$\mathcal{B}_3 \subset \bigcup_{h=0}^2 \overline{\mathcal{M}}^{2,h} \bigcup_{h=0}^1 \overline{\mathcal{M}}^{3,h} \bigcup_{i=1}^2 \overline{\mathcal{M}}^{7,a_i}$$

Considering finite groups of rotations in the space  $\mathbb{R}^3$  and surfaces embedded in  $\mathbb{R}^3$  and invariant by such rotation groups it is possible to show that  $\overline{\mathcal{M}}^{2,0} \cap \overline{\mathcal{M}}^{2,1} \cap \overline{\mathcal{M}}^{2,2} \neq \emptyset$  and  $\overline{\mathcal{M}}^{2,1} \cap \overline{\mathcal{M}}^{3,1} \neq \emptyset$ .

A family of surfaces in  $(\bigcup_{h=0}^2 \overline{\mathcal{M}}^{2,h}) \cap \overline{\mathcal{M}}^{3,0}$  is uniformized by the kernel of  $\theta : \Delta \rightarrow C_6 = \langle a : a^6 = 1 \rangle$ , where  $\Delta$  is a Fuchsian group with signature  $(0; 2, 3, 3, 6)$  and  $\theta(x_1) = a^3$ ,  $\theta(x_2) = a^4$ ,  $\theta(x_3) = a^4$ ,  $\theta(x_4) = a$ .

Finally there are exactly two surfaces of genus 3 having automorphisms of order 7. One is the Klein quartic  $K$ , where  $\text{Aut}(K) = \text{PSL}(2, \mathbb{Z}_7)$ , hence with order two and three automorphisms, and the other one is a surface with automorphism group  $\mathbb{Z}_{14}$ , hence admitting an involution.  $\square$

**Theorem 4** ([?])  $\mathcal{B}_4$  is connected.

Sketch of the proof. It is more involved. For instance the number of equisymmetric strata for genus 4 is 41 (see [?]).

The prime integers  $p$  such that  $C_p$  acts on a surface of genus 4 are: 2, 3 and 5. Let  $X$  be a Riemann surface of genus 4 where there is  $C_p \leq \text{Aut}(X)$ , we denote by  $h$  the genus of  $X/C_p$ . There are three possible actions of order two classified by  $h$  and giving the equisymmetric strata:  $\overline{\mathcal{M}}^{C_2, h}$ , where  $h = 0, 1, 2$ . Two actions of order three determined by the genus of the orbit space, producing the strata  $\overline{\mathcal{M}}^{3, h}$ ,  $h = 1, 2$  and two classes of actions of order three with  $h = 0$ :  $\overline{\mathcal{M}}^{3, 0, i}$ ,  $i = 1, 2$ . Finally there are three actions of order 5 groups, all of them with  $h = 0$ ,  $\overline{\mathcal{M}}^{5, 0, i}$ ,  $i = 1, 2, 3$ . Hence:

$$\mathcal{B}_4 \subset \bigcup_{h=0}^2 \overline{\mathcal{M}}^{2,h} \bigcup_{i=1}^2 \overline{\mathcal{M}}^{3,0,i} \bigcup_{h=1}^2 \overline{\mathcal{M}}^{3,h} \bigcup_{i=1}^3 \overline{\mathcal{M}}^{5,0,i}$$

Using the Singerman list of non-maximal signatures of Fuchsian group it is possible to establish the following inclusions:  $\overline{\mathcal{M}}^{3,2} \subset \overline{\mathcal{M}}^{2,1}$ ,  $\overline{\mathcal{M}}^{5,0,2} \subset \overline{\mathcal{M}}^{2,2}$ ,  $\overline{\mathcal{M}}^{5,0,3} \subset \overline{\mathcal{M}}^{2,2}$ . Thus

$$\mathcal{B}_4 \subset \overline{\mathcal{M}}^{2,0} \cup \overline{\mathcal{M}}^{2,1} \cup \overline{\mathcal{M}}^{2,2} \cup \overline{\mathcal{M}}^{3,0,1} \cup \overline{\mathcal{M}}^{3,0,2} \cup \overline{\mathcal{M}}^{3,1} \cup \overline{\mathcal{M}}^{5,0,1}$$

We denote by  $\mathcal{F}^G$  a family of Riemann surfaces with group of automorphisms isomorphic to  $G$ . Now the connectedness is a consequence of the following facts:

