A Parametric Method for Modeling of Time Varying Spectral Properties

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Abstract

The problem to track time-varying properties of a signal is studied. The somewhat contradictory notion of “time-varying spectrum” and how to estimate the “current” spectrum in an on-line fashion is discussed. The traditional concepts and relations between time and frequency resolution are crucial for this problem. An adaptive estimation algorithm is used to estimate the parameters of a time-varying autoregressive model of the signal. It is shown how this algorithm can be equipped with a feature such that the time-frequency resolution trade-off favors quick detection of changes at higher frequencies and has slower adaptation at lower frequencies. This should be an attractive feature and similar to, for example, what wavelet transform techniques achieve for the same problem.

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1 Introduction

It is a basic problem in many applications to study and track the time-varying properties of various signals. This is at the heart of adaptation and detection mechanisms, and there is a rich literature on this subject, e.g., [13] and [10].

In many contexts it is very attractive to describe the signal characteristics in the frequency domain, i.e. its spectral properties. The spectrum is itself an averaged, time-invariant concept and generalization to a “time-varying” spectrum is somewhat tricky. One aspect of this problem lies in the well known frequency-time uncertainty relation, i.e. that the frequency resolution depends on the time span.

We will argue that it is natural to demand a quicker response, i.e. better time resolution from the adaptive algorithm, at the high frequency than at the low frequency end. In other words we seek a frequency dependent time resolution of our algorithm. This, as such, is nothing new. A typical use of the wavelet transform is exactly to have different trade-offs between time and frequency resolution in different frequency bands.

From this perspective we shall examine current parametric adaptation algorithms and see if they can offer this desired feature. It will turn out that the most used adaptation algorithms – Least Mean Squares (LMS) and Recursive Least Squares (RLS) – do not give this kind of trade-off: The time-window for RLS is frequency independent, while for LMS it depends on the level of the spectrum (not the frequency).

The major point of this contribution is, however, that a frequency-time trade-off of the desired type can be achieved also in parametric modeling. The key is to use a Kalman-filter based algorithm with a carefully tailored state noise covariance matrix.

The paper is organized as follows: In Section 2 we discuss the notion of a “time-varying” spectrum and how it can be formalized, and in Section 3 we make some brief comments on methods for non-parametric spectrum modeling. Section 4 then deals with techniques for parametric spectrum modeling, and in Section 5 we show how the time and frequency resolution can be characterized in terms of a frequency dependent window size. Based on these observations we then, in Section 6, propose an algorithm where a desired trade-off between time and frequency resolution can be included. The proposed technique is then illustrated on both simulated data and real speech data in
Section 7. Finally some conclusions are given in Section 8.

2 Time-Varying Spectra

Consider a signal \( y(t) \), which we for this discussion take to be observed in discrete time:

\[
y(t) \quad t = 0, 1, \ldots
\]  

(1)

One of the most successful ways to describe the properties of \( y(t) \) is to study its spectrum

\[
\Phi_y(\omega) = \sum_{k=-\infty}^{\infty} R_y(k) e^{-ik\omega}
\]  

(2)

where

\[
R_y(k) = \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} y(t) y(t-k)
\]  

(3)

assuming that the limit exists for all \( k \). There is of course a huge literature on how to estimate and utilize spectra. See for example [7].

Now, the spectrum is inherently a time-invariant property, or a time-averaged property. If the signal has time-varying properties - whatever that means - they won’t show up in (2), other than in a time-averaged fashion. Nevertheless we may want to capture "time-varying properties" in spectral terms, at least intuitively. There are many attempts to describe such time varying spectra,

\[
"\Phi_y(\omega,t)"
\]  

(4)

from simple spectrograms (using spectral estimates computed from finite and moving blocks of observed data) to sophisticated transforms of various kinds. Lately there has been a substantial interest in the wavelet transforms also as a means to capture some variant of (4). We shall briefly comment on some of these approaches in the next section.

