A Finite Element Model Updating Formulation Using Frequency Responses and Eigenfrequencies

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1. ABSTRACT

A novel frequency and modal domain formulation of the model updating problem is presented. Deviations in discrete frequency responses and eigenfrequencies, between the model to be updated and a reference model, constitute the criterion function. A successful updating thus results in a model with the reference's input-output relations at selected frequencies. The formulation is demonstrated to produce a criterion function with a global minimum having a large domain of attraction with respect to stiffness and mass variations. The method relies on mode grouping and uses a new extended modal assurance criterion number (eMAC) for identifying related modes. A quadratic objective with inexpensive evaluation of approximate Hessians give a rapid convergence to a minimum by the use of a regularized Gauss-Newton method. Physical bounds on parameters and complementary data, such as structural weight, are treated by imposing set constraints and linear equality constraints. Efficient function computation is obtained by model reduction using a moderately sized base of modes which is recomputed during the minimization. Statistical properties of updated parameters are discussed. A verification example show the performance of the method.

2. NOTATION

This paper conforms to the notation proposed by DTA/NAFEMS. Minor deviations from the proposed notation may exist. Symbols not defined by DTA/NAFEMS are given in the running text at first occurrence.

3. INTRODUCTION

The task of creating finite element models (FEM) that are valid for their planned application is nowadays common but not always easy. Depending on the required accuracy more or less validation of the FEM is usually made before the model is used for analyses. Such validation may be driven by authority or contractor requirements, internal corporate quality assurance policies or the engineer's practice and common sense. It is often found during validation, that the model lacks, to a sufficient accuracy, the properties of reference data. Therefor FEM updating naturally becomes a part of the validation process.

In aeronautics, FEM updating is usually initiated after the first, but also subsequent, ground vibration tests of a new aircraft. In this case the reference model data, to which the FEM is compared, are obtained from tests. The most common procedure of obtaining reference data, also for other structures, is based on testing. The reference model will therefor be abbreviated as XM, associating to experimental model.

Aircraft models, developed early in the development phase, tends to be very big (with respect to number of degrees-of-freedom) since their use is mostly aimed at sizing. Therefor there is often a strive for building reduced size models with similar low-frequency characteristics as the more complex ones. Usually these are established in the modal domain, but it may be useful to obtain reduced size models using physical parameters. In this case the more complex FEM may be used as reference model (XM) to which the small FEM should be correlated. Also in the development of new methods for model updating it is common that analytical models are used as XM with known physical property data to which a perturbed FEM shall be adjusted.

In the past ten years a significant research effort has been made on model updating. The methods developed basically divide into two categories, direct methods and iterative, sensitivity based methods. While the direct methods has the advantage of reproducing the test results it suffers from the approximations made during the required mode shape expansion and/or matrix reduction and also that the engineer's physical insight cannot be fully exploited. Sensitivity based methods aim at minimizing the difference between measurements and corresponding FEM quantities using a general parametrization, under physically motivated constraints. This is made by nonlinear programming methods.

Much of the early work was directed towards parametrizing the FEM coefficient matrices. This has been found to be of limited use to the engineer who needs to substantiate the outcome of the update. He also wants to understand the model parameter updates to help him in future modeling efforts. Furthermore, it often occurs that the real structure undergoes minor changes. These are normally represented by a modified FEM without a verifying test. The revision of coefficient matrices correlated to earlier tests may be cumbersome in this case. Recent papers, of more practical use to the engineer, therefor deals with parameters that have a clear physical interpretation.

Choosing a good parametrization, or model structure, may be difficult since different such structures may represent models with large parameter differences even if all correlate well to the XM. An unsuccessful parametrization or insufficient or inaccurate test data may make the model updating problem ill-conditioned. It is therefor important that numerical methods used for model updating could deal with such problems.

The reproduction of correct input/output relations is a primary requirement for a good model and therefor frequency response function (FRF) based methods are appealing. Studies based on such are numerous and includes the studies by Lin and Ewins[1] and Balmès[2]. Larsson and Sas[3] presented a FRF based method and also highlighted important numerical aspects. Lammens[4] elaborates on damping considerations and damping models that are related to FRF based methods. Reviews on model updating in general have been written by Mottershead and Friswell[5] and Ibrahim[6].

This paper deals with general parametrization, *i.e.* no distinction is made between parameters with direct physical interpretation and more mathematically oriented such as elements in governing matrices. A FRF based formulation is used and an efficient numerical procedure for its solution is presented.

4. BASIC THEORY: SYSTEM INPUT/OUTPUT RELATIONS

The input-output relation of a linear second-order time-invariant system, with state excitation v and observation noise w, can be written using matrix notation as

$$M\ddot{x} + G\dot{x} + Kx = \bar{B}u + v \qquad y = \bar{C}\ddot{x} + w \tag{1a,b}$$

Here x is the displacement vector, u is the nonzero partition of global load vector F and y is the output vector. $\bar{B}$ and $\bar{C}$ are force distribution and output selection matrices. An individual element of y may generally represent any physical quantity that depends linearly on the state and the input. In the formulation used here however, the output is assumed to be an acceleration vector to simplify the presentation and justified on the basis that we mostly deal with experimental data gathered with accelerometers. The presence of noise must, of course, be considered in a practical situation. Pre-test planning and post-test analysis of test data should be carried out to diminish its impact on the data to be used.

