Asymptotic Variance Expressions for a Frequency Domain Subspace Based System Identification Algorithm*

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Abstract

This paper deals with the analysis of a frequency domain identification algorithm. The algorithm identifies state-space models given samples of the frequency response given at equidistant frequencies. A first order perturbation analysis is performed revealing an explicit expression of resulting transfer function perturbation. Stochastic analysis show that the estimate is asymptotically (in data) normal distributed and an explicit expression of the resulting variance is given. Monte Carlo simulations illustrates the validity of the variance expression also for the non-asymptotic case.

Keywords: Identification; Subspace Method; Stochastic Analysis; Variance Expression.

1 Introduction

The present paper deals with a frequency domain based identification problem. In this formulation, the experimental data are taken to be the noisy values of the frequency response of the system at a given set of frequencies. In a number of applications cases frequency domain data are better suited for identification than the time-domain data [18, 14].

Frequency domain subspace algorithms [7, 11, 12, 17, 16] are based on the famous Ho and Kalman realization algorithm [4] or Kung's version [8]. The realization algorithms [4, 8] find a minimal state-space realization given a finite

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sequence of Markov parameters. The Markov parameters or impulse response coefficients of the system are estimated from the inverse discrete Fourier transform (DFT) of the frequency response data. The approach taken in [7] is exact only if the system has a finite pulse response and therefore for lightly damped systems yields very poor estimates. In [17, 16], the inverse DFT technique is combined with a subspace identification step yielding the true finite-dimensional system in spite of the bias introduced by the inverse DFT and possessed by the estimated Markov parameters.

The topic of this paper is to present and analyze a variation of the algorithm [16] which identify the B and D matrices in a different way. The algorithm will be analyzed under the the assumption of bounded errors as well as using a stochastic perspective of the errors. In the stochastic analysis we will show strong consistency of the method and show that the asymptotic estimate is normal distributed. An explicit expression of the asymptotic variance will also be given.

The paper is outlined as follows. In the next section we formulate the identification problem and in the third section the identification algorithm is presented. The fourth section is devoted to error analysis. First the case of bounded errors is considered followed by a first order perturbation analysis. Last stochastic analysis is performed where consistency and asymptotic normality is derived. In Section 6 a Monte Carlo simulation example is given illustrating the validity of the derived asymptotic variance expression. The paper is concluded in Section 7.

2 Problem formulation

Assume that the true system G is a stable multivariable linear time-invariant discrete time system with input/output properties characterized by the impulse response coefficients g_k through the equation

$$y(t) = \sum_{k=0}^{\infty} g_k u(t-k) \quad (1)$$

where $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^m$ and $g_k \in \mathbb{R}^{p \times m}$. If the system is of finite order n it can be described by a state-space model

$$\begin{aligned} x(t+1) &= A_0 x(t) + B_0 u(t) \\ y(t) &= C_0 x(t) + D_0 u(t) \end{aligned} \quad (2)$$

where $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^m$, and $x(t) \in \mathbb{R}^n$. The state-space model (2) is a special case of (1) with

$$g_k = \begin{cases} D_0, & k = 0 \\ C_0 A_0^{k-1} B_0, & k > 0 \end{cases} \quad (3)$$

The frequency response of (1) is

$$G(e^{j\omega}) = \sum_{k=0}^{\infty} g_k e^{-j\omega k} \quad (4)$$

which for the state-space model (2) can be written as

$$G(e^{j\omega}) = C_0(e^{j\omega}I - A_0)^{-1}B_0 + D_0. \quad (5)$$

The problem formulation is then: Given a finite number, possibly noisy, samples

$$G_k = G(e^{j\omega_k}) + n_k, \quad \omega_k \in [0, \pi] \quad (6)$$

of the frequency response of the system, find a finite dimensional state-space system (2) of order n , denoted by $\hat{G}$, such that the true system and the identified model are “close”. In (6) $G(e^{j\omega_k})$ represents the system frequency function (4) and n_k denotes the noise. Closeness between systems is quantified by the distance between the true and estimated transfer functions and is given by

$$\|\hat{G} - G\|_\infty = \sup_{\omega} \bar{\sigma}(\hat{G}(e^{j\omega}) - G(e^{j\omega})). \quad (7)$$

Here $\bar{\sigma}(A)$ denote the largest singular value of the matrix A and $(\cdot)^H$ the complex conjugate and transpose. Since $\bar{\sigma}(A) \leq \|A\|_F$ we notice that

$$\|\hat{G} - G\|_\infty \leq \sup_{\omega} \|\hat{G}(e^{j\omega}) - G(e^{j\omega})\|_F.$$

3 The Algorithm

If the impulse response coefficients (3) are given, well-known realization algorithms can be used to obtain state-space realizations [4, 20, 8, 6]. The algorithm to be presented is closely related to these results but does not require the true impulse response coefficients to be known.

Assume that frequency response data G_k on a set of uniformly spaced frequencies, $\omega_k = \frac{\pi k}{M}$, $k = 0, \dots, M$ are given. Since G is a transfer function with a real valued impulse response (1), frequency response data on $[0, \pi]$ can be extended to $[0, 2\pi]$ which forms the first step of the algorithm.

Algorithm 1

1.

$$G_{M+k} := G_{M-k}^*, \quad k = 1, \dots, M-1 \quad (8)$$

where $(\cdot)^*$ denotes complex conjugate.

2. Let $\hat{h}_i$ be defined by the $2M$ -point Inverse Discrete Fourier Transform (IDFT)

$$\hat{h}_i = \frac{1}{2M} \sum_{k=0}^{2M-1} G_k e^{j2\pi ik/2M}, \quad i = 0, \dots, q+r-1. \quad (9)$$

3. Let the block Hankel matrix $\hat{H}_{qr}$ be defined as

$$\hat{H}_{qr} = \begin{pmatrix} \hat{h}_1 & \hat{h}_2 & \dots & \hat{h}_r \\ \hat{h}_2 & \hat{h}_3 & \dots & \hat{h}_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{h}_q & \hat{h}_{q+1} & \dots & \hat{h}_{q+r-1} \end{pmatrix} \in \mathbb{R}^{qp \times rm} \quad (10)$$

with number of block rows $q > n$ and block columns $r \geq n$. The size of $\hat{H}_{qr}$ is limited by $q+r \leq M$.

4. Let the singular value decomposition of $\hat{H}_{qr}$ be

$$\hat{H}_{qr} = \begin{pmatrix} \hat{U}_s & \hat{U}_o \end{pmatrix} \begin{pmatrix} \hat{\Sigma}_s & 0 \\ 0 & \hat{\Sigma}_o \end{pmatrix} \begin{pmatrix} \hat{V}_s^T \\ \hat{V}_o^T \end{pmatrix} \quad (11)$$

where $\hat{\Sigma}_s$ contains the n largest singular values.

5.

$$\hat{A} = (J_1 \hat{U}_s)^\dagger J_2 \hat{U}_s \quad (12)$$

$$\hat{C} = J_3 \hat{U}_s \quad (13)$$

6.

$$\hat{B} = (I - \hat{A}^{2M}) \hat{\Sigma}_s \hat{V}_s^T J_4 \quad (14)$$

$$\hat{D} = \hat{h}_0 - \hat{C} \hat{A}^{2M-1} (I - \hat{A}^{2M})^{-1} \hat{B}. \quad (15)$$

where

$$J_1 = \begin{pmatrix} I_{(q-1)p} & 0_{(q-1)p \times p} \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0_{(q-1)p \times p} & I_{(q-1)p} \end{pmatrix} \quad (16)$$

$$J_3 = \begin{pmatrix} I_p & 0_{p \times (q-1)p} \end{pmatrix}, \quad J_4 = \begin{pmatrix} I_m \\ 0_{(r-1)m \times m} \end{pmatrix} \quad (17)$$

and I_i denotes the $i \times i$ identity matrix, $0_{i \times j}$ denotes the $i \times j$ zero matrix and $X^\dagger = (X^T X)^{-1} X^T$ denotes the Moore-Penrose pseudo-inverse of the full-rank matrix X of size $q \times r$ and $q > r$.