We can think of \( \Phi_y(\omega, t) \) as a "snapshot" of the signal’s frequency contents at time moment \( t \). It is clear, though, that due to the uncertainty relationship between time and frequency there will be problems to interpret what a "momentary frequency" might be. Let us here introduce a formal definition of \( \Phi_y(\omega, t) \) that in itself is non-contradictory.
We shall assume that the signal \( y(t) \) is generated from a stationary signal source \( e(t) \) as an AR-process:

\[
A_t(q)y(t) = e(t)
\]  

or, in longhand,

\[
y(t) = -a_1(t)y(t-1) - \cdots - a_n(t)y(t-n) + e(t)
\]  

where

\[
A_t(q) = 1 + a_1(t)q^{-1} + \cdots + a_n(t)q^{-n}
\]

Here \( e(t) \) is white noise with variance \( r_2 \) and \( q^{-1} \) is the inverse shift operator. For the signal \( y(t) \), generated by (6), we define the momentary spectrum as

\[
\Phi_y(\omega, t) = \frac{r_2}{|A_t(e^{i\omega})|^2}
\]

In [8] the authors use the term *instantaneous spectrum* for this quantity. This is an exact definition of a momentary spectrum, but the question is whether (8) captures what we intuitively have in mind with the concept "spectrum". We can make two rather obvious observations around this:

- "A quick change" in the spectrum at low frequencies is rather to be interpreted as a high frequency component in the signal.

- To be perceived as a variation of the spectrum at a certain frequency the rate of change must be significantly slower (a factor 10 or so) than the frequency itself.

All this is of course well in agreement with well-known practical ways of handling "time-varying spectra". In amplitude- or frequency-modulation, the modulating signal must change much slower than the carrier. That will also allow the signal to pass with the carrier through the band pass filters designed for the carrier.

The bottom line of this discussion is thus: While (6)-(8) make perfect sense as a formal definition, it is only meaningful as a definition of "time-varying spectra" if the time variation of \( A_t(q) \) is such that \( \Phi_y(\omega, t) \) changes significantly slower than the frequency \( \omega \) in question.
3 Non-Parametric Spectrum Modeling

Among all proposed approaches to spectrum estimation, see for example [6, 7, 4], the simplest one is based on the squared magnitude of a Fourier transform of the signal, commonly referred to as the *periodogram*,

\[ \hat{\Phi}_y(\omega) = |\mathcal{F}(y)|^2. \]

The simplest way to extend this approach to achieve a time-frequency representation of a signal \(y(t)\) is to use the Short-Time Fourier Transform (STFT)

\[ Y^{STFT}(\omega, t) = \int y(\tau)w_N(t - \tau)e^{-j\omega \tau}d\tau \] (9)

That is, we take the Fourier transform of the time-windowed signal, where the time window has compact support on an interval of length \(N\). Thus, we here use \(N\) data points to form the estimate.

A related but conceptually different method that recently has met considerable interest is the Wavelet Transform (WT). Surveys of the wavelet theory are provided by [2, 12, 3]. The *basic wavelet* \(w_N(t)\) is a bandpass function of effective time width \(N_0\). If we assume that the Fourier transform of \(w_N(t)\) is essentially concentrated to its center frequency, say \(\omega_0\), then we have a time-frequency representation,

\[ Y^{WT}(\omega, t) = \int y(\tau)\sqrt{\frac{\omega}{\omega_0}}w_N\left(\frac{\omega}{\omega_0}(\tau - t)\right)d\tau. \] (10)

That is, the window size is frequency dependent

\[ N(\omega) = \frac{\omega_0 N_0}{\omega} \]

so that we have a narrower time window at higher frequencies.

4 Parametric Spectrum Modeling

Introducing the regression vector

\[ \varphi(t) = (-y(t-1), \ldots, -y(t-n))^T \] (11)
and the parameter vector

\[ \theta(t) = (a_1(t), a_2(t), \ldots, a_n(t))^T \]  \hspace{1cm} (12)

the signal \( y(t) \) can be expressed as the linear regression

\[ y(t) = \varphi(t)\theta(t) + e(t) \]  \hspace{1cm} (13)

If we introduce

\[ W(\omega) = (e^{i\omega}, \ldots, e^{in\omega})^T \]  \hspace{1cm} (14)

the momentary spectrum can be written

\[ \Phi_y(\omega, t) = r_2^T \frac{1}{1 + W^*(\omega)\theta(t)} \]  \hspace{1cm} (15)

where * denotes complex conjugated transpose. It is now clear that particular assumptions about the rate of change in the parameters

\[ \Delta \theta(t) = \theta(t) - \theta(t - 1) \]  \hspace{1cm} (16)

lead to different properties of the change in the momentary spectrum

\[ \Delta \Phi_y(\omega, t) = \Phi_y(\omega, t) - \Phi_y(\omega, t - 1). \]  \hspace{1cm} (17)

Below some different assumptions on \( \Delta \theta(t) \) will be examined.

