IDENTIFICATION ASPECTS OF INTERSAMPLE INPUT BEHAVIOR

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Abstract. In this contribution aspects of inter-sample input signal behavior are examined. The starting point is that parametric identification always is performed on basis of discrete-time data. This is valid for identification of discrete-time models as well as continuous-time models. The usual assumptions on the input signal are: i) it is band-limited, ii) it is piecewise constant or iii) it is piecewise linear. One point made in this paper is that if a discrete-time model is used, the best possible (in the model structure) adjustment to data is made. This is independent of the assumption on the input signal. However, a transformation of the obtained discrete model to a continuous one is not possible without additional assumptions on the input signal. The other point made is that the frequency functions of the discrete models very well coincide with the frequency functions of the discretized continuous time models and the continuous time transfer function fitted in the frequency domain.

Key Words. System identification; discrete time systems; frequency domain; inter-sample assumptions.

1 INTRODUCTION

Parametric identification of time-continuous systems is always performed using discrete-time data. The output of a dynamical system at a sampling instant depends of course on the input at all previous times, not only its values at the sampling instants. To obtain a non-ambiguous description of the output at the sampling instants one must thus be able to reconstruct the input at all times from its values at the sampling instants. In this contribution some aspects of this problem are studied.

In the next section some variants of how to reconstruct the input between the samples will be reviewed. In Section 3 the two most common ones in the time domain are treated: viz zero-order hold and first-order hold assumptions on the input signal. The issues are illustrated using Schoukens’ electrical machine data (Schoukens and Pintelon, 1991) in Section 4.

2 RECONSTRUCTING THE TIME-CONTINUOUS INPUT AND RELATING CONTINUOUS-TIME AND DISCRETE TIME DESCRIPTIONS

When and how can the time-continuous input be reconstructed exactly from the sampled values only? How does this affect the discrete-time (sampled) representation of the system that underlies all time-domain estimation methods? These are questions that are dealt with in this section.

Suppose that a time-continuous, causal linear system is given by

$$G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_n}{s^n + a_1 s^{n-1} + \ldots + a_n}$$

with the impulse response

$$g(\tau),$$

or the time state space representation of the transfer function

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where the "direct term" is $D = 0$, if and only if $b_0 = 0$. Given sampled values of the output $y(kT)$, $k = 1, \ldots, N$, the following relation can be stated (assuming $u(t) \equiv 0$ for $t < 0$)

$$y(kT) = \int_0^{kT} g(kT - \tau)u(\tau)d\tau$$

Suppose also that the input $u(\tau), 0 \leq \tau \leq NT$ can be reconstructed from the sampled values
If the reconstruction is linear, the reconstruction rule is given by
\[ u(\tau) = \sum_{k=1}^{N} r_k(\tau)u(kT) \quad (4) \]
and if it in addition is causal it can be written as
\[ u(\tau) = \sum_{k=1}^{\lfloor \tau/T \rfloor} r_k(\tau)u(kT) \quad (5) \]
where \( \lfloor x \rfloor \) denotes the integer part of \( x \). In the linear case the following relation is obtained from (2) and (5)
\[ y(kT) = \sum_{\ell=1}^{N} \hat{g}(k, \ell)u(\ell T) \quad (6) \]
where the summation only goes to \( k \) in the causal case (5). Depending on the reconstruction rule this may or may not be written as a finite dimensional (possibly non-causal) system
\[ y(kT) = \frac{\hat{B}(q)}{\hat{A}(q)}u(kT) \quad (7) \]
where \( \hat{B} \) and \( \hat{A} \) are polynomials in the shift operator \( q \):
\[ q u(kT) = u(kT + T) \]
and also the discrete-time version takes a simple form (7).

An alternative is to assume that the input is \textit{piecewise linear}, i.e.
\[ u(\tau) = u(kT) + \frac{u(kT + T) - u(kT)}{T} (\tau - kT) \quad (9) \]
for \( k \leq \tau/T < k + 1 \)

This reconstruction is also linear. It is however causal only for \( \tau = \ell T, \ell = 1, \ldots, N \), but this still leads to a causal representation of the sampled system (6), (7) at the sampling instants.

The two approaches (8) and (9) are often called \textit{zero-order hold} (zoh) and \textit{first-order-hold} (foh) respectively, and they will be further discussed in the next section.