In the estimation of B and D Algorithm 1 differ from the algorithm presented in [16] where

$$\hat{B}, \hat{D} = \arg \min_{B, D} \sum_{k=0}^{2M-1} \left\| G_k - D - \hat{C} (e^{j\omega_k} I - \hat{A})^{-1} B \right\|_F^2. \quad (18)$$

After the IDFT step 2, Algorithm 1 differ from Kung's [8] in two ways, the estimation of B and D , and in the selection of state-space basis. The difference in B and D stems from the fact that Algorithm 1 uses the coefficients $\hat{h}_k$ from the IDFT instead of the impulse response g_k .

The realization given by the algorithm using noise free data is balanced in the sense that the q -block row observability matrix

$$\mathcal{O}_i = \begin{pmatrix} \hat{C} \\ \hat{C} \hat{A} \\ \vdots \\ \hat{C} \hat{A}^{q-1} \end{pmatrix} \quad (19)$$

and the r -block column controllability matrix

$$\mathcal{C}_r = \begin{pmatrix} \hat{B} & \hat{A} \hat{B} & \dots & \hat{A}^{r-1} \hat{B} \end{pmatrix} \quad (20)$$

satisfy

$$\mathcal{O}_q^T \mathcal{O}_q = I, \quad \mathcal{C}_r \mathcal{C}_r^T = (I - \hat{A}^{2M}) \hat{\Sigma}_s^2 (I - \hat{A}^{2M})^T \quad (21)$$

As M, q and r jointly tend to infinity, the products (21) will converge to the observability and controllability Grammians, respectively, and the diagonal elements of $\hat{\Sigma}_s$ will converge to the Hankel singular values of the system.

By using $\hat{U}_s$ in (12) and (13) instead of $\hat{U}_s \hat{\Sigma}_s^{1/2}$ as in Kung's algorithm results in a different state-space basis. However, the identified transfer functions are still the same if noise-free data is used.

3.1 Practical Aspects

When facing a practical identification problem many models of different orders are estimated and compared in order to find a suitable “best” model. In the presented algorithms most of the computational effort lies in the SVD factorization (11). Given the factorization (11), all models of order less than q are easily obtained from the rest of the algorithms by letting n range from 1 to $q - 1$. Hence, the choice of appropriate model order can easily be accomplished by direct comparison of a wide range of models with different orders at a low computational cost.

Often stable models are sought for various reasons. The algorithms, as they stand, do not guarantee stability of the estimated models when using a finite number of noisy frequency data. However, with an additional step which projects the unstable eigenvalues of $\hat{A}$ into the unit disc, stability can be ensured for Algorithm 1. Transform $\hat{A}$ to the complex Schur form. Project any diagonal elements (eigenvalues) λ_i which lie outside the unit disc into the unit disc by $1/\lambda_i$. Eigenvalues on the unit circle can be moved into the unit disc by changing the magnitude of the eigenvalue to $1 - \epsilon$ for some small ϵ . Finally transform $\hat{A}$ back into its original form before proceeding further.

3.2 Noise Free Case

Let us now assume that $n_k = 0$ in (6), *i.e.* noise free measurements. In the sequel we will denote by X the noise free counterpart of the symbol $\hat{X}$. Furthermore assume that the frequency responses G_k origin from an n th order system with a state-space representation (A_0, B_0, C_0, D_0) .

Algorithm 1 is based on the fact that the resulting coefficients $\hat{h}_i$ from the finite IDFT of the frequency response has a Hankel matrix with the same range space as the observability matrix of the true system. It is well known that the Hankel matrix of the impulse response has this property, [4, 8] but much less known that the Hankel matrix (10) also has this property. A and C are thus estimated as in Kung's algorithm. However, the estimates of B and D has to take into account that the Hankel matrix is not based on the impulse response. The factor $(I - \hat{A}^{2M})$ in (14) accomplishes this.

The usefulness of Algorithm 1 in the case of finite M follows from the following key theorem.

Theorem 1 *Let G be an n th order stable discrete time system represented by (2). Then $n+2$ noise-free equidistant frequency response measurements of G on $[0, \pi]$ are sufficient to identify a state-space realization with a transfer function equal to G by Algorithm 1.*

Proof: Denote by $\mathcal{O}_0$ and $\mathcal{C}_0$ the q block row extended observability (19) and r block column controllability (20) matrices from the system realization (A_0, B_0, C_0, D_0) . Let $\rho(A)$ denote the spectral radius [5] of A . Since G is a stable transfer function, it can be represented by the following Taylor series

$$\begin{aligned} G(z) &= D_0 + C_0(zI - A_0)^{-1}B_0 = D_0 + \sum_{k=1}^{\infty} C_0 A_0^{k-1} B_0 z^{-k} \\ &= \sum_{k=0}^{\infty} g_k z^{-k}, \quad \rho(A) < |z|. \end{aligned}$$

Notice that h_k defined by (9) can be written as

$$\begin{aligned} h_k &= \frac{1}{2M} \sum_{s=0}^{2M-1} \sum_{i=0}^{\infty} g_i e^{j2\pi s(k-i)/2M} \\ &= \sum_{i=0}^{\infty} g_{k+2iM} \\ &= C_0 A_0^{k-1} \left(\sum_{i=0}^{\infty} A_0^{2iM} \right) B_0 = C_0 A_0^{k-1} (I - A_0^{2M})^{-1} B_0, \quad k > 0 \end{aligned}$$

and therefore H_{qr} can be factored as

$$H_{qr} = \mathcal{O}_0 (I - A_0^{2M})^{-1} \mathcal{C}_0 \quad (22)$$

From the dimensions of the factors in (22) it is clear that H_{qr} has a maximal rank n . Furthermore since the system is stable $\rho(A) < 1$, $(I - A^{2M})$ is always of rank n . Minimality of the system also implies that both $\mathcal{C}_0$ and $\mathcal{O}_0$ are of rank n and hence also H_{qr} if $r \geq n$ and $q > n$. In (11) then $\Sigma_o = 0$ and the column range spaces of H_{qr} , $\mathcal{O}_0$ and U_s will be equal. A valid extended observability matrix $\mathcal{O}$ is then given by U_s since there exists a non-singular $n \times n$ matrix S such that

$$\mathcal{O} = U_s = \mathcal{O}_0 S.$$

U_s is thus an extended observability matrix from a state-space realization (A, B, C, D) which is similar to the original realization (A_0, B_0, C_0, D_0) . (12) and (13) will yield A and C matrices which are related to the original realization as

$$A = S^{-1} A_0 S \quad C = C_0 S$$

as shown in [8].

According to the estimated A and C we notice

$$\begin{aligned} H_{qr} &= \mathcal{O}_0 (I - A_0^{2M})^{-1} \mathcal{C}_0 \\ &= \mathcal{O} S^{-1} (I - A_0^{2M})^{-1} \mathcal{C}_0 \\ &= \mathcal{O} (I - S^{-1} A_0^{2M} S)^{-1} S^{-1} \mathcal{C}_0 \\ &= \mathcal{O} (I - A^{2M})^{-1} \mathcal{C} \end{aligned}$$

From this it follows that

$$\Sigma_s V_s^T = (I - A^{2M})^{-1} \mathcal{C}.$$

Hence, we obtain

$$(I - A^{2M}) \Sigma_s V_s^T J_4 = B$$

which proves (14).

From the IDFT (9) we have

$$h_0 = g_0 + \sum_{i=1}^{\infty} g(2iM) = D_0 + \sum_{i=1}^{\infty} C_0 A_0^{2iM-1} B_0 = D_0 + C A^{2M-1} (I - A^{2M})^{-1} B$$

which proves (15). Then the state-space realizations (A, B, C, D) and (A_0, B_0, C_0, D_0) are similar and have equal transfer functions. Letting $q = n + 1$, $r = n$, $M = n + 2$, we satisfy the condition $q + r < 2M$ which completes the proof. $\square$

3.3 Uniqueness of the Realization

The particular realization obtained by the algorithm is not completely unique. The non-uniqueness origins from the SVD (11). If the singular values Σ_s are assumed to be distinct and in a descending order the only non-uniqueness origins from possible sign changes in the left and right singular vectors. A unique SVD can in this case be imposed with the additional constraint that the first non-zero element in each of the left singular vectors (column vectors of $\hat{U}_s$) to be positive. If some of the singular values are equal a more involved procedure can be undertaken. A possible way is to use a modified version of the procedure outlined in the proof of Theorem 2.1 in [15]. For all practical identification applications there are however no need for an unique realization since the goal is only to obtain the transfer function.

