Time-subspace projection for bias-correction in system identification

P. Carrette

Department of Electrical Engineering
Linköping University, S-581 83 Linköping, Sweden
WWW: <http://www.control.isy.liu.se>
Email: carrette@isy.liu.se

March 11, 1998

Report no.: LiTH-ISY-R-2018

Accepted at CDC-97

Technical reports from the Automatic Control group in Linköping are available by anonymous ftp at the address [ftp.control.isy.liu.se](ftp://ftp.control.isy.liu.se). This report is contained in the compressed postscript file 2018.ps.Z.

Time-subspace projection for bias-correction in system identification

P. Carrette* - carrette@isy.liu.se

Department of Electrical Engineering, Linköping University,
S-58183 Linköping, Sweden

Abstract

We present results concerning the parameter estimates obtained by prediction error methods in the case of input signals that are insufficiently rich when considered locally in time. As is intuitively obvious, the data located in time intervals where the system excitation is poor carry only an incomplete information about the system input-to-output (I/O) dynamics. In noise undermodeling situations, this leads to “local” model parameters presenting large bias outside the related excitation subspace. We here propose to decrease this bias error in taking into account the parameter estimates only in the system excitation subspaces associated to the different time intervals.

Keywords: system identification, time-subspace excitation, direction-of-arrival techniques

1 Introduction

In this paper, we consider an identification problem whose objective is to estimate the I/O dynamics of a single input single output system by the use of an ARX model structure that is able to represent the system I/O dynamics exactly but not its disturbance dynamics. Furthermore, we are interested in situations where the data measurements originate from system excitations that exhibit insufficiently rich characteristics when considered locally in time. More precisely, the system input would be the concatenation of sequences that are exciting of order possibly less than the number of model parameters. Such input signal can be viewed as being composed of successive sine-waves of different frequencies.

If we consider the identification data located within the time intervals corresponding to these input sine-waves then, apart from transient effects, the system input contribution to each of these data subsets may exhibit only a partial information on the system I/O

*This work was supported by the Swedish Research Council for Engineering Sciences.

dynamics. That is to say that, due to noise undermodeled dynamics, the model parameter estimate that is evaluated by use of a particular data subset only shows up a large deviation in the “null space” of the deterministic contribution to this data subset. The reason for this is that no system input is found in that “null space” so that it is related to system disturbance excitations that do not bring any knowledge of the corresponding system I/O dynamics.

In order to overcome this subspace bias problem, we propose to cancel out the model parameter components originating from the “null space” of the data subset. This is as if the estimated model I/O dynamics were only trusted within the pass-bands where system input excitation is present at the current time interval. Repeating this operation for each data subset, one would be left with improved model parameters that could be rearranged to lead to a model estimate that includes all the information on the system I/O dynamics. As a whole, such an improving method could be related to fictitious data measurements exhibiting larger signal-to-noise ratio: in each time period, the system disturbance is filtered out to only contribute in the pass-bands corresponding to the related sine-wave excitations.

2 System, model and identification data

The true system is a scalar stable SISO system written as

$$A_0(q)\tilde{y}(t) = B_0(q)u(t) + C_0(q)e(t) \quad (1)$$

where $(u(t), \tilde{y}(t))$ is the system I/O data pair and $e(t)$ is a zero-mean white-noise with bounded moments while $A_0(q) = 1 + a_{01}q^{-1} + \dots + a_{0n_a}q^{-n_a}$ and $B_0(q) = b_{01}q^{-1} + \dots + b_{0n_b}q^{-n_b}$. The $C_0(q)$ dynamics stands for a monic stable rational function in q^{-1} .