One can of course continue to define second- and higher-order interpolation rules, splines etc., for the reconstruction of the input. This is straightforward conceptually, but could typically lead to non-causal representations.

Now, what are the practical aspects of these reconstructions? If one is in full control of the identification experiments it is natural to either let the input be band-limited (constructing it as a sum of sinusoids) or to be piecewise constant. The choice is dictated by practical considerations and partly by tradition in different areas.

A further question is what happens when the input cannot be reconstructed from its sampled values. (The same issues apply when an incorrect reconstruction rule is applied, e.g. foh when the input really is band-limited).

The answer will be that the model makes the best out of the actually applied reconstruction rule. Any input reconstruction error will be treated as noise and lumped together with other disturbances. One can thus fairly easily evaluate how serious this problem is: Check how big, for example, the discrepancy between the true input and a linearly interpolated one are, and compare that with other disturbances present.

### 3 Zero-Order Hold and First-Order Hold Relations

Assume that a system is given on the form
\[ x = Ax + Bu \]
\[ y = Cx + Du \quad (10) \]
with \( x \in \mathbb{R}^n, y \in \mathbb{R} \) and \( u \in \mathbb{R} \). If the system is driven by a piecewise constant input signal (zoh) as in (8) or a piecewise linear input signal (foh)
as in (9), the corresponding sampled system can be described as

\[ x(t+1) = Fx(t) + G_1u(t) + G_2u(t+1) \]
\[ y = Cx + Du \]  \hspace{1cm} (11)

with

\[ F = e^{AT}, G_1 = \int_0^T e^{Ad}Bdt, \quad G_2 = 0. \]

in the piecewise constant case and

\[ F = e^{AT}, G_1 = \int_0^T \frac{t}{T} e^{A(T-t)}Bdt, \]
\[ G_2 = \int_0^T (1 - \frac{t}{T}) e^{A(T-t)}Bdt \]

in the piecewise linear case. Clearly, (11) with \( G_2 \neq 0 \) can be put into the standard form (10) of the same order as if \( G_2 = 0 \).

The transfer function for the sampled system is described by

\[ H(z) = C(zI-F)^{-1}(G_1 + zG_2) + D \]

The general discrete input-output relation can then be written

\[ A(z)y(t) = B(z)u(t) \]  \hspace{1cm} (12)

with \( A(q) = q^n + a_1q^{n-1} + \ldots + a_n \) and \( B(q) = b_0q^n + b_1q^{n-1} + \ldots + b_n \). The relations between the coefficients \( a_i, b_i \) and the coefficients in \( F, C, G_1, G_2 \) and \( D \) or the original continuous coefficients in \( A, B, C, D \) are complicated. However, it can be stated that if the assumption that the input signal is of zero-order type is used and the continuous system with \( D = 0 \) is given, then the coefficient \( b_0 \) in the discrete-time model is zero. If \( D \neq 0 \) or if a first order hold is used then usually \( b_0 \neq 0 \).

The transition from continuous-time models to discrete-time models is unambiguous, when the input-signal assumption is fixed. The transition from discrete-time models to continuous time models, however, is more complicated. To calculate the continuous-time model knowledge about the input-signal assumption has to be incorporated. Nothing can be said only on the basis of discrete-time model coefficients.

See Fig. 1 for a illustration of the relations between the continuous-time and discrete-time models and corresponding input signal assumptions.

![Fig. 1. Possible transitions from continuous time models to discrete time models and vice versa.](image)

The methods tried out were the following:

1. A discrete-time output error method, using the oe command in SITB in a straightforward fashion. No time-delay was assumed, and models of different orders were tried out. They are called oe1du, oe2du etc., and have the form of the type (second order example)

\[ \frac{b_0 + b_1q^{-1} + b_2q^{-2}}{1 + a_1q^{-1} + a_2q^{-2}}. \]

2. A continuous-time state space model, parameterized in canonical form. The model is of output error character and has a direct term:

\[ \dot{x} = \begin{bmatrix} \theta_1 & \theta_2 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \]

![CONTINUOUS TIME DISCRETE TIME](image)