Viewing the parameter vector \( \theta(t) \) as the state vector in a dynamical system, where the state equation, from equation (16), is given by

\[ \theta(t) = \theta(t - 1) + \Delta \theta(t) \]  \hspace{1cm} (18)

and the observation equation is (13) the parameter estimation problem can be seen as a state estimation problem. The optimal parameter estimate is then given by the Kalman filter. See for example [10]. The Kalman filter applied to the system defined by (18) and (13) results in the update equation

\[ \dot{\theta}(t) = \dot{\theta}(t - 1) + K(t)\varepsilon(t) \]  \hspace{1cm} (19)

\[ \varepsilon(t) = y(t) - \varphi^T(t)\dot{\theta}(t - 1) \]  \hspace{1cm} (20)
where
\[
K(t) = \frac{P(t-1)\varphi(t)}{\hat{r}_2 + \varphi^T(t)P(t-1)\varphi(t)}
\] (21)

and
\[
P(t) = P(t-1) - \frac{P(t-1)\varphi(t)\varphi^T(t)P(t-1)}{\hat{r}_2 + \varphi^T(t)P(t-1)\varphi(t)} + \hat{R}_1
\] (22)

The design variables \(\hat{R}_1\) and \(\hat{r}_2\) denote the assumed covariance matrix of the parameter variations and the assumed driving noise variance respectively, and a key issue when applying the Kalman filter algorithm is to assign suitable values to these variables.

Another common choice of gain vector \(K(t)\) is given by the recursive least squares (RLS) method, see [10]. Here the parameter estimate is updated according to (19) and the gain vector is
\[
K(t) = \frac{P(t-1)\varphi(t)}{\lambda + \varphi^T(t)P(t-1)\varphi(t)}
\] (23)

and
\[
P(t) = \frac{1}{\lambda} [P(t-1) - \frac{P(t-1)\varphi(t)\varphi^T(t)P(t-1)}{\lambda + \varphi^T(t)P(t-1)\varphi(t)}]
\] (24)

The variable \(\lambda\) denotes the so called forgetting factor, which is used to control the length of the update step in the algorithm, and it is typically chosen in the interval \(0.9 \leq \lambda \leq 1\).

Finally, the third algorithm to be studied is the so called least mean squares (LMS) algorithm. See, for example, [13] for details. The LMS algorithm corresponds to choosing
\[
K(t) = \gamma \varphi(t)
\] (25)

where \(\gamma\) is a positive scalar.

It should also be emphasized that the RLS and LMS algorithms can be interpreted as special cases of the Kalman filter algorithm with particular \textit{ad hoc} choices of the matrix \(\hat{R}_1\). This is discussed in [9].
5 Time Windows in Parametric Spectrum Modeling

It is not immediate how to define a time window for a parametric model of the spectrum. Our definition is based on matching the uncertainty for the optimal estimate in a time-window of length $N(\omega)$, assuming time-invariance, to the uncertainty of an adaptive estimate. As in the non-parametric case above, we choose to study a spectral factor of the spectrum, which we denote $Y(\omega, t)$ to stress the resemblance to the non-parametric approach. A parametric estimate of the spectral factor is hence given by

$$Y(\omega, t) = \frac{\sqrt{r_2}}{1 + W^*(\omega)\theta(t)},$$

where $\theta(t)$ is obtained from some parametric estimation method.

In the off-line (OL) time-invariant case it can be shown, see [1] and [11], that asymptotically in $n$ and $N$ the variance of the optimal estimate using $N$ data points, i.e.

$$Y^{OL}(\omega) = \frac{\sqrt{r_2}}{1 + W^*(\omega)\theta_N}$$

is given by

$$\text{Var} Y^{OL}(\omega) \approx \frac{n}{N} \Phi_y(\omega)$$

(27)

Similar asymptotic variance expressions for adaptive algorithms are derived in [5]. The assumptions are that the model order $n$ is large and the adaptation is “slow”, which means that $\hat{R}_1$, i.e. the assumed covariance matrix of the parameter variation $\Delta \theta(t)$, in the Kalman filter and the step size $\gamma$ in LMS are small and that the forgetting factor $\lambda$ is close to one. For details concerning the RLS and LMS algorithms and the Kalman filter interpretation of the parameter estimation problem we refer to, for example, [10].