4 Error Analysis

In the previous section, we demonstrated that an n th order system can be recovered by a simple algorithm involving only the IDFT, SVD, and simple algebraic manipulations using $n + 2$ frequency response measurements. The assumption that the system which generated the data is finite dimensional with a known order is not realistic and we will admit the possibility of some “small unmodeled dynamics” which is not captured by any finite dimensional model set. First, we will derive expressions of the perturbation of the transfer function and demonstrate that the proposed algorithm is robust to unmodeled dynamics and noise. Secondly, we analyze the algorithm by stochastic analysis and show consistency as well as derive an asymptotic variance expression of the estimated transfer function. In this section we will drop all subscripts denoting the size of a matrix.

In the noisy case we can write

$$\hat{H} = H + \Delta H$$

where ΔH represents the Hankel matrix of the noise part. In general $\hat{H}$ will be of full rank ($= \min(qp, mr)$) because of the perturbation matrix ΔH . If the largest singular value of ΔH is significantly smaller than the smallest singular value of H the n largest singular values $\hat{\Sigma}_s$ of $\hat{H}$ and corresponding left singular vectors $\hat{U}_s$ will be close to the unperturbed counterparts and the estimated system will be close to the true system. The SVD of the identification algorithm will thus have a noise threshold and when the noise level increases over this level the resulting estimates will not be reliable since the singular vectors in $\hat{U}_s$ might

change places. In the next section we will discuss how the estimated system is perturbed when ΔH_{qr} is “small”.

4.1 Bounded Noise

Suppose that frequency response data G_k as in (6) where G is the finite dimensional nominal model is given and n_k captures unmodeled dynamics and noise. First we consider the case when n_k is uniformly bounded

$$\|n\|_\infty := \sup_k \|n_k\|_F \leq \epsilon. \quad (23)$$

The noise term n_k will yield an additive perturbation on the coefficient h_k and we find

$$\hat{h}_k = h_k + \Delta h_k$$

where Δh_k are the IDFT of n_k

$$\Delta h_k = \frac{1}{2M} \sum_{i=0}^{2M-1} n_i e^{j2\pi k i / (2M)}. \quad (24)$$

The perturbed Hankel matrix is given by perturbed Hankel matrix as

$$\hat{H} = H + \Delta H \quad (25)$$

where H is the unperturbed Hankel matrix and ΔH is the perturbation matrix with elements

$$[\Delta H]_{i,j} = \Delta h_{i+j-1}. \quad (26)$$

The norm of ΔH is bounded by

$$\|\Delta H\|_F \leq \sqrt{qr} \epsilon.$$

Let the singular value decompositions of the unperturbed Hankel matrix be as

$$H = \begin{pmatrix} U_s & U_o \end{pmatrix} \begin{pmatrix} \Sigma_s & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_s^T \\ V_o^T \end{pmatrix}. \quad (27)$$

Denote by $\sigma_k(H)$ the n singular values of H ordered in a non-increasing order. In the identification algorithm the SVD plays an important role or more precisely the left singular vectors U_s of the Hankel matrix. The following result will describe how the noise will influence this quantity.

Lemma 1 *Let $\hat{H} = H + \Delta H$ and $\|\Delta H\|_F \leq \epsilon$.*

Let $\hat{U}_s$ be given by (11) and U_s by (27), and assume that

$$\epsilon < \sigma_n(H)/4. \quad (28)$$

Then $\exists$ a nonsingular matrix $T \in \mathbb{R}^{n \times n}$ such that

$$\hat{U}_s = (U_s + U_o P) T \quad (29)$$

and

$$\|P\|_F \leq \frac{4}{\sigma_n(H)} \epsilon. \quad (30)$$

Proof. Let

$$\begin{pmatrix} U_s^T \\ U_o^T \end{pmatrix} \Delta H \begin{pmatrix} V_s & V_o \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = E. \quad (31)$$

From (31) we notice that $\|E\|_F = \|\Delta H\|_F \leq \epsilon$ and also $\|E_{ij}\|_2 \leq \|E_{ij}\|_F$ since the spectral norm is upper bounded by the Frobenius norm. Therefore it is clear that

$$\begin{aligned} \delta &= \sigma_n(H) - \|E_{11}\|_2 - \|E_{22}\|_2 \geq \sigma_n(H) - \|E_{11}\|_F - \|E_{22}\|_F \\ &\geq \sigma_n(H) - 2\epsilon \geq \frac{1}{2}\sigma_n(H) > 0 \end{aligned} \quad (32)$$

by assumption (28). We also have

$$\frac{\|E_{21} \ E_{12}^T\|_F}{\delta} \leq \frac{\epsilon}{\frac{1}{2}\sigma_n(H)} \leq \frac{1}{2}.$$

Then by Theorem 8.3.5 in [3] there exists a matrix P satisfying

$$\|P\|_F \leq \frac{2\epsilon}{\delta} \leq \frac{4\epsilon}{\sigma_n(H)}$$

such that $\text{range}(\hat{U}_s)$ and $\text{range}(U_s + U_o P)$ are equal. Since the range spaces are equal and U_s is of full rank there exists a unique non-singular matrix T such that (29) is satisfied. $\square$

It is clear from the result above that the resulting subspace spanned by $\hat{U}_s$ is close to the subspace spanned by U_s if the noise level is low. Since only the subspaces are close we had to introduce the matrix T in order to relate $\hat{U}_s$ and U_s directly. This *a priori* unknown T represents the “unknown” basis resulting from the identification. This would lead one to expect a difficulty in bounding the transfer function perturbation. However, we will show that this fact does not represent any problem since different state-space basis do not alter the resulting transfer function.

Theorem 2 *Let the frequency data be given by (6) with the noise uniformly bounded $\|n_k\|_\infty \leq \epsilon$ and let G be a stable n th order linear system with transfer function $G(z)$. Let $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ be the identified state-space model given by Algorithm 1 and let $\hat{G}(z)$ be the corresponding transfer function. Then $\exists \epsilon_0, c > 0$ such that $\forall \epsilon \leq \epsilon_0$*

$$\sup_{\|n_k\|_\infty \leq \epsilon} \sup_{|z|=1} \|\hat{G}(z) - G(z)\|_F \leq c\epsilon \quad (33)$$

Proof. First we bound the perturbation of the estimated state-space matrices with the aid of a change of basis represented by the unknown but non-singular T from Lemma 1. These bounds are then used to finally bound the transfer function itself. Throughout the proof we let c denote a bounded real constant which might have different values.