The deterministic characteristic of the input signal means that each $u(t)$ sample is independent of the system disturbance in (1). In view of this, it is convenient to denote the system output as $\tilde{y}(t) = y(t) + y_e(t)$ in order to distinguish between its deterministic part coming from the system input, i.e. $y(t) = [B_0(q)/A_0(q)]u(t)$, and that taking into account the system disturbance contributions, i.e. $y_e(t) = [C_0(q)/A_0(q)]e(t)$. Note that $y(t) = E(\tilde{y}(t))$ for the system disturbance is zero-mean.

We choose to identify this system by use of an ARX model structure [2] of the form

$$A(q)\tilde{y}(t) = B(q)u(t) + \varepsilon(t) \quad (2)$$

where $A(q) = 1 + a_1q^{-1} + \dots + a_{n_a}q^{-n_a}$ and $B(q) = b_1q^{-1} + \dots + b_{n_b}q^{-n_b}$, while $\varepsilon(t)$ stands for the model prediction error.

As the degrees of the polynomials constituting the I/O dynamics of the system and of the model are identical (i.e. n_a and n_b , respectively), the system I/O dynamics can be modeled exactly. By contrast, the system noise-to-output dynamics does not belong to the model set (except in the trivial situation where $C_0(q) = 1$).

As mentioned above, our aim is to identify the system I/O dynamics, i.e. $B_0(q)/A_0(q)$ as

accurately as possible from open-loop data, despite the fact that the system disturbance dynamics is undermodeled.

At any sample time t , an output prediction $\hat{y}(t|\theta)$ can be associated with the model equation (2) by the relation

$$\hat{y}(t|\theta) = B(q)u(t) - (A(q) - 1)\tilde{y}(t) \quad (3)$$

where $\theta = [a_1, \dots, a_{n_a}, b_1, \dots, b_{n_b}]^T$ is called the model parameter vector. With the regressor vector $\tilde{\varphi}(t) = [-\tilde{y}(t-1), \dots, -\tilde{y}(t-n_a), u(t-1), \dots, u(t-n_b)]^T$, we can rewrite the system and the model equation as

$$\tilde{y}(t) = [\tilde{\varphi}(t)]^T \theta_0 + \varepsilon_0(t) \quad \text{and} \quad \hat{y}(t|\theta) = [\tilde{\varphi}(t)]^T \theta \quad (4)$$

where $\theta_0 = [a_{01}, \dots, a_{0n_a}, b_{01}, \dots, b_{0n_b}]^T$ (with a_{0n_a} and b_{0n_b} assumed nonzero) is the system parameter vector, also called the true parameter vector, and $\varepsilon_0(t)$ stands for the unmodeled part of the system disturbance, i.e. $\varepsilon_0(t) = C_0(q)e(t)$ also identical to the output prediction error evaluated at θ_0 .

Similarly to the system output, it is possible to separate the regressor vector into two parts as $\tilde{\varphi}(t) = \varphi(t) + \varphi_e(t)$ in which $\varphi(t) = E(\tilde{\varphi}(t))$ denotes the deterministic contribution to $\tilde{\varphi}(t)$.

Remark 1 In vector form, we write the equation (4) as:

$$\tilde{y} = \tilde{\Phi} \theta_0 + \varepsilon_0 \quad \text{and} \quad \hat{y}(\theta) = \tilde{\Phi} \theta \quad (5)$$

where $\tilde{y} = [\tilde{y}(1), \dots, \tilde{y}(N)]^T$ is a column containing the (N -length) realization of the system output. Similarly, $\hat{y}(\theta)$ is the predicted output column at θ and ε_0 is the unmodeled system disturbance column while the matrix $\tilde{\Phi} = [\tilde{\varphi}(1), \dots, \tilde{\varphi}(N)]^T$ denotes the regressor matrix. We also have $\tilde{\Phi} = \Phi + \Phi_e$ with $\Phi = E(\tilde{\Phi})$. $\square$

Finally, let us state the characteristics of the data set, i.e. $Z^N = \{(\tilde{y}(t), u(t)), t = 1 \dots N\}$. In view of the system disturbance in (1), the output data are seen to constitute a N -length realization of the random signal $\tilde{y}(t)$.