In the following it will be assumed that the stiffness matrix K and the mass matrix M both depend on the selected general parametrization θ. The non-parametrized viscous damping matrix G is assumed to be of Caughy[7] type, *i.e.* the system's normal modes also diagonalizes the damping matrix. Thus only modal dampings will be considered.

When the low frequency dynamics of lightly damped systems is essential, which is mostly the case for structural dynamicists, the model order of the systems may be reduced. This can be made without significant loss of accuracy, by the use of the low frequency eigenmodes of the corresponding undamped system. Collecting the truncated set of mass-orthonormalized modes in the mode matrix Φ, the reduced-order noise-free model is

$$\ddot{q} + \mathrm{diag}\,(2\zeta_r\omega_r)\,\dot{q} + \mathrm{diag}(\omega_r^2)q = \Phi^T\bar{B}u \qquad y = \bar{C}\Phi\ddot{q} \tag{2a,b}$$

The corresponding frequency domain equation, with the transfer function H relating the input to the output, is

$$Y = \left[C\,(sI - A)^{-1}B + D\right]u \equiv H\,(s)\,U \tag{3a}$$

$$A = \begin{bmatrix} -\mathrm{diag}(2\zeta_r\omega_r) & -\mathrm{diag}(\omega_r^2) \\ \mathrm{I} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} \Phi^T\bar{B} \\ 0 \end{bmatrix} \tag{3b,c}$$

$$C = \bar{C}\Phi\left[-\mathrm{diag}(2\zeta_r\omega_r) \;\; -\mathrm{diag}(\omega_r^2)\right] \qquad D = \bar{C}\Phi\Phi^T\bar{B} \tag{3d,e}$$

The modal properties ω_r and Φ originates from the large eigenproblem $K\Phi = M\Phi\Omega^2$, which may be expensive to compute for a large FEM. A reduction of the problem using $\underline{M} = \Psi^T M\Psi$ and $\underline{K} = \Psi^T K\Psi$, may then be introduced using a truncated eigenmode set in the mode matrix Ψ. This mode set may be computed initially and kept constant during the updating or be recomputed at regular occasions. We have noted, during numerical experimentation, that by recomputing the reduction matrix after each major iteration of the optimization, the convergence properties are better than when it is held constant.

5. MODEL UPDATING FORMULATION

The basic requirement for the criterion function to be used is that, when at minimum, the FEM and XM input/output relations are well correlated. Also since model updating, using incomplete XM, is an iterative process by nature, such criterion function is sought that decreases monotonic to the global minimum over a large parameter domain surrounding it. Moreover, to admit the use of powerful Gauss-Newton iterative methods, it is beneficial if Hessian information can be provided and produced with little computational effort. It will be showed, by example, that the FRF based criterion function suggested here fulfills these requirements.

The magnitude of the system's transfer function is a very rapidly varying function close to resonances. This imply that when computing transfer matrix differences, by any matrix norm, a very steep decrease will occur when the FEM get close to the XM, *i.e.* when the system's poles and residues correlate. The properties of the transfer functions are also such that when evaluated for a FEM more distant from the XM, the attraction to the XM configuration will be smaller, or stated otherwise, the gradient will decrease with the distant from the true configuration. To improve the attraction properties, artificial damping may be introduced. This can be made by evaluating the transfer function not for purely imaginary $s = \mathrm{i}\omega$, but adding to it a small fixed real number, say s_ζ. The transfer matrix, corresponding to Equation (3a), with supplementary artificial damping thus is

$$H(\mathrm{i}\omega) = C[(s_\zeta + \mathrm{i}\omega)I - A]^{-1}B \tag{4}$$

To furthermore increase the attraction far away from the minimum it has been found that resonance deviation terms should be included. A quadratic criterion function, also fulfilling the requirement of providing easily obtainable approximate Hessians, thus is

$$W(\theta) = \sum_{r=1}^{m} w_r^H \left\|{}_F H_r - {}_X H_r\right\|_F^2 + \sum_{r=1}^{m} w_r^f \left|({}_F\omega_r - {}_X\omega_r)\right|^2 \equiv w(\theta)^T w(\theta) \tag{5}$$

Here ${}_F H_r$ and ${}_X H_r$ are the transfer matrices of the FEM and XM and $\|\cdot\|_F$ is the Frobenius norm. How to create the transfer matrices, by use of Equations (3) and (4), should be obvious. If each FRF and resonance frequency component of the XM is polluted with independent zero mean Gaussian noise, and the scalings w_r^H and w_r^f are in accordance with the inverse of the variance, then the minimization of $W(\theta)$ is the maximum-likelihood estimate of the unknown parameters.