Then, for RLS, LMS and the Kalman filter

$$\text{Var} Y^{RLS}(\omega, t) \approx \frac{n(1 - \lambda)}{2} \Phi_y(\omega, t)$$

(28)

$$\text{Var} Y^{LMS}(\omega, t) \approx \frac{n\gamma}{2} \Phi_y^2(\omega, t)$$

(29)

$$\text{Var} Y^{KF}(\omega, t) \approx \frac{n}{2} \Phi_y^{3/2}(\omega, t) \sqrt{\frac{1}{n} W^*(\omega)\hat{R}_1 W(\omega)} \frac{\hat{r}_2}{\hat{r}_2}$$

(30)
respectively. The derivations and the approximations involved are detailed in [5]. Comparing these variances to (27) gives effective time windows as follows:

\[
N_{RLS}(\omega) = \frac{2}{1 - \lambda} \\
N_{LMS}(\omega) = \frac{2}{\gamma \Phi_y(\omega)} \\
N_{KF}(\omega) = \frac{2}{\sqrt{\Phi_y(\omega) \sqrt{\frac{1}{n} W^*(\omega) \hat{R}_1 W(\omega)}}}.
\]

Choosing \(\lambda\) close to one in RLS hence corresponds to using a large set of data in the parameter estimation and obtaining high accuracy (low variance) in the spectrum estimate. Running the LMS algorithm means that we use a data window with frequency dependent width. In a region with high signal energy, i.e., where \(\Phi_y(\omega)\) is large, the window is short. We stress that for LMS the time-window depends on the signal the algorithm is applied to while it for the RLS algorithm only depends on the design variable \(\lambda\). Also for the Kalman filter the spectrum of the observed signal affects the window width since \(N(\omega)\) is inversely proportional to the square root of the signal spectrum. We however also see that the design variable \(\hat{R}_1\) can be given a frequency domain interpretation, and this will give us a method to affect the width of the equivalent data window. The following example illustrates a simple non-adaptive choice of \(\hat{R}_1(t)\) which gives a frequency dependent window in (33).

**Example 1** Assume that \(\hat{R}_1\) is a Toeplitz matrix

\[
\hat{R}_1 = \begin{pmatrix}
1 & -0.4 & 0 & \ldots & 0 \\
-0.4 & 1 & -0.4 & 0 & \ldots & 0 \\
0 & -0.4 & 1 & -0.4 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}
\]

Then we have

\[
\frac{1}{n} W^*(\omega) \hat{R}_1 W(\omega) = 1 - \frac{n-1}{n} 0.8 \cos(\omega) \approx 1 - 0.8 \cos(\omega).
\]

This shows that \(N(\omega)\) contains a frequency dependent, but signal independent, factor for this particular choice of \(\hat{R}_1\). This choice of \(\hat{R}_1\) is ad-hoc, and a more systematic design is detailed below.
6 Shaping the Time Window in the Kalman Filter

6.1 Transforming spectral variations to $\hat{R}_1$

Before the proposed algorithm is presented, we will give a theorem for how a model of the spectral variation should be transformed to a suitable value of $\hat{R}_1$ in the Kalman filter. This means that we assume how the quantity

$$p_\phi(\omega, t) \triangleq E \left| \Delta \Phi y(\omega, t) \right|^2$$

is distributed over frequency and transform this variation back to parameter variations, represented by $\hat{R}_1$.

**Theorem 1** Suppose the spectral variations $p_\phi(\omega, t)$ are specified independently at the frequencies $\omega_1, \ldots, \omega_n$. The parameter noise covariance matrix $\hat{R}_1 = E(\Delta \theta \Delta \theta^T)$ of an AR($n$) model which gives the desired spectral variation is then approximately for small $|p_\phi(\omega, t)|$ given by

$$\hat{R}_1(t) = \Psi_{\theta(t-1)}^{-T} \hat{R}_\phi \Psi_{\theta(t-1)}^{-1}$$

where

$$\Psi_{\theta(t-1)} = (\psi_{\theta(t-1)}(\omega_1), \ldots, \psi_{\theta(t-1)}(\omega_n))$$

$$\hat{R}_\phi = \text{diag}(p_\phi(\omega_1, t), \ldots, p_\phi(\omega_n, t))$$

$$\psi_{\theta(t-1)}(\omega) = \frac{\overline{W(\omega)}(1 + W^T(\omega)\theta(t - 1)) + W(\omega)(1 + W^*(\omega)\theta(t - 1))}{1 + W^*(\omega)\theta(t - 1)}$$

and $W(\omega)$ is defined in (14). Furthermore $\overline{\cdot}$ denotes complex conjugate.