Let (A, B, C, D) denote the state-space realization of the true system which is a result from applying noise free data to Algorithm 1. First consider the

estimation of the A matrix (12). Let ϵ be small enough in order to use the result from Lemma 1. Then $\hat{U}_s = (U_s + U_o P)T$. Therefore

$$\hat{A} = (J_1(U_s + U_o P)T)^\dagger J_2(U_s + U_o P)T$$

which can be written

$$T\hat{A}T^{-1} = (J_1(U_s + U_o P))^\dagger J_2(U_s + U_o P)$$

where we used the fact that $(XT)^\dagger = T^{-1}(X)^\dagger$. Hence we can write

$$\begin{aligned} \|T\hat{A}T^{-1} - A\|_F &= \|(J_1(U_s + U_o P))^\dagger J_2(U_s + U_o P) - (J_1 U_s)^\dagger J_2 U_s\|_F \\ &\leq \|(J_1(U_s + U_o P))^\dagger J_2 U_o P\|_F + \|(J_1(U_s + U_o P))^\dagger - (J_1 U_s)^\dagger\|_F \|J_2 U_s\|_F \end{aligned}$$

Since U_s is an extended observability matrix of an n th order minimal system $J_1 U_s$ is of full rank. The matrix $J_1(U_s + U_o P)$ will thus also be of full rank for all sufficiently small values of ϵ . We can thus use Lemma 4 in appendix to obtain

$$\begin{aligned} \|T\hat{A}T^{-1} - A\|_F &\leq \|(J_1(U_s + U_o P))^\dagger J_2 U_o P\|_F \\ &\quad + \|(J_1(U_s + U_o P))^\dagger\|_2 \|(J_1 U_s)^\dagger\|_2 \|J_1 U_o P\|_F \|J_2 U_s\|_F \end{aligned}$$

By using Lemma 5 in appendix it follows that the pseudo-inverse is bounded for all sufficiently small ϵ and $\|P\|_F \leq c\epsilon$. All this yields

$$\|T\hat{A}T^{-1} - A\|_F \leq c\epsilon, \quad \forall \epsilon < \epsilon_A, \quad (34)$$

for some $\epsilon_A > 0$. For convenience let $\Delta A = T\hat{A}T^{-1} - A$.

For $\hat{B}$ from (14) we obtain

$$\hat{B} = (I - \hat{A}^{2M})\hat{U}_s^T \hat{H} J_4 = T^{-1}(I - (A + \Delta A)^{2M})TT^T(U_s + P^T U_o^T)(H + \Delta H)J_4$$

Since $\hat{U}_s^T \hat{U}_s = I = T^T(U_s^T + P^T U_o^T)(U_s + U_o P)T = T^T + T^T P^T P T$ we can write $T^T = T^{-1} - T^T P^T P$. Also we can write¹

$$(A + \Delta A)^{2M} = A^{2M} + \sum_{k=1}^{2M} \binom{2M}{k} A^{2M-k} \Delta A^k$$

With this we get

$$\begin{aligned} \hat{B} &= T^{-1} \left[I - A^{2M} + \sum_{k=1}^{2M} \binom{2M}{k} A^{2M-k} \Delta A^k \right] \\ &\quad \times (I - TT^T P^T P)(U_s^T + P^T U_o^T)(H + \Delta H)J_4 \\ &= T^{-1}B + O(\epsilon) \end{aligned}$$

Hence we obtain

$$\|T\hat{B} - B\|_F = O(\epsilon) \leq c\epsilon, \quad \forall \epsilon < \epsilon_B, \quad (35)$$

for some $\epsilon_B > 0$. The $\hat{C}$ matrix is straightforward

$$\hat{C} = J_3(U_s + U_o P)T$$

¹This is actually abuse of notation unless ΔA and A commutes.

which directly gives

$$\|\hat{C}T^{-1} - C\| \leq c\epsilon, \quad \forall \epsilon < \epsilon_A. \quad (36)$$

Let $\Delta B := T\hat{B} - A$ and $\Delta C := \hat{C}T^{-1} - C$. For $\hat{D}$ we have

$$\hat{D} = D + CA^{2M-1}(I - A^{2M})^{-1}B + \Delta h_0 - \hat{C}\hat{A}^{2M-1}(I - \hat{A}^{2M})^{-1}\hat{B}$$

and we obtain

$$\begin{aligned} \|\hat{D} - D\|_F &\leq \|\Delta h_0\|_F \\ &+ \|CA^{2M-1}(I - A^{2M})^{-1}B \\ &- (C + \Delta C)(A + \Delta A)^{2M-1}(I - (A + \Delta A)^{2M})^{-1}(B + \Delta B)\|_F \end{aligned}$$

The last term above requires some further study and we obtain

$$\begin{aligned} &\|CA^{2M-1}(I - A^{2M})^{-1}B \\ &- (C + \Delta C)(A + \Delta A)^{2M-1}(I - (A + \Delta A)^{2M})^{-1}(B + \Delta B)\|_F \\ &\leq \|\Delta C\|_F \|(A + \Delta A)^{2M-1}(I - (A + \Delta A)^{2M})^{-1}(B + \Delta B)\|_F \\ &+ \|C((A + \Delta A)^{2M-1} - A^{2M-1})(I - (A + \Delta A)^{2M})^{-1}(B + \Delta B)\|_F \\ &\quad + \|CA^{2M-1}(I - (A + \Delta A)^{2M})^{-1}\Delta B\|_F \\ &+ \|CA^{2M-1}(I - A^{2M})^{-1}((A + \Delta A)^{2M} - A^{2M})(I - (A + \Delta A)^{2M})^{-1}B\|_F \end{aligned} \quad (37)$$

We notice that

$$\|(A + \Delta A)^{2M} - A^{2M}\|_F = \left\| \sum_{k=1}^{2M} \binom{2M}{k} A^{2M-k} \Delta A^k \right\|_F \leq c\epsilon.$$

Since $(I - A^{2M})$ is nonsingular Lemma 5 is applicable to conclude that the inverses in expression (37) are bounded by some constant for all sufficiently small values of ϵ . Therefore we can conclude

$$\|\hat{D} - D\|_F \leq c\epsilon, \quad \forall \epsilon < \epsilon_D \quad (38)$$

for some constant $\epsilon_D > 0$. We have now established that the distances between the estimated state-space matrices converted to some basis and the state-space matrices of the true system indeed are bounded by ϵ whenever ϵ is sufficiently small.

Consider the transfer function error $\hat{G}(z) - G(z)$ which has a norm which can be written as

$$\begin{aligned} \|\hat{G}(z) - G(z)\|_F &= \|\hat{D} + \hat{C}(zI - \hat{A})^{-1}\hat{B} - D - C(zI - A)^{-1}B\|_F \\ &= \|\hat{D} + \hat{C}T^{-1}(zI - T\hat{A}T^{-1})^{-1}T\hat{B} - D - C(zI - A)^{-1}B\|_F \end{aligned}$$

for any nonsingular matrix $T \in \mathbb{R}^{n \times n}$. This expression can be bounded as

$$\begin{aligned} \|\hat{G}(z) - G(z)\|_F &\leq \|\hat{D} - D\|_F + \|\Delta C(zI - \hat{A})^{-1}\hat{B}\|_F \\ &+ \|C(zI - \hat{A})^{-1}\Delta B\|_F + \|C(zI - A)^{-1}\Delta A(zI - \hat{A})^{-1}B\|_F \end{aligned}$$

Since G is a stable system $(zI - A)$ is nonsingular for all $|z| = 1$. Again we can use Lemma 5 to conclude that the norm of $(zI - T\hat{A}T^{-1})$ is bounded by some constant. By using the bounds (34), (35), (36) and (38) we obtain (33) with $\epsilon_0 = \min(\epsilon_A, \epsilon_B, \epsilon_D)$ which concludes the proof. $\square$

4.2 First Order Perturbation Analysis

The perturbation analysis will be performed using a first order expansion. The analysis will then be valid whenever ϵ is sufficiently small. The route we will follow below is inspired by similar types of analysis of array signal processing algorithms [9, 10]. Although $\hat{U}_s$ is highly nonlinear in the data $\hat{H}$ we seek a first order perturbation matrix ΔU_s which is linear in the perturbation ΔH . An obvious but not tractable way is to find ΔU_s by differentiation. Instead a *backward error analysis* [10] is employed to find ΔU_s . The following lemma gives the expression of the first order perturbation of the left singular vectors U_s .