The fundamental and, to the authors' knowledge, new idea is that a comparison of the FEM and XM transfer function matrices can be made on a mode by mode basis. This is since the transfer matrix $H(s)$ is a matrix superposition of contributions from decoupled modes. If furthermore, the comparison is made at a single fixed frequency, possibly different for the individual transfer matrix terms, the comparison may be non-expensive to evaluate. Natural choices of such frequencies, where the associated modes of a well correlated FEM should contribute the most, are the XM resonance frequencies. It may also be noted that the transfer function expression when approximated with the contribution from a single mode is very simple and is thus well suited for gradient calculations. An element h_{jk} of the transfer function difference, evaluated at the rth resonance of the XM and including only the corresponding mode, is

$$h_{jk} = \frac{{}_F\phi_{jr}\ {}_F\phi_{kr}}{{}_F\omega_r^2 - {}_X\omega_r^2 + 2i\ {}_X\zeta_r\ {}_X\omega_r\ {}_F\omega_r} - \frac{{}_X\phi_{jr}\ {}_X\phi_{kr}}{2i\ {}_X\zeta_r\ {}_X\omega_r^2} \tag{6}$$

Here it may be noted that the modal damping has been assumed to be equal for the two models. If artificial damping s_ζ is introduced it should be included in ${}_X\zeta_r$.

When the modal contribution to the transfer functions, at the observed resonances, of the two models correlate one can assume that the FEM well represents the dynamics of the tested structure. Should however unresolved multiple modes be present in the XM, the assumption is invalid. Care must thus be taken such that repeated modes are not included in the XM.

Figures 1 and 2 illustrate the fundamental idea using reduced data from the verification test example. In Figure 1a one transfer function, synthesized using modal data from a fictitious test, is shown. Some system modes may be lacking since they are not observable in the test data or neglected during the experimental modal analysis. Figure 1b shows the corresponding transfer function of the FEM. One notes that the modal density is higher here. Figure 2 shows two examples of the transfer functions synthesized from pairs of FEM and XM modes. The frequency samples of the FRF used in the criterion function, at the XM resonances, are highlighted with circles.

The monotonic decrease property of the criterion function, relying on eMAC (see Section 7) based mode pairing, is shown in Figure 3. Here a few of the 17 variables of the given verification example have been varied, in turn, from 50% to 200% of the true parameter value while the rest have been set to their correct values. The two stiffness and inertia parameters with the most influence on the criterion function have been selected. The function behaves similarly for variation of variables not shown. Figure 3 also shows the non-monotonic behavior when the mode tracking fails using standard MAC mode pairing. The pairing of modes is seen to be crucial.

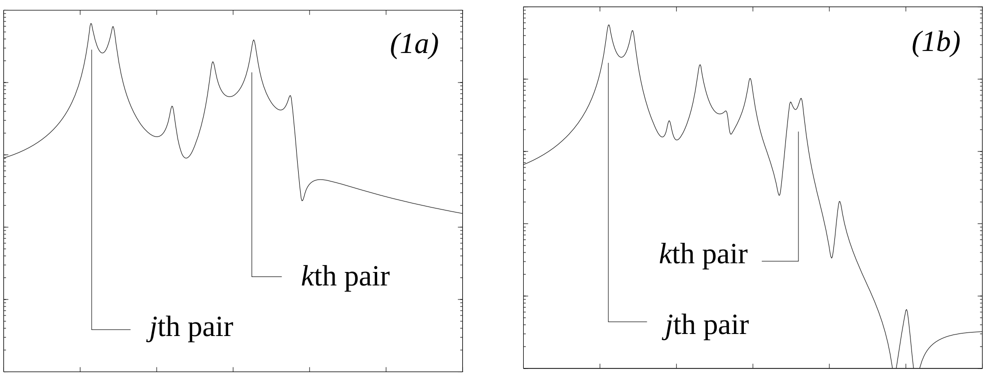


Figure 1. (1a) Synthesized transfer function magnitude of the XM. (1b) Magnitude of transfer function of the FEM using average modal damping from the XM. Marked modes are paired based on the best MAC correlation.

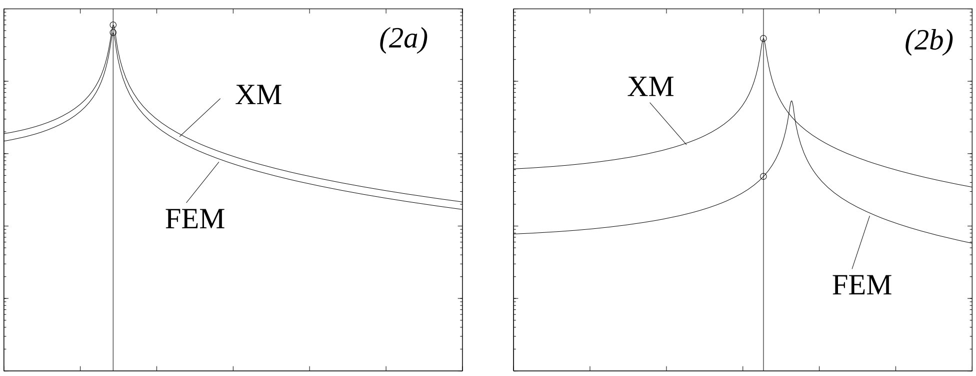


Figure 2. (2a) Transfer function contribution from the jth pair of modes of the FEM and XM. (2b) Transfer function contribution from the kth pair of modes. Both pair of curves should agree for well correlated FEM and XM. Circles and vertical lines indicate the XM resonance frequencies at which the functions are evaluated.