**Proof 1** Using a first order Taylor expansion we can express the momentary spectrum as

$$\Phi y(\omega, t) = \Phi y(\omega, t - 1) + \psi_{\theta(t-1)}^T(\omega) \Delta \theta(t)$$

where $\psi_{\theta(t-1)}(\omega)$ is the gradient of $\Phi y(\omega)$ with respect to the parameter vector $\theta$ evaluated at $\theta(t - 1)$. Then recalling (17) gives

$$\Delta \Phi y(\omega, t) = \psi_{\theta(t-1)}^T(\omega) \Delta \theta(t)$$

10
and consequently
\[ | \Delta \Phi_y(\omega, t) |^2 = \psi_{\hat{\theta}(t-1)}^T(\omega) \Delta \theta(t) \Delta \theta^T(t) \psi_{\theta(t-1)} \]  \hspace{1cm} (43)

Thus, we have
\[ p_\theta(\omega, t) = \psi_{\hat{\theta}(t-1)}^T(\omega) \hat{R}_1 \psi_{\theta(t-1)}(\omega) \]  \hspace{1cm} (44)

Since independence is assumed, we obtain a diagonal matrix $\hat{R}_\phi$ given by
\[ \hat{R}_\phi = \text{diag}(p_\phi(\omega_1, t), \ldots, p_\phi(\omega_n, t)) \]  \hspace{1cm} (45)

By evaluating the vector $\psi_{\theta(t-1)}$ in these frequency points and using (44) we get the equation
\[ \hat{R}_\phi = \Psi_{\theta(t-1)}^T(\omega) \hat{R}_1 \Psi_{\theta(t-1)} \]  \hspace{1cm} (46)

where
\[ \Psi_{\theta(t-1)} = (\psi_{\theta(t-1)}(\omega_1), \ldots, \psi_{\theta(t-1)}(\omega_n)) \]  \hspace{1cm} (47)

From equation (46), the corresponding parameter covariance matrix $\hat{R}_1$ in (37) follows.

Finally, writing
\[ \Phi_y(\omega) = \frac{\hat{r}_2}{(1 + W^*(\omega)\theta(t-1))(1 + W^T(\omega)\theta(t-1))} \]  \hspace{1cm} (48)

it is easily checked that $\psi_y = \frac{d}{d\theta} \Phi_y$ is given by equation (40).

Remark: The vector $\psi_y$ is a function of the true unknown parameter vector $\theta(t-1)$ but a feasible alternative is to evaluate $\psi_y$ in the current parameter estimate $\hat{\theta}(t-1)$.

6.2 The proposed algorithm

As have been argued previously in this paper, the frequency dependent time window in the wavelet transform is intuitively very appealing. On the other hand, parametric methods have found many practical applications where the wavelet transform cannot be applied. If we interpret parametric methods as special choices of $\hat{R}_1$ in the Kalman filter, a disadvantage is that $\hat{R}_1$ has to be tuned from case to case and that there is no generically good choice of it.

In this section we will try to combine the advantages of the wavelet transform and the parametric methods. We propose to use the same assumptions as in the wavelet
transform. That is, the spectral variations in the interval \([\pi/4, \pi/2]\) is half as fast as in the interval \([\pi/2, \pi]\), and so on. This is done by picking out the center frequencies in each interval,

\[
\omega_i = \frac{3\pi}{2^n}, \quad i = 1, 2, \ldots, n,
\]

and letting

\[
p_\phi(\omega, t) = \frac{C}{2^m}.
\]

The matrix \(\hat{R}_1(t)\) is then formed as

\[
\hat{R}_1(t) = \Psi^{-T}_{\hat{\theta}(t-1)} \hat{R}_\phi \Psi^{-1}_{\hat{\theta}(t-1)}
\]

where

\[
\hat{R}_\phi = C \text{diag}(\frac{1}{2^n}, \frac{1}{2^n}, \ldots, \frac{1}{2^n})
\]