1.  $\overline{\mathcal{M}}^{5,0,1} \subset \overline{\mathcal{M}}^{2,0} \cap \overline{\mathcal{M}}^{2,2}$
2. The existence of  $\mathcal{F}^{C_6 \times C_2}$  such that  $\mathcal{F}^{C_6 \times C_2} \subset \overline{\mathcal{M}}^{2,1} \cap \overline{\mathcal{M}}^{2,2} \cap \overline{\mathcal{M}}^{3,0,2}$ .
3. The existence of  $\mathcal{F}^{D_6}$  such that  $\mathcal{F}^{D_6} \subset \overline{\mathcal{M}}^{2,2} \cap \overline{\mathcal{M}}^{3,0,1}$ .
4. The existence of  $\mathcal{F}^{D_3 \times C_3}$  such that  $\mathcal{F}^{D_3 \times C_3} \subset \overline{\mathcal{M}}^{3,0,2} \cap \overline{\mathcal{M}}^{3,1}$ .
5. The existence of  $\mathcal{F}^{D_3 \times D_3}$  such that  $\mathcal{F}^{D_3 \times D_3} \subset \overline{\mathcal{M}}^{3,0,1} \cap \overline{\mathcal{M}}^{3,2}$ .  $\square$

The following result present further results obtained recently:

**Theorem 5** [?]  $\mathcal{B}_5, \mathcal{B}_6$  are connected with the exception of an isolated point,  $\mathcal{B}_7$  is connected and  $\mathcal{B}_8$  is connected with the exception of two isolated points.  $\square$

## 4 Isolated strata of dimension $> 0$

The isolated strata of the equisymmetric stratification are defined as follows:  $\mathcal{M}_g^H$  is an isolated stratum if and only if  $\overline{\mathcal{M}}_g^{H, a_H} \cap \overline{\mathcal{M}}_g^{G, a_G} = \emptyset$ ,  $(G, a_G) \neq (H, a_H)$ . Note that in order to be isolated strata must be  $H = C_p$  with  $p$  a prime. The isolated strata of smaller dimension, i.e. the isolated points has been studied by R. Kulkarni in 1991. Kulkarni shows that the isolated points appear in  $\mathcal{B}_g$  when  $2g + 1$  is an odd prime distinct from 7 ([?])

After the results in Section 3 is natural to ask if  $\mathcal{B}_g$  is connected up isolated points. But the answer is negative by constructing isolated strata of dimension 1. We present the following complete result:

**Theorem 6** ([?]) *The branch locus  $\mathcal{B}_g$  of the moduli space of Riemann surfaces of genus  $g$  contains isolated connected components of (complex) dimension 1 if and only if  $g = p - 1$ , with  $p$  a prime  $\geq 11$ .*

Sketch of the proof. Let  $s$  be the signature  $(0; p, p, p, p)$ , note that  $\dim \mathbf{T}_s = 1$ . Let  $\Delta$  be a group with signature  $(0; p, p, p, p)$  and  $\theta : \Delta \rightarrow C_p = \langle \gamma : \gamma^p = 1 \rangle$  defined by  $\theta(x_1) = \gamma, \theta(x_2) = \gamma^i, \theta(x_3) = \gamma^j, \theta(x_4) = \gamma^{p-1-i-j}$ , where  $1 < i < j$  are integers \*\*\*. Then  $\ker \theta$  is a surface group of genus  $p - 1$ , i.e. isomorphic to the fundamental group of a surface of genus  $p - 1$ . The inclusion  $i_\theta : \ker \theta \rightarrow \Delta$  produces  $i_\theta(\mathbf{T}_s) \subset \mathbf{T}_g$  and the *isolated one-dimensional strata are the image of  $i_\theta(\mathbf{T}_s)$  by  $\mathbf{T}_g \rightarrow \mathcal{M}_g$* . By the way of construction it is also shown that these are the only possible one dimensional isolated strata.

The following result shows that there are isolated strata of large dimension.