6. GRADIENTS

The quadratic criterion function (5) is seen to be composed of two types of terms, transfer function and resonances frequency terms. For the first type of terms we have

$$\frac{\partial}{\partial\theta_i}(\|{}_FH_r - {}_XH_r\|_F^2) = 2\mathrm{Re}\sum_j\sum_k h_{jk}^* h'_{jk} \tag{7}$$

Here a prime denotes the partial derivative with respect to the *i*th parameter θ_i. The idea of evaluating the transfer functions only at resonance of the XM model results in a simple expression for the gradient of h_{jk}

$$h'_{jk} = \frac{{}_F\phi'_{jr}\,{}_F\phi_{kr} + {}_F\phi_{jr}\,{}_F\phi'_{kr}}{{}_F\omega_r^2 - {}_X\omega_r^2 + 2i\,{}_X\zeta_r\,{}_X\omega_r\,{}_F\omega_r} - \frac{2\,{}_F\phi_{jr}\,{}_F\phi_{kr}\,{}_F\omega_r'\,({}_F\omega_r + i\,{}_X\zeta_r\,{}_X\omega_r)}{({}_F\omega_r^2 - {}_X\omega_r^2 + 2i\,{}_X\zeta_r\,{}_X\omega_r\,{}_F\omega_r)^2} \tag{8}$$

The expression involves gradients of modal quantities which may be computed[8] as

$${}_F\omega_r' = \left[{}_F\phi_r^T K' - {}_F\omega_r^2 M')\,{}_F\phi_r\right] / 2\,{}_F\omega_r \qquad {}_F\phi_r' = \sum_j a_{jr}\,{}_F\phi_j \tag{9a,b}$$

$$a_{jr} = \left[{}_F\phi_j^T(K' - {}_F\omega_r^2 M')\,{}_F\phi_r\right] / ({}_F\omega_r^2 - {}_F\omega_j^2) \qquad j \neq r \tag{9c}$$

$$a_{jr} = -\frac{1}{2}\,{}_F\phi_j^T M'\,{}_F\phi_r \qquad j = r \tag{9d}$$

Other methods for computing the modal gradients are normally preferred[9,10], since the sum in (9b) usually involves many terms if not truncated. However, if model reduction techniques are applied, the number of modes may be limited to (say) a few hundred and the above so-called modal method of computing the mode gradient is feasible. Gradients of the transfer function terms of the criterion function may then be evaluated by computationally efficient methods. The calculation of the resonance frequency gradient terms of the criterion function should be obvious.

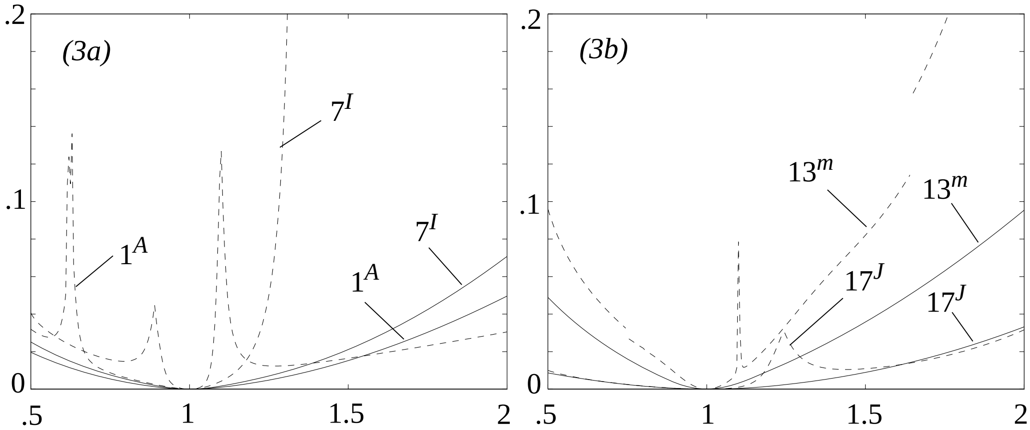


Figure 3. Criterion function vs. selected normalized stiffness and inertia parameters. (3a) shows behavior for stiffness variations for eMAC paired modes (solid) and MAC paired (dashed). (3b) shows similar curves for inertia property variation. Unitary normalized parameters represent correct model. Parameter notation as in Verification Example.

7. MODE PAIRING

Pairing of modes often resorts to maximizing the trace of the MAC (Modal Assurance Criterion) matrix where the individual elements are

$$MAC_{jk} = \cos^2(\mathrm{ang}({}_X\varphi_j, {}_F\varphi_k)) \tag{10}$$

Here ang defines the angle between the two sub-spaces spanned by ${}_X\varphi_j$ and ${}_F\varphi_k$. By rearranging the sequence of analytical modes, by some sorting algorithm, a sequence can be found that fulfills the maximum trace criterion and defines the best pairing of XM and FEM modes. The sorting itself constitutes a complex combinatoric problem and its solution in not further elaborated upon here.

It is well known that eigenvectors corresponding to closely spaced eigenvalues are highly sensitive to perturbations in the mass and stiffness matrices. Also for small perturbations the mode shapes may change drastically. Should this occur, the pairing under the maximum trace criterion becomes cumbersome since it may well be that during such parameter changes the paired FEM mode sequence flip-flops. This imply that a comparison of associated eigenvalues to the paired modes is not meaningful.