Lemma 2 *The perturbed subspace spanned by $\hat{U}_s$ is spanned by $U_s + U_o P$ where P is a matrix with a norm of the order of $\|\Delta H\|$. Furthermore a first order term expression of $\hat{U}_s$ is given by*

$$\hat{U}_s = U_s + \Delta U_s$$

where

$$\Delta U_s = U_o U_o^T \Delta H V_s \Sigma_s^{-1}. \quad (39)$$

Proof. See [10, 9]. $\square$

The derived perturbation expression has the following two properties. It is exactly orthogonal to the unperturbed space and hence

$$\Delta U_s^T U_s = 0.$$

The perturbed subspace is orthonormal up to first order

$$(U_s + \Delta U_s)^T (U_s + \Delta U_s) = I + O(\|\Delta H\|_F^2).$$

If we assume $U_s = \mathcal{O}$ we obtain the alternative expression

$$\Delta U_s = U_o U_o^T \Delta H ((I - A^{2M})^{-1} \mathcal{C})^\dagger$$

since

$$\Sigma_s V_s^T = (I - A^{2M})^{-1} \mathcal{C}.$$

With the aid of this lemma it is straightforward to derive the corresponding first order perturbation expressions for the estimated system matrices $\hat{A}$ (12), $\hat{B}$ (14), $\hat{C}$ (13) and $\hat{D}$ (13).

We will in the sequel encounter the inverse of a perturbed matrix and present the following lemma.

Lemma 3 *Assume $X = Y^{-1}$. Let $\hat{Y} = Y + \Delta Y$ and $\hat{Y}^{-1} = \hat{X} = X + \Delta X$. Then*

$$\Delta X = -Y^{-1} \Delta Y Y^{-1}$$

is a first order approximation of the perturbation of the inverse.

Proof. We have $\hat{Y}^{-1} = \hat{X}$. Hence $(Y + \Delta Y)(X + \Delta X) = I$. A first order approximation is then

$$Y\Delta X = -\Delta YX = -\Delta YY^{-1}$$

which concludes the proof. $\square$

In what follows we assume that (A, B, C, D) represents the corresponding state-space realization which is obtained from Algorithm 1 when $n_k = 0$. The only uniqueness imposed is that the extended observability matrix must satisfy

$$\mathcal{O}^T \mathcal{O} = I$$

which is guaranteed by the algorithm since $\mathcal{O} = U_s$.

Theorem 3 *Consider the Hankel matrix $\hat{H} = H + \Delta H$ where ΔH is the perturbations of the matrix. The first order perturbations of the estimated system matrices $\hat{A}, \hat{B}, \hat{C}$ and $\hat{D}$ from $\hat{H}$ using the algorithm (11)–(15) can be described by*

$$\begin{aligned}\hat{A} &= A + \Delta A \\ \hat{B} &= B + \Delta B \\ \hat{C} &= C + \Delta C \\ \hat{D} &= D + \Delta D\end{aligned}$$

where

$$\Delta A = (J_1 U_s)^\dagger (J_2 \Delta U_s - J_1 \Delta U_s A) \quad (40)$$

$$\Delta B = [(I - A^{2M})U_s^T \Delta H - 2MA^{2M-1} \Delta A U_s^T H] J_4 \quad (41)$$

$$\Delta C = J_3 \Delta U_s \quad (42)$$

$$\begin{aligned}\Delta D &= \Delta h_0 \\ &\quad - [\Delta C A^{2M-1} + (2M-1)C A^{2M-2} \Delta A] (I - A^{2M})^{-1} B \\ &\quad - C A^{2M-1} (I - A^{2M})^{-1} \\ &\quad \times [2MA^{2M-1} \Delta A (I - A^{2M})^{-1} B + \Delta B]\end{aligned} \quad (43)$$

and ΔU_s is given by (39). Furthermore, the first order perturbation of the transfer function is given by

$$\hat{G}(z) = G(z) + \Delta G(z)$$

where

$$\begin{aligned}\Delta G(z) &= \Delta D + \Delta C(zI - A)^{-1} B + \\ &\quad C(zI - A)^{-1} \Delta A(zI - A)^{-1} B + C(zI - A)^{-1} \Delta B.\end{aligned} \quad (44)$$

Proof. In this proof all equality signs are valid up to first order. From the algorithm we have $\hat{A} = (J_1 \hat{U}_s)^\dagger J_2 \hat{U}_s$. Hence we obtain

$$J_1(U_s + \Delta U_s)(A + \Delta A) = J_2(U_s + \Delta U_s). \quad (45)$$

By using the fact that $J_1 U_s A = J_2 U_s$ and truncate the left-hand side of (45) up to first order terms yields

$$J_1 U_s \Delta A + J_1 \Delta U_s A = J_2 \Delta U_s$$

which proves (40).

$\hat{B}$ is given by $\hat{B} = (I - \hat{A}^{2M})\hat{\Sigma}_s\hat{V}_s^T J_4$. Notice that $\hat{U}_s^T \hat{H} = \hat{\Sigma}_s\hat{V}_s^T$. Using this equality we obtain $\hat{B} = (I - \hat{A}^{2M})\hat{U}_s^T \hat{H} J_4$. For $\hat{A}^{2M}$ we have $(A + \Delta A)^{2M} = A^{2M} + 2MA^{2M-1}\Delta A$ up to first order terms. Using this yields

$$\begin{aligned} B + \Delta B &= (I - (A + \Delta A)^{2M})(U_s + \Delta U_s)^T (H + \Delta H) J_4 \\ &= (I - A^{2M})U_s^T H J_4 + [(I - A^{2M})U_s^T \Delta H - 2MA^{2M-1}\Delta A U_s^T H \\ &\quad + (I - A^{2M})\Delta U_s^T H] J_4 \\ &= (I - A^{2M})U_s^T H J_4 + \\ &\quad [(I - A^{2M})U_s^T \Delta H - 2MA^{2M-1}\Delta A U_s^T H] J_4 \end{aligned}$$

where the last equality follows from the fact that $\Delta U_s^T H = 0$. $\hat{C} = J_3 \hat{U}_s$ directly gives

$$C + \Delta C = J_3 U_s + J_3 \Delta U_s.$$

$\hat{D}$ is given by (15). By using Lemma 3 we can simplify the factor

$$\begin{aligned} (I - (A + \Delta A)^{2M})^{-1} &= (I - A^{2M} - 2MA^{2M-1}\Delta A)^{-1} \\ &= (I - A^{2M})^{-1} + \\ &\quad (I - A^{2M})^{-1} 2MA^{2M-1}\Delta A (I - A^{2M})^{-1}. \end{aligned}$$

which occurs in (15). A simple collection of the first order terms of (15) proves (43). The transfer function is given by

$$\hat{G}(z) = D + \Delta D + (C + \Delta C)(zI - A - \Delta A)^{-1}(B + \Delta B).$$

By applying Lemma 3 to $(zI - A - \Delta A)^{-1}$ and collecting the first order terms we arrive at (44). $\square$

Remark 1 *Although we derived the expressions under the assumption that (A, B, C, D) is the noise free realization this particular choice of state-space basis do not influence the perturbation expression given in the theorem. This can easily be seen by introducing a orthonormal similarity transformation in the expression of $\Delta G(z)$. It is also possible to drop the requirement of orthonormality of the similarity transformation. However this would lead to a much more complex expression for (41).*

4.3 Stochastic Noise

If we now adopt the view on n_k as being stochastic variables, the estimated transfer function $\hat{G}(z)$ will also be a stochastic variable. The aim of this section is to first show that the elements of ΔH converge w.p. 1 to zero as M tends to infinity. This implies that the norm $\|\Delta H\|_F$ also tends to zero and strong consistency follows. Furthermore we will show that the estimate is asymptotically normal and derive the variance of the estimate.

4.3.1 Assumptions

Let us assume that the noise term n_k has the form

$$n_k = \alpha_k + j\beta_k$$

and α_k and β_k are assumed to be zero mean independent random variables with equal variance $E\{\alpha_k \alpha_k^T\} = E\{\beta_k \beta_k^T\} = \frac{1}{2}R_k$ such that

$$E\{n_k n_k^H\} = R_k. \quad (46)$$

Furthermore assume that all terms have a common bound

$$R_k \leq R. \quad (47)$$

In (46) $(\cdot)^H$ denotes the complex conjugate and transpose of a complex matrix. We also assume that $E\{n_k n_l^H\} = 0$, $\forall l \neq k$, *i.e.* the noise term for different frequencies are independent. For more information on complex noise models see [1, 18]. These noise assumptions are rather weak and, for example, valid asymptotically if the frequency response is obtained as the empirical transfer function estimate, see [13].