**Theorem 7** *Let  $p$  be a prime and  $d > 1$  be an integer such that  $\frac{(d+2)(d+1)}{2} \not\equiv 0 \pmod{p}$ ,  $p > (d+2)^2$ , then there are isolated equisymmetric strata of dimension  $d$  in  $\mathcal{M}_{\frac{(d+1)(p-1)}{2}}$ .*

Proof. We denote  $g = \frac{(d+1)(p-1)}{2}$ . Let  $s$  be the signature  $(0; p, d+3, p)$ . Let  $\Delta$  be a group with signature  $s$  and  $\theta : \Delta \rightarrow C_p = \langle \gamma : \gamma^p = 1 \rangle$  defined by  $\theta(x_i) = \gamma^i$ ,  $i = 1, \dots, d+2$  and  $\theta(x_{s+3}) = \gamma^{-\frac{(d+2)(d+1)}{2}}$ . Then  $\ker \theta$  is a surface group of genus  $g$ . The inclusion  $i_\theta : \ker \theta \rightarrow \Delta$  produces  $i_\theta(\mathbf{T}_s) \subset \mathbf{T}_g$  and we want to show that the *image of  $i_\theta(\mathbf{T}_s)$  by  $\mathbf{T}_g \rightarrow \mathcal{M}_g, \mathcal{M}^{C_p, \theta}$* . If  $\mathcal{M}^{C_p, \theta}$  is not isolated then there in a surface  $X$  in  $\mathcal{M}^{C_p, \theta}$  admitting an automorphism group  $G \not\cong C_p$ . Since by [?]  $C_p$  is normal in  $G$ , then there is an action of  $G/C_p$  on  $X/C_p$  producing a finite automorphism  $\alpha$  of  $\pi_1 O(X/C_p)$  such that  $\theta \circ \alpha$  must be  $\beta \circ \theta$ , where  $\beta$  is an automorphism of  $C_p$ , i.e.  $\theta \circ \alpha(x) = (\theta(x))^j$ , where  $j \in \{1, \dots, p-1\}$ . By the way of construction of  $\theta$  and the condition that  $p > (d+2)^2$  such an automorphism does not exist.

## 5 On the connectedness of the branch locus of the moduli space of Riemann surfaces considered as Klein surfaces.

Let  $X$  be a surface of genus  $g > 1$  and  $r_i : \pi_1(X) \rightarrow Isom^+(\mathcal{H}) = PSL(2, \mathbb{R})$ ,  $i = 1, 2$  be two representations, with  $r_i(\pi_1(X))$  discrete subgroups of  $PSL(2, \mathbb{R})$  and  $\mathcal{H}/r_i(\pi_1(X))$  is homeomorphic to  $X$ . The Fuchsian groups  $r_1(\pi_1(X))$  and  $r_2(\pi_1(X))$  uniformize equivalent Klein surfaces if there is  $g \in Isom^\pm(\mathcal{H})$ , such that  $r_1(\pi_1(X)) = g(r_2(\pi_1(X)))g^{-1}$ . The space of classes of representations  $r : \pi_1(X) \rightarrow PSL(2, \mathbb{R})$ , such that  $\mathcal{H}/r(\pi_1(X)) \simeq X$ , by conjugation in  $Isom^\pm(\mathcal{H})$  is the Teichmüller space  $\mathbf{T}_g^K$ .

The group  $Aut^\pm(\pi_1(X))/Inn(\pi_1(X)) = Mod_g^\pm$  acts by composition on  $\mathbf{T}_g^K$  and we define the Moduli space of Riemann surfaces considered as Klein surfaces by  $\mathcal{M}_g^K = \mathbf{T}_g^K / Mod_g^\pm$ .

The projection  $\mathbf{T}_g^K \rightarrow \mathcal{M}_g^K$  is a regular branched covering with *branch locus  $\mathcal{B}_g^K$* , in other words  $\mathcal{M}_g^K$  is an orbifold with singular locus  $\mathcal{B}_g^K$ . Note that there is a two fold branched covering  $\mathcal{M}_g \xrightarrow{2:1} \mathcal{M}_g^K$ .

**Theorem 8**  *$\mathcal{B}_g^K$  is connected for every  $g \geq 2$ .*

Sketch of the proof *The branch loci admits a cover  $\mathcal{B}_g^K = \bigcup \overline{\mathcal{M}}^{C_p, a}$ . In each  $\overline{\mathcal{M}}^{C_p, a}$  there are Riemann surfaces  $X$  with  $Aut^\pm(X) \cong D_p$  and such that  $X/Aut^\pm(X)$  is a surface with boundary, then such surfaces have reflections. Now the theorem follows from the fact that the locus of Riemann surfaces with reflections (real locus) is connected. ([?], [?]).  $\square$*

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