To overcome the mode tracking problem when the eigenvalues becomes repeated or almost coalesce, the use of an extended MAC (eMAC) matrix is suggested. To create the eMAC, one creates blocks of eigenmodes each containing modes with eigenvalues that are close to each other (ultimately all considered low frequency eigenmodes), here called multiple modes. Call each such block ${}_F\bar{\Phi}_j = [{}_F\bar{\varphi}_j \;\; {}_F\bar{\varphi}_{j+1} \ldots]$ and repeat each such block with its multiplicity, *i.e.* ${}_F\bar{\Phi}_{j+1} = {}_F\bar{\Phi}_j$ etc.. In the case the eigenvalue is not multiple, its eigenmode itself constitutes an eigenmode block, *i.e.* ${}_F\bar{\Phi}_j = {}_F\bar{\varphi}_j$. The eMAC matrix elements may now be defined in analogy with (10) as

$$eMAC_{jk} = \cos^2(\mathrm{ang}({}_X\bar{\varphi}_j, {}_F\bar{\Phi}_k)) \tag{11}$$

A re-sequence algorithm can now be applied for defining the best possible pairing of experimental modes to a subspace defined by the mode blocks.

If it is found that a pairing to a block of multiple modes is suggested by the algorithm, there still remains the question of which mode and eigenvalue to compare with the corresponding of the XM. It is here suggested that a linear combination of the FEM modes that underlay the block should be used. The combination that most closely adapts to the XM mode, in least square sense, is

$${}_F\bar{\phi}_j = {}_F\bar{\Phi}_j a \qquad {}_F\phi_j = {}_F\Phi_j a \qquad a = {}_F\bar{\Phi}_j^{\dagger}\,{}_X\bar{\varphi}_j \tag{12a-c}$$

Here the row dimensions of the modes ${}_F\bar{\phi}_j$ and ${}_F\phi_j$ are consistent with the XM and FEM respectively and ${}_F\Phi_j$ is the mode matrix of full FEM row dimension. The vector a holds the combination coefficients a_k. An approximation to the eigenvalue may now be computed, using ${}_F\phi_j$ as an assumed mode, by use of the Rayleigh-quotient as

$$\tilde{\omega}_j^2 = \frac{{}_F\phi_j^T K \, {}_F\phi_j}{{}_F\phi_j^T M \, {}_F\phi_j} = \frac{\Sigma a_k^2 \omega_k^2}{\Sigma a_k^2} \tag{13}$$

Since the number of FEM modes often is very large and the number of measurement locations is limited, a truncation of the analytical modes prior to the eMAC calculation is crucial for successful pairing. Using only those modes with eigenvalues in a frequency range well embracing the observed resonance frequencies of the XM is usually sufficient in order to get a correct pairing provided the FEM is not too distant from the XM.

For multiple FEM modes the gradient computations are more complex since they involve gradients of the combination coefficients of equation (12c). The probably least cumbersome method of computing the associated transfer function differences is then by finite difference calculations, suitable for parallel processing.

8. OPTIMIZATION METHOD

The minimization of the criterion function (5) is a non-linear least squares problem. Methods for solving such problems are iterative and each iteration step usually is taken as the solution to a related linear least squares (LLS) problem.

Since the Hessian of $w(\theta)$ is too expensive to compute for large scale problems, we use a Gauss-Newton type method that only need the gradient of $w(\theta)$. By linearizing $W(\theta)$ and adding a regularization term we end up with a sequence of LLS problems

$$\delta\theta_k = \arg\min\{\|w(\theta_k) + w'(\theta_k)\,\delta\theta\|_2^2 + \mu_k^2\|\delta\theta\|_2^2\} \quad \theta_{k+1} = \theta_k + \alpha_k\delta\theta_k \quad \text{(14a,b)}$$

This is the well-known Levenberg-Marquardt method[11] where the matrix w' is the gradient of the error vector w with respect to all parameters. If $\mu_k = 0$ and $\alpha_k = 1$, equation (14) describes the Gauss-Newton iterations. When $\mu_k = 0$ and α_k are determined by line search along the search direction $\delta\theta_k$ it is known as the damped Gauss-Newton method. The Gauss-Newton direction $\delta\theta_k$ is computed under the assumption that the criterion function decreases as a quadratic function.

If the criterion function is insensitive to some parameters or directions in the parameter space, w' is singular or nearly singular. In this case the (damped) Gauss-Newton scheme breaks down since $\delta\theta_k$ will be dominated by these directions. The use of $\mu_k > 0$ will then regularize the solution and prevent the insensitive parameters from dominating the search direction. By letting μ_k increase, the norm of $\delta\theta_k$ decreases and becomes more and more parallel to the steepest descent direction.