Furthermore

\[
\Psi_{\hat{\theta}(t-1)} = (\psi_{\hat{\theta}(t-1)}(\frac{3\pi}{2^n}), \ldots, \psi_{\hat{\theta}(t-1)}(\frac{3\pi}{2^n}))
\]

where

\[
\psi_{\hat{\theta}(t-1)}(\omega) = \hat{r}_2 \frac{2 \text{Re}(\tilde{W}(\omega)(1 + W^T(\omega) \hat{\theta}(t-1)))}{|1 + W^*(\omega) \hat{\theta}(t-1)|^4}
\]

The only parameters left to choose are the constant \(C\) in the spectral variation and the measurement noise variance \(\hat{r}_2\). From the construction of \(\hat{R}_1(t)\) it can be seen that it is the ratio between \(C\) and \(\hat{r}_2\) that affect the algorithm gain. Hence one of these parameters can be fixed while the other may be used as the user knob to tune the trade-off between time and frequency resolution.

Since the norm of \(\hat{R}_1(t)\) may vary very much, it has turned out from simulations that it is very difficult to tune the filter in this way. A better choice is to use

\[
\hat{R}_1(t) = \mu \frac{\hat{R}_1(t)}{\|\hat{R}_1(t)\|},
\]

The influence of \(C\) and \(\hat{r}_2\) is now eliminated and a new design parameter, \(\mu\), is introduced. This new design parameter can be used to adjust the base-length of the window, while the shape is given by \(\hat{R}_1(t)\), and it has turned out to be as simple to tune as the counterpart in the Kalman filter with constant \(\hat{R}_1\).

We are now ready to summarize the proposed algorithm:
• Select frequencies \( \omega_i = \frac{3\pi}{2^n} \), \( i = 1, 2, \ldots, n \) where \( n \) is the model order.

• Form the matrix
  \[
  \hat{R}_\Phi = \text{diag}(\frac{1}{2^2}, \frac{1}{2^4}, \ldots, \frac{1}{2^{2n}})
  \]

• Select \( \mu \).

• At each time step carry out the following steps.
  
  - Form the matrix
    \[
    \Psi_{\hat{\theta}(t-1)} = (\psi_{\hat{\theta}(t-1)}(\frac{3\pi}{2^1}), \ldots, \psi_{\hat{\theta}(t-1)}(\frac{3\pi}{2^n}))
    \]
    using the current parameter estimate.
  
  - Compute
    \[
    \hat{R}_1(t) = \Psi_{\hat{\theta}(t-1)}^{-T} \hat{R}_\Phi \Psi_{\hat{\theta}(t-1)}^{-1}
    \]
  
  - Compute
    \[
    \hat{\hat{R}}_1(t) = \mu \frac{\hat{R}_1(t)}{\| \hat{R}_1(t) \|}
    \]
  
  - Update the parameter estimates according to equations (19), (21) and (22) using \( \hat{R}_1(t) \).

7 Numerical Illustrations

In this section we will first illustrate the performance of the new algorithm on the simplest possible example and then try to analyze the result on a real speech signal and compare it to other methods.

Example 2 We start with the simplest possible example to focus on the principles of the method.

• The test signal is generated by a second order AR model, which is also the chosen model structure in the filtering. Hence, we here allow perfect modeling.

• The poles of the AR model are to start with located close to the unit circle, and then suddenly moved towards the origin. In this way, we can compare how the tracking ability of the adaptive parametric algorithm performs, compared to for instance RLS, by varying the resonance frequency.
• The poles of the three tested models are shown in Figure 1. The resonance frequencies are $3\pi/4$, $3\pi/8$ and $3\pi/16$, which are the three highest center frequencies in the perfect wavelet transform. The magnitude of the poles are changed at time 100 from 0.99 to 0.90.

• 100 simulations were performed for 200 data, and the mean parameter estimate was computed.

• The first plot in Figure 2 shows the magnitude of the poles in the estimated AR model. As seen, the step response is slower for lower frequencies. As comparison, the second plot shows the same estimate for the RLS algorithm with forgetting factor 0.98 and the third plot shows the standard Kalman filter with $R_1 = 10^{-4} \cdot I$. Here, there is no visible frequency dependence.