4.3.2 Consistency

In this section we show that the described method is strongly consistent. *i.e.* the limiting transfer function estimate is equal to the true transfer function with probability one as the number of frequency points M tend to infinity.

Theorem 4 *Let G be a stable linear system of order n and let G_k be given by (6) where n_k satisfies the assumptions (46) and (47). Let $\hat{G}$ denote the transfer function obtained by Algorithm 1. Then*

$$\lim_{M \rightarrow \infty} \sup_{|z|=1} \|\hat{G}(z) - G(z)\|_F = 0, \quad \text{w.p.1} \quad (48)$$

Proof. The elements of ΔH given by (24) are all sums of zero mean independent random variables with a common bound on the second moments. Applying Theorem 5.1.2 in [2] we directly obtain

$$\lim_{M \rightarrow \infty} \Delta h_k = 0, \quad \text{w.p. 1}$$

which implies

$$\lim_{M \rightarrow \infty} \|\Delta H\|_F = 0, \quad \text{w.p. 1.}$$

The result (48) now follows from Theorem 2. $\square$

4.3.3 Asymptotic Distribution and Variance

The first order perturbation analysis has given us expressions of the estimation error dependence of the matrix ΔH and consequently the sums Δh_k . We have

$$\hat{G}(z) - G(z) = \Delta G(z) + O(\|\Delta H\|^2) \quad (49)$$

where, as shown in Theorem 3, $\Delta G(z)$ is a linear function of the noise n_k . This property can be utilized to prove asymptotic normality of the transfer function estimate.

Theorem 5 *Let G be a stable linear system of order n and let G_k be given by (6) where n_k satisfies the assumptions (46) and (47) and have bounded moments of order six. Let $\hat{G}$ denote the transfer function obtained by Algorithm 1. Then*

$$\sqrt{M}(\hat{G}(z) - G(z)) \in AsN(0, P(z)), \text{ as } M \rightarrow \infty \quad (50)$$

where

$$P(z) = \lim_{M \rightarrow \infty} E M \Delta G(z) \Delta G(z)^H \quad (51)$$

and $\Delta G(z)$ is given by (44). If we modify the algorithm to guarantee estimated transfer functions satisfying

$$\sup_{|z|=1} \|\hat{G}(z)\| < \infty. \quad (52)$$

the covariance of the estimated transfer function satisfies

$$M E (\hat{G}(z) - E \hat{G}(z))(\hat{G}(z) - E \hat{G}(z))^H \rightarrow P(z), \text{ as } M \rightarrow \infty \quad (53)$$

Proof. We will conduct the proof under the assumption that the transfer function is scalar valued. In the following let c , possibly with one or more indices, denote a bounded real or complex constant. From Theorem 3 we know we can write

$$\sqrt{M} \Delta G(z) = \sqrt{M} \frac{1}{2M} \sum_{i=1}^q \sum_{j=1}^r \sum_{k=0}^{2M-1} c(i, j, k, z) n_k = \frac{1}{\sqrt{M}} \sum_{k=0}^{2M-1} c(k, z) n_k.$$

Let

$$X_{Mk} = \frac{c(k, z) n_k}{\sqrt{M}}$$

and

$$S_M = \sum_{k=0}^{2M-1} X_{Mk}.$$

Furthermore, using the properties of the noise n_k we obtain

$$E X_{Mk} = 0, \quad E |X_{Mk}|^2 = \frac{1}{M} |c(k, z)|^2 R_k < \infty, \quad \forall M, k. \quad (54)$$

For the third order moment of X_{Mk} we obtain

$$\gamma_{Mk} = E |X_{Mk}|^3 = \frac{1}{M^{3/2}} E |c(k, z) n_k|^3 \leq \frac{c}{M^{3/2}}, \quad \forall M, k$$

where the last equality follows from the assumption of bounded sixth moments and Jensen's inequality [2]. This implies that

$$\lim_{M \rightarrow \infty} \sum_{k=0}^{2M-1} \gamma_{Mk} = 0. \quad (55)$$

(54) and (55) together with Theorem 7.1.2 in [2] implies that

$$\sqrt{M} \Delta G(z) \in AsN(0, P(z)). \quad (56)$$

Consider now the term $\sqrt{M}O(\|\Delta H\|_F^2)$. Let

$$X_M = \sqrt{M}\|\Delta H\|_F^2 = \frac{1}{M^{3/2}} \sum_{i=0}^{2M-1} \sum_{j=0}^{2M-1} c(i, z)c(j, z)^H n_i n_j^H.$$

which gives the expectation

$$E|X_M|^3 = \frac{1}{M^{9/2}} \sum_{i_1=0}^{2M-1} \dots \sum_{i_6=0}^{2M-1} c(i_1, z) \dots c(i_6, z)^H n_{i_1} \dots n_{i_6}^H.$$

In this sum only terms in which there exists an index pairing such that each pair have equal index yield non-zero values. The number of such terms are asymptotically of order N^3 and together with the assumption on bounded moments of order 6 we conclude that

$$E|X_M|^3 \leq \frac{cM^3}{M^{9/2}} = \frac{c}{M^{3/2}}$$

holds asymptotically. Chebyshev's inequality then gives

$$P(|X_M| > \epsilon) \leq \frac{E|X_M|^3}{\epsilon^3} \leq \frac{c}{M^{3/2}}, \forall \epsilon > 0$$

and consequently

$$\lim_{M \rightarrow \infty} \sum_{k=0}^{2M-1} P(|X_M| > \epsilon) < \infty, \forall \epsilon.$$

Applying the Borel-Cantelli Lemma [2] gives

$$\lim_{M \rightarrow \infty} X_M = 0, \text{ w.p. } 1$$

which implies that $\sqrt{M}O(\|\Delta H\|_F^2) \rightarrow 0$ w.p. 1 as $M \rightarrow \infty$ which proves (50).

The asymptotic distribution result above does not necessarily imply (53). The rest of the proof will be devoted to this issue.

The perturbation expression (49) is valid for all sufficiently small norms of ΔH . When ΔH becomes large some small singular values of the n largest in (11) will change places. As shown in Lemma 1 the perturbation expression of the singular vectors are valid as long as

$$\sigma_n(H) > 4\|\Delta H\|_F.$$

Divide the event space Ω into two complementary sets

$$\Omega_1 = \{\omega \mid \sigma_n(H) > 4\|\Delta H\|_F\}$$

$$\Omega_2 = \overline{\Omega}_1$$

where ω is the elementary event variable. To bound the probability of $\overline{\Omega}_2$ we have

$$E\|\Delta H\|_F^4 \leq \frac{c}{M^2}$$

where the inequality follows from the bounded sixth moments. Hence by Chebyshev's inequality we have

$$P(\|\Delta H\|_F > \sigma_n(H)/4) \leq \frac{E \|\Delta H\|_F^4}{(\sigma_n(H)/4)^4} \leq \frac{c}{M^2} \quad (57)$$

For events belonging to Ω_1 the expression

$$\hat{G}(z) = G(z) + \Delta G(z) + O(\|\Delta H\|_F^2)$$

holds and for Ω_2

$$\|\hat{G}(z)\| \leq c$$

from assumption (52).