The conditioning of w' critically depends on the chosen parametrization and actuator/sensor locations. Although the problem may have a well conditioned solution, w' may eventually be ill-conditioned during the iterative optimization. Since an iteration step involves a possibly ill-conditioned LLS, numerically stable methods should be used. The LLS problem (14) cast in a form suitable for orthogonal transformation to be solved by numerically stable QR-factorization is

$$\delta\theta_k = \arg\min\left\|\begin{pmatrix} w'(\theta_k) \\ \mu_k I \end{pmatrix}\delta\theta + \begin{pmatrix} w(\theta_k) \\ 0 \end{pmatrix}\right\|_2 \equiv \arg\min\|A\delta\theta + b\|_2 \qquad (15)$$

There is no general guideline for the selection of a proper μ_k. Our suggestion is to choose $\mu_k = \varepsilon\|w'(\theta_k)\|_2^2$, where ε is a small number (say $\varepsilon = \sqrt{u}$ where u is the machine precision). This choice of μ_k may be interpreted as an approximated truncation of the singular value decomposition of A, where the singular values below μ_k are truncated[11].

We have found that an extensive line-search along the search direction $\delta\theta_k$ does not pay off. Instead we use a simple fitting of a quadratic interpolation function with interpolation points at the beginning, mid and end of the search range. The search range is $0 < \alpha_k \leq 1$ or bounded by the parameter set constraint. The line-search minimum is taken as either the minimum of the quadratic function (if it is convex and has minimum in the search range) or the end-point of the search range whichever gives the smallest criterion function value. A terminated line-search ends one major iteration cycle.

Linear equality constraints and active set constraints, cast as $L\theta = l$, may be imposed[12] by projecting the search direction $\delta\theta_k$ onto directions $\delta\theta_k^a$ admitted by the constraints as

$$\delta\theta_k^a = \left[I - L^T(LL^T)^{-1}L\right]\delta\theta_k \tag{16}$$

Complementary physical insight, such as knowing the structural weight to be kept constant, may thus be easily accounted for.

9. MEASUREMENT NOISE AND QUALITY OF PARAMETER ESTIMATE

Errors in the XM data affect the outcome of the model updating exercise. Error analysis forms an important part in order to quantify the impact these errors have on the estimated model parameters. Recall that the criterion can be written as $W(\theta) = w(\theta)^T w(\theta)$, where $w(\theta)$ is the vector of modeling errors and let $\hat{\theta}$ be its minimizing argument. Denote by $\overline{W}(\theta)$ the criterion function using noise-free XM data and let θ_0 denote the unique minimizing argument to $\overline{W}$. Then

$$e = w(\theta_0) - \overline{w}(\theta_0) \tag{17}$$

is a vector of the measurement noise. Assume that the criterion is scaled such that each component e_i of e is an independent Gaussian random variable with variance σ^2. If the criterion function is sufficiently smooth it follows that $\hat{\theta} \to \theta_0$ as $\sigma^2 \to 0$ with probability 1.

Let the deviation between the true parameters and the estimate be $\delta = \hat{\theta} - \theta_0$. Since $\hat{\theta}$ is a minimizing argument, $W'(\hat{\theta}) = 0$. The mean value theorem yields

$$0 = W'(\theta_0 + \delta) = W'(\theta_0) + W''(\theta^*)\,\delta \tag{18}$$

where θ^* is between θ_0 and $\hat{\theta}$. This implies that the deviation is

$$\delta = -[W''(\theta^*)]^{-1} W'(\theta_0) \tag{19}$$

Here the function gradient is

$$W'(\theta_0) = 2w'(\theta_0)^T w(\theta_0) = 2\overline{w}'(\theta_0)^T(e + \overline{w}(\theta_0)) = 2\overline{w}'(\theta_0)^T e \tag{20}$$

since $\overline{W}'(\theta_0) = 0$. The statistical variability of the parameters can thus be quantified by

$$\begin{aligned} \lim_{\sigma^2 \to 0} \frac{E\delta\delta^T}{\sigma^2} &= \lim_{\sigma^2 \to 0} \frac{4}{\sigma^2} E\{[W''(\theta^*)]^{-1} \overline{w}'(\theta_0)^T e e^T \overline{w}'(\theta_0)\,[W''(\theta^*)]^{-1}\} \\ &= 4\,[\overline{W}''(\theta_0)]^{-1} \overline{w}'(\theta_0)^T \overline{w}'(\theta_0)\,[\overline{W}''(\theta_0)]^{-1} \end{aligned} \tag{21}$$

Here E denotes the expectation operator. Consequently an approximation of the covariance matrix of the estimate is

$$\mathrm{Cov}(\hat{\theta}) \approx 4\sigma^2\,[\overline{W}''(\theta_0)]^{-1} \overline{w}'(\theta_0)^T \overline{w}'(\theta_0)\,[\overline{W}''(\theta_0)]^{-1} \tag{22}$$

If the model structure is able to exactly model the noise-free XM data, *i.e.* we have a so-called consistent updating problem, then the variance expression can be further simplified. In this case we have

$$\overline{W}''(\theta_0) = 2w'(\theta_0)^T w'(\theta_0) \tag{23}$$

since $\overline{w}(\theta_0) = 0$. The covariance then is

$$\mathrm{Cov}(\hat{\theta}) \approx \sigma^2 \left[\overline{w}'(\theta_0)^T \overline{w}'(\theta_0)\right]^{-1} \tag{24}$$

In practice θ_0 is not known. However, $\hat{\theta}$ could be used to evaluate Equation (22) since $\hat{\theta} \to \theta_0$ when $\sigma^2 \to 0$. Recall that $w'(\theta)$ is already used in the minimization and consequently $\mathrm{Cov}(\hat{\theta})$ is obtained for free. When we deal with consistent updating, the noise variance σ^2 can be estimated by $\hat{\sigma}^2 = W(\hat{\theta})/(n-N)$, where n is the number of terms in W and N is the number of estimated parameters.