Moreover, the input data record originates from the concatenation of parts of *persistently exciting* sequences, of order possibly less than n (see [2]). More precisely, we assume that the matrix Φ , i.e. the deterministic contribution to the model regressor matrix, is full column rank and can be partitioned into several *irreducible* row-blocks Φ_i in such a way that :

$$\Phi = [\dots, \Phi_i^T, \dots]^T \quad (6)$$

with $\Phi_i \in \mathcal{R}^{N_i \times n}$, $N_i \geq n$ and $\text{rank}(\Phi_i) = n_i \leq n$. The irreducibility of the row-block Φ_i stands for the fact that there does not exist any $\Phi_s, \Phi_l \subset \Phi$ such that $\Phi_s \subset \Phi_i \subset \Phi_l$ with $\text{rank}(\Phi_s) < n_i = \text{rank}(\Phi_l)$.

With the help of the singular value decomposition (SVD)[3], we can express each row-block Φ_i as

$$\Phi_i = U_i \Sigma_i V_i^T$$

where $\Sigma_i = \text{diag}(\sigma_{i1} \cdots \sigma_{in_i})$ with $\sigma_{ij} > 0$ the j -th (in decreasing order) singular value of Φ_i and $U_i \in \mathcal{R}^{N_i \times n_i}$ while $V_i = (v_{ij}, j \in [1, n_i]) \in \mathcal{R}^{n \times n_i}$ together with $U_i^T U_i = V_i^T V_i = I_{n_i}$. We also denote by P_i the orthogonal projector onto the orthogonal complement of the null space of Φ_i , i.e. $P_i = V_i V_i^T$.

It is important to note that the row-partitioning of the matrix Φ can be related to subspace (or frequency) excitation properties of the system input $u(t)$ that are local in time. This originates from the fact that $[\varphi(t)]^T x$ (with $x \in \mathcal{R}^n$) constitutes a particular filtering of the system input, i.e.

$$\begin{aligned} [\varphi(t)]^T x &= (-X_y(q)[B_0(q)/A_0(q)] + X_u(q))u(t) \\ &= G_x(q)u(t) \end{aligned} \quad (7)$$

where $X_y(q) = x_1 q^{-1} + \cdots + x_{n_a} q^{-n_a}$ and $X_u(q) = x_{n_a+1} q^{-1} + \cdots + x_{n_a+n_b} q^{-n_b}$. Thus, for any singular partition Φ_i (with $n_i < n$), we have that $\Phi_i x = 0$ when $P_i x = 0$. That is to say that, in the time interval determined by the row bounds of this i -th partition, there exists filters that cancel out the system input signal. So, these filters, i.e. $G_x(q)$'s with $P_i x = 0$, indicate a frequency dead-zone for this time interval: this is the concept of local time-frequency excitation.

Hence, the system input signal results from the concatenation of possibly insufficiently exciting subsignals but, as a whole, this signal is sufficiently exciting because it is associated to a full rank matrix Φ .

Remark 2 It is worth defining row-blocks of the regressor matrix $\tilde{\Phi}$ correspondingly to those in the matrix Φ , i.e. $\tilde{\Phi}_i = \Phi_i + \Phi_{e,i}$ for each i . Similarly for $\tilde{y}_i$ and $\varepsilon_{0,i}$ belonging to the column $\tilde{y}$ and ε_0 , respectively. $\square$

3 Bias of the LS solution

The parameter estimation approach used in this paper is the LS estimate of the linear model (5): it consists of minimizing the mean square of the model prediction errors, i.e. $\varepsilon(t|\theta) = y(t) - \hat{y}(t|\theta)$, over all possible values of the parameter vector θ . The LS solution $\tilde{\theta}$ is

$$\tilde{\theta} = \underset{\theta \in \mathcal{R}^n}{\text{argmin}} \sum_t \varepsilon^2(t|\theta) / 2N \quad (8)$$