The variance of the estimated transfer function is given by

$$\int_{\Omega} M |\hat{G}(z) - E \hat{G}(z)|^2 d\omega$$

By dividing the integral over the two event sets we obtain asymptotically for Ω_2

$$\lim_{M \rightarrow \infty} \int_{\Omega_2} M |\hat{G}(z) - E \hat{G}(z)|^2 d\omega \leq \lim_{M \rightarrow \infty} \frac{c}{M} = 0$$

where the inequality follows from (52) and (57). For the event set Ω_1 we thus obtain

$$\lim_{M \rightarrow \infty} \int_{\Omega_1} M |\Delta G(z) + O(\|\Delta H\|_F^2) - E O(\|\Delta H\|_F^2)|^2 d\omega = P(z)$$

which concludes the proof. $\square$

Remark 2 *The expressions given in Theorem 5 are valid for single-input multi-output transfer functions, i.e. when G_k is a column vector. The general case can be handled analogously by vectorizing the matrices using the `vec` operator.*

4.3.4 Explicit Expression of the Asymptotic Variance

First we notice that the asymptotic covariance between elements Δh_k is

$$\begin{aligned} R_{\Delta h}(k-l) &= \lim_{M \rightarrow \infty} E M \Delta h_k \Delta h_l^T \\ &= \lim_{M \rightarrow \infty} E \frac{1}{4M} \sum_{t=0}^{2M-1} \sum_{s=0}^{2M-1} n_t n_s^H e^{j2\pi(k-l)s/(2M)} \\ &= \lim_{M \rightarrow \infty} \frac{1}{4M} \sum_{t=0}^{2M-1} R_t e^{j2\pi(k-l)t/(2M)} \end{aligned} \quad (58)$$

which can be interpreted as the inverse DFT of the sequence of noise covariances R_k . To simplify notation denote by $\overline{E}$ the operator

$$\overline{E} = \lim_{M \rightarrow \infty} EM \quad (59)$$

where E is the expectation operator.

The asymptotic variance is given by the following expression

$$\begin{aligned}
P(z) &= \overline{E} \{ \Delta G(z) \Delta G(z)^H \} \\
&= \overline{E} \{ (\Delta D + \Delta C(zI - A)^{-1} B + C(zI - A)^{-1} \Delta A(zI - A)^{-1} B \\
&\quad + C(zI - A)^{-1} \Delta B)(\dots)^H \} \\
&= \overline{E} \{ \Delta D \Delta D^T + \Delta C(zI - A)^{-1} B(\dots)^H \\
&\quad + C(zI - A)^{-1} \Delta A(zI - A)^{-1} B(\dots)^H \\
&\quad + C(zI - A)^{-1} \Delta B(\dots)^H \\
&\quad + 2 \operatorname{Re} [\Delta D [\Delta C(zI - A)^{-1} B + C(zI - A)^{-1} \Delta A(zI - A)^{-1} B \\
&\quad + C(zI - A)^{-1} \Delta B]^H \\
&\quad \times C(zI - A)^{-1} \Delta A(zI - A)^{-1} B B^T (zI - A)^{-H} \Delta C^T \\
&\quad + C(zI - A)^{-1} \Delta A(zI - A)^{-1} B \Delta B^T (zI - A)^{-H} C^T \\
&\quad + \Delta C(zI - A)^{-1} B \Delta B^T (zI - A)^{-H} C^T] \} \\
&= R_{\Delta h}(0) + V_1 + V_2 + V_3 + 2 \operatorname{Re} (V_4 + V_5 + V_6 + V_7 + V_8 + V_9)
\end{aligned} \tag{60}$$

where

$$\begin{aligned}
V_1 &= \overline{E} \{ \Delta C(zI - A)^{-1} B(\dots)^H \} \\
&= J_3 U_o U_o^T \overline{E} \{ \Delta H C^\dagger (zI - A)^{-1} B \\
&\quad \times B^T (zI - A)^{-H} (C^T)^\dagger \Delta H^T \} U_o U_o^T J_3^T
\end{aligned} \tag{61}$$

$$\begin{aligned}
V_2 &= \overline{E} \{ C(zI - A)^{-1} \Delta A(zI - A)^{-1} B(\dots)^H \} \\
&= C(zI - A)^{-1} (J_1 \mathcal{O})^\dagger [\\
&\quad J_2 U_o U_o^T \overline{E} \{ \Delta H C^\dagger (zI - A)^{-1} B \\
&\quad \times B^T (zI - A)^{-H} (C^T)^\dagger \Delta H^T \} U_o U_o^T J_2^T \\
&\quad - 2 \operatorname{Re} \{ J_1 U_o U_o^T \overline{E} \{ \Delta H C^\dagger A(zI - A)^{-1} B \\
&\quad \times B^T (zI - A)^{-H} (C^T)^\dagger \Delta H^T \} U_o U_o^T J_2^T \} \\
&\quad + J_1 U_o U_o^T \overline{E} \{ \Delta H C^\dagger A(zI - A)^{-1} B \\
&\quad \times B^T (zI - A)^{-H} A^T (C^T)^\dagger \Delta H^T \} U_o U_o^T J_1^T \\
&\quad] (\mathcal{O}^T J_1^T)^\dagger (zI - A)^{-H} C^T
\end{aligned} \tag{62}$$

$$\begin{aligned}
V_3 &= \overline{E} \{ C(zI - A)^{-1} \Delta B(\dots)^H \} \\
&= C(zI - A)^{-1} \mathcal{O}^T \overline{E} \{ \Delta H J_4 J_4^T \Delta H^T \} \mathcal{O} (zI - A)^{-H} C^T
\end{aligned} \tag{63}$$

$$\begin{aligned}
V_4 &= \overline{E} \{ \Delta D [\Delta C(zI - A)^{-1} B]^H \} \\
&= \overline{E} \{ \Delta h_0 B^T (zI - A)^{-H} (C^T)^\dagger \Delta H^T \} U_o U_o^T J_3^T
\end{aligned} \tag{64}$$

$$\begin{aligned}
V_5 &= \overline{E} \{ \Delta D [C(zI - A)^{-1} \Delta A(zI - A)^{-1} B]^H \} \\
&= [\overline{E} \{ \Delta h_0 B^T (zI - A)^{-H} (C^T)^\dagger \Delta H^T \} U_o U_o^T J_2^T \\
&\quad - \overline{E} \{ \Delta h_0 B^T (zI - A)^{-H} A^T (C^T)^\dagger \Delta H^T \} U_o U_o^T J_1^T] \\
&\quad \times (\mathcal{O}^T J_1^T)^\dagger (zI - A)^{-H} C^T
\end{aligned} \tag{65}$$

$$\begin{aligned}
V_6 &= \overline{E} \{ \Delta D [C(zI - A)^{-1} \Delta B]^H \} \\
&= \overline{E} \{ \Delta h_0 J_4^T \Delta H^T \} \mathcal{O} (zI - A)^{-H} C^T
\end{aligned} \tag{66}$$

$$\begin{aligned}
V_7 &= \overline{E} \{ C(zI - A)^{-1} \Delta A(zI - A)^{-1} B B^T (zI - A)^{-H} \Delta C^T \} \\
&= C(zI - A)^{-1} (J_1 \mathcal{O})^\dagger \\
&\quad [J_2 U_o U_o^T \overline{E} \{ \Delta H C^\dagger (zI - A)^{-1} B B^T (zI - A)^{-H} (C^T)^\dagger \Delta H^T \} \\
&\quad - J_1 U_o U_o^T \overline{E} \{ \Delta H C^\dagger A(zI - A)^{-1} B \\
&\quad \times B^T (zI - A)^{-H} (C^T)^\dagger \Delta H^T \} U_o U_o^T J_3^T]
\end{aligned} \tag{67}$$

$$\begin{aligned}
V_8 &= \overline{E} \{ C(zI - A)^{-1} \Delta A (zI - A)^{-1} B \Delta B^T (zI - A)^{-H} C^T \} \\
&= C(zI - A)^{-1} (J_1 \mathcal{O})^\dagger [J_2 U_o U_o^T \overline{E} \{ \Delta H C^\dagger (zI - A)^{-1} B J_4^T \Delta H^T \} \\
&\quad - J_1 U_o U_o^T \overline{E} \{ \Delta H C^\dagger A (zI - A)^{-1} B J_4^T \Delta H^T \}] \mathcal{O} (zI - A)^{-H} C^T
\end{aligned} \tag{68}$$

$$\begin{aligned}
V_9 &= \overline{E} \{ \Delta C (zI - A)^{-1} B \Delta B^T (zI - A)^{-H} C^T \} \\
&= J_3 U_o U_o^T \overline{E} \{ \Delta H C^\dagger (zI - A)^{-1} B J_4^T \Delta H^T \} \mathcal{O} (zI - A)^{-H} C^T
\end{aligned} \tag{69}$$

In the expressions given above we have deliberately removed any factors A^{2M} although the limit is not “completely evaluated”. However this is only abuse of notation and do not change the final result. In the expressions two expectations occur frequently. For simplicity assume single input single output case. Then these expressions are given as follows.