• We remark that the relative difference in the convergence properties of the three methods depends on the tuning parameters, and it is not interesting in this context where we investigate frequency dependences.

Example 3 The purpose of this example is to compare parametric and non-parametric methods for spectral analysis and test the new algorithm on a real signal.

• We will examine a speech signal where the letter $s$ is pronounced like “ess”. The signal is shown in Figure 3. First we have silence for about 1000 samples, then the e-sound for about 1500 samples followed by the high-frequency dominated s-sound and then silence again.

• Figure 3 shows also the spectrogram computed using a hamming window of width 30 for three segments of the speech signal, corresponding to silence (with some quantified measurement noise), e- and s-sound. We note that the e-sound has two frequency peaks at 0.2 and 1.5 rad/s, respectively, and the s-sound is dominated by the peak at 3 rad/s.

• We will now examine how some of the discussed methods are able to find these spectral peaks and what the time responses look like. It should be noticed that we will not try to optimize the different tuning parameters to get as good tracking performance as possible, because that is another rather subjective matter.
Instead, the design parameters in the parametric methods will be tuned to get an approximate window size of 2000 for the lowest resonance peak for the e-sound.

- The three plots to the right in Figure 4 show the result of
  - RLS with forgetting factor 0.99,
  - the Kalman filter with $R_1 = \frac{1}{16} 10^{-4} \cdot I$ and $R_2 = 1$,
  - the new algorithm with $\|R_1\|/R_2 = 10^{-4}$.

The resonance peak is in the interval $[\pi/8, \pi/4]$ where the window size is increased a factor $4^2$ compared to the highest frequencies, which explains the factor 1/16 in the Kalman filter. As seen, this peak frequency is tracked almost identically by these three methods. Note that the new Kalman filter algorithm is faster than the standard formulation to track the high frequency peaks.

- The left column of plots in Figure 4 shows the non-parametric methods
  - DFT with sliding rectangular window of width 2000,
  - wavelet transform using a 16-points approximation of the ideal high-pass filter as mother wavelet.

For the windowed DFT, the time-resolution is better compared to the parametric methods. On the other hand, the frequency resolution is poorer.

- An interesting question is how to control the tracking ability in the wavelet transform. The current plot is not fair to compare to the other methods, because of the very good time resolution and poor frequency resolution. Obviously, the only parameter which influences the tracking ability in the wavelet transform is the sampling period – at least if we consider the mother wavelet as given. In the highest octave, $[\pi/2, \pi]$, the time window width is approximately 2. Thus, in the third octave containing the peak frequency in the e-sound, the window width is $2^3 = 8$. To get the desired window width of 2000, we would have to increase the sample rate a factor 256!
8 Conclusions

In this contribution, we have focused on the time and frequency resolution of several parametric methods for spectral estimation, with the terminology used in the non-parametric context. In the parametric approach, we computed the spectrum from a recursively estimated AR model. It was shown that the time windows – that is, the effective number of samples used to compute the spectrum at a certain frequency – for common adaptive methods as LMS, RLS and the Kalman filter are inherently frequency independent. The time resolution depends only on the design parameters and the spectrum itself.

We have argued that the time resolution should increase with higher frequencies, similar to the wavelet transform. The proposed method is based on the Kalman filter interpretation of parameter estimation, where the state noise covariance matrix is adapted recursively. This new algorithm was compared to other approaches for both a simulated signal and a speech signal.

Finally we would like to stress that we by the proposed method offer a default choice of $\hat{R}_1$ which is not ad hoc, in contrast to the RLS and LMS algorithms.

References


Figure 1: Pole configuration of the three test signals before (close to the unit circle) and after the spectral change. * - poles at frequency $3\pi/16$. o - poles at frequency $3\pi/8$. x - poles at $3\pi/4$. 
Figure 2: The magnitude of the estimated poles.

Upper figure: The new algorithm. 1 - poles at frequency $3\pi/16$. 2 - poles at $3\pi/8$. 3 - poles at $3\pi/4$. Middle figure: The RLS algorithm. Lower figure: Standard Kalman filter.
Figure 3: The speech signal of “ess” and time-invariant spectrograms for three hand-picked segments.

Figure 4: The speech signal and time-frequency representations. The two lower figures in the left column contain non-parametric methods using windowed DFT and wavelet transform, and the right column parametric methods using RLS, Kalman filter and the frequency selective Kalman filter.