A singular or almost singular matrix $w'(\theta_0)^T w'(\theta_0)$ is an indication that the estimation problem is ill-posed or the number of estimated parameters is too large. In order to obtain a well-posed problem, the number of parameters has to be reduced, in accordance with the null-space of $w'(\theta_0)^T w'(\theta_0)$, or the minimization should be carried out using the regularization $\mu_k > 0$. Regularization can be seen as constraining the solution and result in the covariance

$$\mathrm{Cov}(\hat{\theta}) \approx \sigma^2 \left[\overline{w}'(\theta_0)^T \overline{w}'(\theta_0)\right]^{\dagger} \tag{25}$$

Here the pseudo-inverse should be evaluated by setting the singular values of $\overline{w}'(\theta_0)^T \overline{w}'(\theta_0)$ that are smaller than μ_k to zero.

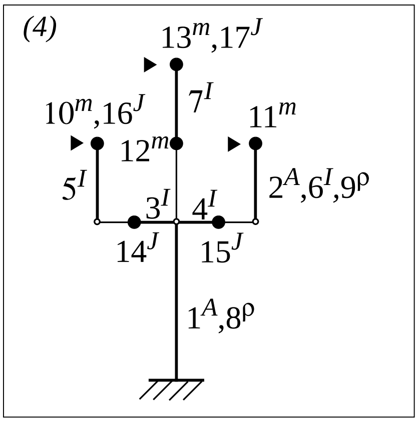
10. SCALING

Since the magnitude of the transfer functions at different resonances may vary over many decades, a scaling of individual transfer function terms may be appropriate. To smoothen out the spread over decades, it has been proposed that the logarithm of the transfer functions should be used instead. However, it has been found that not only do the logarithm function smoothen out peaks in the transfer functions but it also tends to minus infinity for transfer elements that tend to zero. Small transfer function elements thus contribute much to the criterion function, which is undesirable. Instead we have here set the weight factors w_r^H such that all weighted transfer function terms of the criterion function are initially equal.

Criterion function normalization may simplify comparative studies. We have chosen the weight factors w_r^f and w_r^H such that the criterion function is initially set to one. The weight factors have also been set such that the transfer function sum and the frequency deviation sum contribute equally for the initial FEM.

11. VERIFICATION EXAMPLE

A numerical example, previously studied in literature, have been chosen as an illustration. This example was given by Caesar *et.al.*[13]. A plane frame model consisting of nine beam elements, six inertia elements, ten nodes and 27 degrees-of-freedom is studied. The 18 nodal translational accelerations are measured. Three exciter positions are used (see Figure 4) and thus 3x18 transfer functions are obtained. The model updating problem contains some difficulties, such as bad MAC correlation between the FEM and the XM and large model errors in bending stiffness and inertia properties. The initial MAC values of correlated modes varies from 0.442 to 0.919 and the initial parameters are up to 40% off the true values. Frequencies are off by up to 11.4%. A consistent updating is here possible, since the true model is within reach of the selected parametrization. The model properties are given in Reference [13]. Initial values of the parameters, together with correct parameter values and results from the updating procedure are presented in Table 1. Table 1 also gives the parameter type used. The frequency range of interest is here between 0 and 60 Hz and involves 8 structural resonances. Figure 5 and 6 show the iteration history of eigenfrequencies and variables/criterion function. Figure 7 shows examples of the initial, updated and true transfer functions. The updating process was terminated after 10 iterations. No significant improvement was noted after the 6th iteration where the normalized criterion was 0.03%. The subspace for eMAC calculation was generated using all modes in 0-70 Hz range. The model reduction was based on modes up to 280 Hz and involved 17 modes.



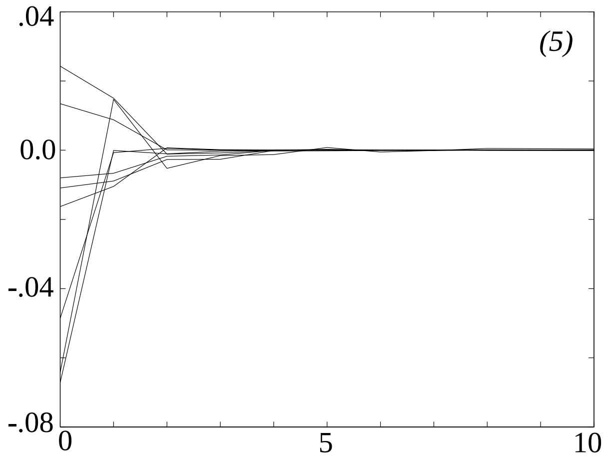


Figure 4. Plane frame finite element model with 9 beam elements and 10 nodes. Nodes instrumented with accelerometers (circles or bullets) and fitted with exciters (arrowheads) are shown. Parameterized beam elements are highlighted and lumped inertia are bulleted. Parameter numbers and types are shown, e.g. 2^A is second parameter (area).