<table>
<thead>
<tr>
<th>xx</th>
<th>Model structure</th>
<th>oe/ss</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>Model order</td>
<td></td>
</tr>
<tr>
<td>t</td>
<td>Type of model</td>
<td>c/d</td>
</tr>
<tr>
<td>p</td>
<td>Pre-processing</td>
<td>u/f</td>
</tr>
<tr>
<td>a</td>
<td>Input assumption</td>
<td>z/f</td>
</tr>
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</table>

Table 1. Table explaining the notational convention regarding the estimated models in this section. oe/ss – output error model/state space model. Model order – degree of \( A(q) \) or dimension of the \( A \)-matrix in (10). c/d – continuous model/discrete model. u/f – unfiltered signals/filtered signals. z/f – zoh assumption/foh assumption.

The methods tried out were the following:
\[ y = [\theta_0 \theta_1] x + \theta_2 u + e \]

and correspondingly for higher order models. The models are fitted to discrete-time data assuming that the input is piecewise constant, using \( \text{ms}2\text{th} \) (in its 'czoh' option) and \( \text{pem} \) of the \text{STB}. The models are denoted by \( \text{ssnduz} \), where \( n \) is the order. The corresponding sampled models are denoted by \( \text{ssnduf} \).

3. Same as above, but assuming that the input is piecewise linear. This is obtained in the \text{STB} as above, but using the 'cfcoh' option in \( \text{ms}2\text{th} \). The models are denoted by \( \text{ssncuz} \), where \( n \) is the order. The corresponding sampled models are denoted by \( \text{ssnduf} \) in the sampled variants.

4. A continuous time input-output model was fitted in the frequency domain to Fourier transformed data. This was obtained in MATLAB as follows (fourth order example):

\[
\begin{align*}
    U &= \text{fft}([\text{ze}(1,2);]) \; ; \; \; V = \text{fft}([\text{ze}(1,1);]) \\
    g &= 0.2999 / 3000 \cdot 2 \pi; \\
    [b, a] &= \text{invfreqs}(g, w, 4, \text{abs}(U) \cdot 2, 20);
\end{align*}
\]

The models are denoted by \( \text{mf}n \), \( n \) being the order (4 in the example above).

5. Same as above, but only using frequencies up to the half Nyquist frequency (i.e. the 750 first points above). These are models denoted by \( \text{mf}nb \), \( n \) being the model order.

6. When modeling the system using second order models, the data is filtered in some cases. A low-pass filter (Chebyshev type) with the cut-off frequency at 0.95 \( \cdot \) \( f_N \), where \( f_N \) is the Nyquist frequency is used.

In Table 2 the resulting models in the first order case are shown. The coefficients of the discrete \( \text{oe} \)-model and the discretized first order continuous models are almost identical.

In Table 3 it is shown how the second order and fourth order models are able to reproduce the validation data.

In Fig. 2-5 the Bode plots of the different 4\(^{th} \) order models are shown.

The following conclusions can be drawn:

- Despite the fact that the frequency domain approach handles the band limited data in a formally correct way, the resulting model still shows some artifacts around the Nyquist frequency. The "strange" behavior of \( \text{mf}4 \) around \( \omega = \pi / T \) is explained by the fact that the phase should be a multiple of \( \pi \) at

<table>
<thead>
<tr>
<th>Model</th>
<th>Fit</th>
<th>Trans. func.</th>
</tr>
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<tbody>
<tr>
<td>oe1du</td>
<td>0.0157</td>
<td>([-0.0022 \quad 0.0000])</td>
</tr>
<tr>
<td>ss1duz</td>
<td>0.0157</td>
<td>([-0.0022 \quad 0.0000])</td>
</tr>
<tr>
<td>ss1duf</td>
<td>0.0157</td>
<td>([-0.0022 \quad 0.0000])</td>
</tr>
</tbody>
</table>

Table 2. Comparison of the first order models. The discrete-time oe-model and the discretized (foh and zoh) continuous-time models are shown. The second column (Fit) shows the mean square error between the simulated and measured data. Note that the measured data is not the same as was used in the estimation of the model. The third column shows the transfer functions of the models, but the argument \( q \) is suppressed.