This vector $\tilde{\theta}$ is classically written in terms of the pseudo-inverse $\tilde{\Phi}^+$ (see e.g. [3]) of the regressor matrix $\tilde{\Phi}$, i.e.

$$\tilde{\theta} = \tilde{\Phi}^+ \tilde{y} = [\tilde{\Phi}^T \tilde{\Phi}]^{-1} (\tilde{\Phi}^T y) = \theta_0 + [\tilde{\Phi}^T \tilde{\Phi}]^{-1} (\tilde{\Phi}^T \varepsilon_0) \quad (9)$$

In the sequel, we refer to the cross-product of the regressor matrix, i.e. $\tilde{\Phi}^T \tilde{\Phi}$, as the *information matrix* of the model estimation.

In order to derive a simple expression for the bias error of this LS solution, i.e. $E(\tilde{\theta} - \theta_0)$ where $E(\cdot)$ denotes the mathematical expectation over the system white-noise distribution, let us introduce the following assumption (see [1, Assumption 2.2]).

Assumption 1 (excitation) *The expectation of the information matrix of the model estimation is such that*

$$\beta\sigma_e^2 \ll \lambda_{\min}(E(\tilde{\Phi}^T \tilde{\Phi}))$$

where σ_e^2 is the variance of the system white-noise and β is related to the cross-correlation of the system disturbance in case σ_e^2 is one (see in [1, Corollary 2.2]). $\square$

This assumption actually reflects situations where the system input energy induced in the regressor matrix globally dominates the system disturbance power.

It has been shown in [1] that this excitation assumption allows us to substitute the expectation of the information matrix, i.e.

$$E(\tilde{\Phi}^T \tilde{\Phi}) = \Phi^T \Phi + N\Delta$$

where $\Delta = E([\varphi_e(t)]^T [\varphi_e(t)])$ for any time t due to the stationarity of the system disturbance, for its actual value $\tilde{\Phi}^T \tilde{\Phi}$ in the expression of $\tilde{\theta}$: we then avoid dealing with the random inverse.

By use of this assumption, the bias error of the LS solution $\tilde{\theta}$ can be written as

$$E(\tilde{\theta} - \theta_0) \approx ([\Phi^T \Phi + N\Delta]^{-1} N) c_e \quad (10)$$

where $c_e = E([\varphi_e(t)]^T \varepsilon_0(t))$ because of the system disturbance. From a 2-norm point of view, we have the following upper-bound

$$\|E(\tilde{\theta} - \theta_0)\|_2 \leq (\|[\Phi^T \Phi + N\Delta]^{-1}\|_2 N) \|c_e\|_2 \quad (11)$$

where the first 2-norm factor of the RHS is identical to the inverse of the smallest eigenvalue of the expected information matrix, i.e. $\lambda_{\min}(\Phi^T \Phi + N\Delta)$.

It is worth making the following comments.

- The vector c_e exhibits the correlation between the random part of the regressor vector and the unmodeled part of the system disturbance. It is nonzero unless $C_0(q) = 1$ for which $\varepsilon_0(t) = e(t)$.
- The upper-bound on the 2-norm of the bias error of LS solution can be seen as a valuable *noise-to-signal* ratio: $\lambda_{\min}(\Phi^T \Phi + N\Delta)$ and $N\|c_e\|_2$ being the “signal” and “noise” components, respectively.
- The final value of this bias error depends on the contributions of the system input (within Φ) to the “signal” component. Due to the system disturbance, it obviously is smaller than $\|c_e\|_2/\lambda_{\min}(\Delta)$.

4 Time-subspace projection of the regressor matrix

We here take advantage of the structure of the deterministic contribution to the regressor matrix in order to provide a “cleaned” version of this latter matrix.

First, let us assume that the structure of the deterministic part of any row-block $\tilde{\Phi}_i$ can be read into the expectation of its associated information matrix.