Figure 5. Iteration history of relative resonance frequency deviation between the FEM and the XM. Approximate frequencies from Rayleigh-quotient are used in plot. Maximum frequency error is less than 0.3% in three iterations.

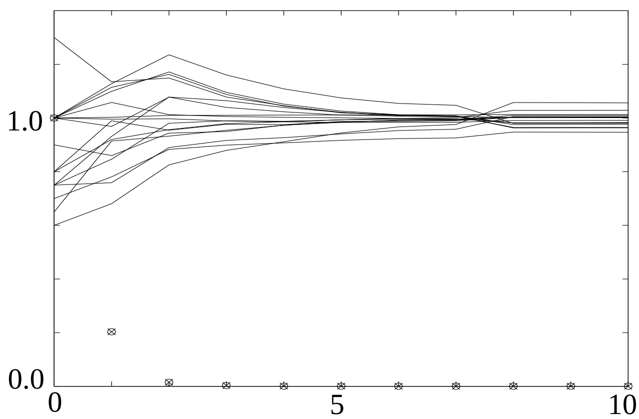


Figure 6. Normalized criterion function (crossed circles) and normalized parameter (solid lines) iteration history. Criterion function normalized to one at zeroth iteration. Parameters are normalized to ones for exact parameters of XM model (possible here since the true model is known). Criterion function decreases to less than 2% in two major iterations.

Table 1. Initial, updated and true parameter values. Note that here we have normalized the parameters against the true parameter values whereas Caesar et.al.[13] normalizes against initial values.

#	Type	Initial	Updated	True
1	Cross sectional area	0.80	1.0018	1.00
2	Cross sectional area	0.75	1.0287	1.00
3	Cross sectional area inertia	1.00	0.9631	1.00
4	Cross sectional area inertia	1.00	1.0023	1.00
5	Cross sectional area inertia	0.75	1.0050	1.00
6	Cross sectional area inertia	0.75	0.9649	1.00
7	Cross sectional area inertia	0.65	1.0025	1.00
8	Element density	1.00	1.0128	1.00
9	Element density	1.00	0.9912	1.00
10	Lumped mass	0.70	0.9474	1.00
11	Lumped mass	1.30	0.9762	1.00
12	Lumped mass	0.90	1.0095	1.00
13	Lumped mass	0.80	1.0025	1.00
14	Lumped moment of inertia	1.00	0.9806	1.00
15	Lumped moment of inertia	1.00	0.9823	1.00
16	Lumped moment of inertia	0.60	1.0571	1.00
17	Lumped moment of inertia	1.00	0.9821	1.00

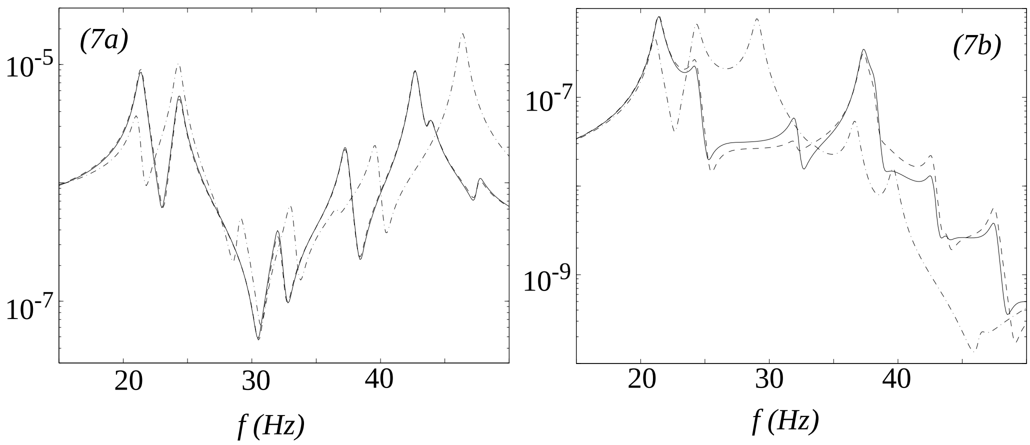


Figure 7. Examples of transfer functions of initial model (dash-dotted), updated model (dashed) and true model (solid). (7a) gives example from set showing best correlation between updated and true models (curves are almost coincident) and (7b) gives example from set showing poorest obtained correlation.

12. CONCLUDING REMARKS

A frequency domain model updating method has been suggested. The inclusion of resonance frequency error to the criterion function has been made to enhance its convergence properties. The formulation has been showed, on a small scale problem, to have a very large domain of attraction to the global minimum. By the use of a quadratic criterion function, with a cheap evaluation of the approximate Hessian matrix and regularization, rapid and stable iterations to the minimum are achieved. A non-extensive quadratic one-dimensional search procedure, with few function evaluations and less accurate line-search minimum determination, has been found to be adequate. A Rayleigh-quotient based technique and a new enhanced MAC number help to solve the mode tracking problem when the eigenfrequencies become closely adjacent. Various problem scalings have been elaborated upon treating the criterion function. Expressions have been derived which quantifies how measurement noise will influence the estimated parameters. A numerical example has demonstrated the performance of the method.

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