<table>
<thead>
<tr>
<th>Model</th>
<th>Fit</th>
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<tbody>
<tr>
<td>oe2du</td>
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<tr>
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</tr>
<tr>
<td>ss2duz</td>
<td>0.0157</td>
</tr>
<tr>
<td>ss2duf</td>
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<td>ss2dfz</td>
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<td>ss2diff</td>
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<td>oe4du</td>
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<tr>
<td>ss4duz</td>
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<td>ss4duf</td>
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</tr>
<tr>
<td>mf4</td>
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</tr>
<tr>
<td>mf4b</td>
<td>0.0391</td>
</tr>
</tbody>
</table>

Table 3. Second and fourth order models tried out are shown, together with their resulting fits when tested against validation data. The syntax for the models can be found in the beginning of this section.

Fig. 2. Good agreement of the discretized continuous models and the frequency function estimate up to the Nyquist frequency. Bode plot showing the following curves: \( \text{mf}4 \), \( \text{ss4duz} \) and \( \text{ss4duf} \).
Fig. 3. Importance of not including the Nyquist frequency when estimating the frequency function directly in the frequency domain. Bode plot showing the frequency functions estimated on the whole discrete Fourier transform $mf4$ (the line with jumpier phase curve), and the frequency function estimated on the first half of the discrete Fourier transform data $mf4b$ (avoiding the peak at the Nyquist frequency, and the fact the the phase has to be a multiple of $\pi$).

$\omega = \pi/T$. This follows since the measured phase shift at $\omega = \pi/T$ can only be a multiple of $\pi$. A comparison between $mf4$ and $mf4b$ shows that it is the inclusion of the frequencies around $\omega = \pi/T$ which gives this behavior. See Fig. 3.

- The sampled data models $ss4duz$ and $ss4duf$ are almost identical and also almost coincide with the continuous-time model $mf4$ up to the Nyquist frequency, see Fig. 2. This confirms:
  - The fit in terms of discrete-time models is identical, and does not depend on what assumptions we have about interpolation rules of the input.
  - The continuous-time model, fitted in the frequency domain really gives a fit of a sampled system! This is due to the Nyquist frequency effect mentioned above.

- One would then expect that also $oe4du$ model would be identical to $ss4duz$ and $ss4duf$. This is not quite the case, as seen in Fig. 4. The reason is that the intrinsically discrete-time model works with poles on the negative real axis. These models have no continuous-time counterpart (the family of nth order discrete-time models is strictly larger than those nth order discrete-time models that can be obtained by sampling (zoh or foh)). Actually, in Table 3 it can be seen that these models, e.g. $oe4du$, that do not have a continuous-time counterpart are clearly better at reproducing data.

- The resulting continuous time models $ss4cuz$ and $ss4cuf$ differ and the first-order hold model is closer to the $mf4b$, that was obtained in the frequency domain, fitted up to $\pi/(2T)$.

Finally, a comparison between the fourth order model and the second order model obtained from filtered data is presented. The Bode plots are shown in Fig. 6.

From Fig 6 it can be concluded that when removing the high-energy component of the input signal near the Nyquist frequency, good results can be obtained by using a lower order model.

Fig. 5. Comparing continuous-time models. From top to bottom; Bode plots of $ss4cuz, ss4cuf$ and $mf4b$.

5 CONCLUSIONS

We have in this contribution studied some identification aspects of inter-sample input behavior. The most important points are that:
Fig. 6. Filtering data, removing the high frequency disturbance, enables finding a good second order model. The models shown are oe2df, ss2dff and oe4du.

- Using discrete-time (sampled) models implies no assumption or prejudice on the continuous-time input, e.g. that it is piecewise constant.

- Some input reconstruction schemes would require non-causal discrete-time models. However, the discrete-time model (estimated by using a prediction error method) in any case does the best possible reconstruction available with the model structure.

- For example, assuming a continuous-time model that is either zero-order hold or first-order hold sampled or a discrete-time model, all lead to the same fitted discrete-time representation, provided it is covered by the model structure. Possible discrepancies, can be due to the fact that first-order hold requires a direct (no-delay) term, and that not all discrete-time models have continuous counterparts.

These aspects have been illustrated on data from an electrical machine.

One could also add that for fast sampled data the assumptions of piecewise constant or piecewise linear inputs can be quite good approximations (reconstruction rules) for any reasonable input signal.

6 REFERENCES