Assumption 2 *The expectation of the information matrix of the row-block $\tilde{\Phi}_i$ is such that*

$$\delta_M \ll \tilde{\lambda}_{in_i}/N_i$$

where $\delta_M = \lambda_{\max}(\Delta)$ and $\tilde{\lambda}_{ij} = \lambda_j(E(\tilde{\Phi}_i^T \tilde{\Phi}_i))$ denotes the j -th (in decreasing order) eigenvalue of $E(\tilde{\Phi}_i^T \tilde{\Phi}_i)$. $\square$

By use of Weyl's inequality [3, p. 211] leading to

$$\tilde{\lambda}_{in_i}/N_i \leq \sigma_{in_i}^2/N_i + \delta_M, \quad (12)$$

this assumption is equivalent to state a large signal-to-noise ratio within the subspaces excited by the system input, i.e. $\mathcal{R}(\Phi_i^T)$. Indeed, similarly to Section 2, it is seen that

$$\begin{aligned} [\tilde{\varphi}(t)]^T x &= G_x(q)u(t) + (-X_y(q)[C_0(q)/A_0(q)])e(t) \\ &= G_x(q)u(t) + H_x(q)e(t) \end{aligned} \quad (13)$$

for any $x \in \mathcal{R}^n$. Obviously, this relation holds for any row blocks $\tilde{\Phi}_i$. So, the inequality (12) together with Assumption 2 implies that

$$\sum_{t \in \mathcal{T}_i} \|G_{x_i}(q)u(t)\|_2^2/N_i \gg \delta_M = \max_{x \in \mathcal{R}} E(\|H_x(q)e(t)\|_2^2)$$

where $x_i \in \mathcal{R}(\Phi_i^T)$ and $\mathcal{T}_i$ contains the rows indices determining the row-block Φ_i .

Moreover, Assumption 2 implies that two clusters can be drawn within the eigenvalue set of $E(\tilde{\Phi}_i^T \tilde{\Phi}_i)$, i.e.

$$\tilde{\lambda}_{in} \leq \dots \leq \tilde{\lambda}_{i(n_i+1)} \leq N_i \delta_M \ll \tilde{\lambda}_{in_i} \leq \dots \leq \tilde{\lambda}_{i1}$$

The larger the gap between these two clusters, the more robust the system input subspace with respect to the system disturbance influences (see e.g. [3, chap. 5]).

Now, let us restrict the effects of the system disturbance within the i -th row-block $\tilde{\Phi}_i$ of the regressor matrix to the subspace excited by the system input contributions. More precisely, the “cleaned” version of this i -th regressor row-block is defined as $\hat{\Phi}_i = \tilde{\Phi}_i \hat{P}_i$ where $\hat{P}_i$ denotes the orthogonal projector onto the subspace associated with the first n_i eigenvalues of the expected information matrix corresponding to $\tilde{\Phi}_i$, i.e.

$$x^T E(\hat{\Phi}_i^T \hat{\Phi}_i) x \geq \tilde{\lambda}_{in_i} \quad \text{for } x = \hat{P}_i x$$

with $\|x\|_2 = 1$. Note that Assumption 2 leads to $\hat{P}_i \approx P_i$ (see e.g. [3, p. 246]). It is also worth mentioning that the proposed projection performs particular filtering of the identification data set. Indeed, by use of the expression (13), we see that projecting the row-block $\tilde{\Phi}_i$ onto the subspace associated to $\hat{P}_i$ is similar to impose that the averaged contributions of the disturbance of the system cancel out in the respective frequency bands (given by $H_x(q)$ with $x \in \text{Ker}(\hat{\Phi}_i)$). This is performed only in the time interval associated to that row block.

By repeating this projection for each i , we end up with a “cleaned” version of the whole regressor matrix, i.e.

$$\hat{\Phi} = [\dots, \hat{\Phi}_i^T, \dots]^T \quad (14)$$

originating from particular “time-frequency” filtering of the identification data that would generate fictitious identification data exhibiting an improved signal-to-noise ratio.