Let X be a matrix of size $r \times r$. For matrix element i, j we obtain

$$\begin{aligned}
[\overline{E} \{ \Delta H X \Delta H^T \}]_{i,j} &= \sum_{k=1}^r \sum_{m=1}^r \overline{E} \{ \Delta h_{i+k-1} x_{k,m} \Delta h_{m+j-1} \} \\
&= \sum_{k=1}^r \sum_{m=1}^r x_{k,m} R_{\Delta h}(i + k - m - j)
\end{aligned} \tag{70}$$

where $[X]_{i,j}$ denotes the element of the matrix X located at row i and column j .

Let X denote a $1 \times r$ row vector which gives the expectation the value

$$[E \{ \Delta h_0 x \Delta H^T \}]_{1,i} = \sum_{k=1}^r x_k R_{\Delta h}(k + i - 1). \tag{71}$$

If we consider the special case when $R_k = \lambda \forall k$ then

$$R_{\Delta h}(k) = \frac{\lambda}{2} \delta(k)$$

where $\delta(k)$ is Kronecker’s delta function. In this case (70) and (71) simplifies to

$$[\overline{E} \{ \Delta H X \Delta H^T \}]_{i,j} = \frac{\lambda}{2} \sum_{k=1}^r \sum_{m=1}^r x_{k,m} \delta(i + k - m - j)$$

and

$$[E \{ \Delta h_0 x \Delta H^T \}]_{1,i} = \frac{\lambda}{2} \sum_{k=1}^r x_k \delta(k + i - 1) = 0, \quad \forall i.$$

5 Verification by Monte Carlo Simulation

We shall in this Section verify the validity of the asymptotic variance expression derived in section 4.3.3 for finite data. We do this by use of so called Monte Carlo simulations wherein we estimate the variance by calculating the sample variance of a large number of noise realizations.

The variance $P(z)$ given by (51) is only valid as M tends to infinity. However it is common to use such an expression [13] also for finite data to get an estimate of the resulting variance. In our case we have approximately

$$E |\hat{G}_M(z) - G(z)|^2 \approx \frac{1}{M} P(z)$$

where subscript M in $\hat{G}_M(z)$ explicitly shows the dependency of the data record length.

As an example the following second order system

$$y(t) = \frac{10^{-2}(6.376 q^2 + 9.140 q + 6.204)}{q^2 - 1.613 q + 0.828} u(t) = G(q) u(t) \quad (72)$$

will be used. The magnitude of the transfer function is depicted in figure 1. Using the transfer function G 101 samples of the frequency response are generated using a linear frequency grid ranging from $\omega_0 = 0$ to $\omega_M = \pi$. Each frequency response sample is corrupted with independent complex Gaussian noise with three different variance levels $R = 1$, $R = 0.1$ and $R = 0.01$. Using different realizations of the noise 3×10000 different estimation data sets is generated.

From the data sets 3×10000 models of second order are estimated using Algorithm 1. Let the estimated models be denoted by $\hat{G}_k$ where subscript k indicates which data set is used in the identification. The estimated variance of the transfer function is evaluated as

$$P_{MC}(\omega_k) = \frac{1}{10000} \sum_{k=1}^{10000} |\hat{G}_k(e^{j\omega_k}) - G(e^{j\omega_k})|^2. \quad (73)$$

The result of the simulations are depicted in Figure 2. The theoretical and simulated values are very close for the two lower noise levels. A somewhat larger discrepancy can be seen from the highest noise level. This is in line with the analysis since the theoretical expression is derived under the assumption that the noise level is sufficiently low. However, a noise level with variance 1 is quite high compared with the maximal magnitude of the transfer function $\|G\|_\infty = 2.55$.

This example clearly shows that the asymptotic variance expression (51) indeed can be used to accurately estimate the resulting variance the estimated transfer function when using Algorithm 1.

6 Conclusions

We have in this paper presented a non-iterative frequency domain state-space identification algorithm. If the frequency data is noise free and is generated by a n th order system we show that only $n + 2$ equidistant frequency samples are required to exactly recover the true system. If the measurements are contaminated with unmodeled dynamics and/or noise upper bounded by ϵ we show that the resulting identification error is upper bounded by ϵ and hence the algorithm is robust. An asymptotic stochastic analysis shows that the algorithm is strongly consistent if each measurement is perturbed by an independent stochastic noise term. Furthermore we show that the resulting estimated transfer function is asymptotically normal distributed and give an explicit expression of the asymptotic variance.

Figure 1: Magnitude of transfer function $G(z)$ defined in (72)

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Figure 2: Result of Monte Carlo Simulations for three different noise levels, $R = 1$, $R = 0.1$ and $R = 0.01$. The solid line shows $P_{MC}(z)$ and the dashed line shows $\frac{1}{M}P(z)$ given by 51.

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A Some Matrix Results

The following lemma describes the resulting perturbation of the pseudo inverse of a perturbed matrix.

Lemma 4 [19] *Let $A \in \mathbb{C}^{m \times n}$, where $m \geq n$ and $\hat{A} = A + \Delta A$. If $\text{rank}(A) = \text{rank}(\hat{A})$, then*

$$\|\hat{A}^\dagger - A^\dagger\|_F \leq \|A^\dagger\|_2 \|\hat{A}^\dagger\|_2 \|\Delta A\|_F. \quad (74)$$

Proof. See Theorem 3.9 in [19]. $\square$

In the next lemma we show that the norm of the perturbed pseudo inverse is always bounded if the perturbation is sufficiently small.

Lemma 5 *Let $A, \Delta A \in \mathbb{R}^{m \times n}$, $m \geq n$, let $\|\Delta A\| = \epsilon$ and let $\|\cdot\|$ be any consistent matrix norm. Also let A be of full rank n , then*

$$\|(A + \Delta A)^\dagger\| \leq c, \quad \forall \epsilon < \epsilon_0 \quad (75)$$

with

$$\epsilon_0 = \sqrt{\|A\|^2 + \|(A^T A)^{-1}\|^{-1}} - \|A\|$$

and

$$c = \frac{\|(A^T A)^{-1}\|}{1 - \|A^T A\|(\epsilon_0 + 2\|A\|)\epsilon_0} (\|A\| + \epsilon_0).$$

Proof. Let $X = A^T A$ and $\Delta X = \Delta A^T \Delta A + \Delta A^T A + A^T \Delta A$. We then get

$$\|(A + \Delta A)^\dagger\| = \|(X + \Delta X)^{-1}(A + \Delta A)^T\| \leq \|(X + \Delta X)^{-1}\| \|A + \Delta A\| \quad (76)$$

Now from the construction of X we conclude that

$$\|\Delta X\| \leq (\|\Delta A\| + 2\|A\|)\|\Delta A\| \leq (\epsilon + 2\|A\|)\epsilon$$

X is of full rank since A is of full rank. $\|X^{-1}\|$ is therefore bounded. Then from the definition of ϵ_0

$$\|X^{-1}\Delta X\| \leq \|X^{-1}\|(\epsilon + 2\|A\|)\epsilon < 1, \quad \forall \epsilon < \epsilon_0.$$

By the use of Theorem 2.3.4 in [3] we obtain

$$\|(X + \Delta X)^{-1}\| \leq \frac{\|X^{-1}\|}{1 - \|X^{-1}\Delta X\|} \leq c_2, \quad \forall \epsilon < \epsilon_0 \quad (77)$$

where $c_2 = \frac{\|X^{-1}\|}{1 - \|X^{-1}\|(\epsilon_0 + 2\|A\|)\epsilon_0}$. By inserting (77) into (76) we obtain

$$\|(A + \Delta A)^\dagger\| \leq c_2(\|A\| + \epsilon) \leq c, \quad \forall \epsilon < \epsilon_0$$

□

Remark 3 *For the inverse of a perturbed square matrix Lemma 5 is also applicable by setting $m = n$.*