Practical implementation

From a practical point of view, we need to locate the different regressor row-blocks as well as their associated projection subspaces from the realization of the regressor matrix $\tilde{\Phi}$ only.

First, in the light of Assumption 2, any row-block $\tilde{\Phi}_i$ can be written as $\tilde{\Phi}_i = \tilde{U}_i \tilde{\Sigma}_i \tilde{V}_i^T$ where $\tilde{\Sigma}_i = \text{diag}(\tilde{\sigma}_{i1} \cdots \tilde{\sigma}_{in})$ with $\tilde{\sigma}_{ij} = \sigma_j(\tilde{\Phi}_i)$ satisfying $\tilde{\sigma}_{ij}^2 \approx \tilde{\lambda}_j$ for $j \leq n_i$ and $\ll \tilde{\lambda}_{n_i}$ for $j > n_i$. The matrices $\tilde{U}_i$ and $\tilde{V}_i = (\tilde{v}_{i1} \cdots \tilde{v}_{in})$ are left-orthogonal. Thus, despite the fact that $\tilde{\Phi}_i$ is generally full column rank, a relevant gap is found in its singular value set. Furthermore, the subspace associated to its first n_i right singular vectors, i.e. $\mathcal{R}([\tilde{v}_{i1} \cdots \tilde{v}_{in_i}])$, is approximately that corresponding to the range of the projector $\hat{P}_i$ [3, chap. 5]. Hence, a good estimate for the “cleaned” $\tilde{\Phi}_i$ is derived in cancelling out the last $(n - n_i)$ singular values of $\tilde{\Phi}_i$.

The consequence of this is that a practical algorithm for constructing the successive $\hat{\Phi}_i$ ’s can be derived on the basis of rank revealing (URV) techniques [4, 5]. This is an iterative algorithm whose steps are as follows. Let us say that $\tilde{\Phi}(t, t_i) = [\tilde{\varphi}(t_i) \cdots \tilde{\varphi}(t)]$ be the currently constructed row-block whose (virtual) rank is n_i , i.e. $\tilde{\sigma}_{in_i}^2(t) \gg \tilde{\sigma}_{i(n_i+1)}^2(t)$. Can $\tilde{\varphi}(t+1)$ be added to $\tilde{\Phi}(t, t_i)$? For this, we should test whether $\tilde{\Phi}(t+1, t_i)$ has rank n_i as well as whether $\tilde{\Phi}(t+1, t-n+2)$ has rank less than n_i . In case the first answer is no, a new row-block is started with $t_{i+1} = t+1$. In case the second answer is yes, $\tilde{\Phi}(t-n+1, t_i)$ forms the i -th row-block and a new row-block is started with $t_{i+1} = t-n+2$. This procedure is performed until the last regressor vector. Then the SVD of each row-block is evaluated and the estimate of $\hat{\Phi}$ is computed.

5 Improvement of the LS solution

In this section, we propose to evaluate the model parameter vector by use of the “cleaned” regressor matrix $\hat{\Phi}$. This is done in substituting $\hat{\Phi}$ for the original $\tilde{\Phi}$ in the expression of the LS solution (9). The corresponding parameter vector is then written as

$$\hat{\theta} = [\hat{\Phi}^T \hat{\Phi}]^{-1} (\hat{\Phi}^T \tilde{y}) = [\hat{\Phi}^T \hat{\Phi}]^{-1} \sum [\hat{\Phi}_i^T \hat{\Phi}_i] \hat{\theta}_i \quad (15)$$

where $\hat{\theta}_i = \hat{\Phi}_i^+ \tilde{y}_i$ stands for the parameter vector evaluated by use of the i -th projected regressor row-block only. This vector $\hat{\theta}_i$ actually lies in the range of the corresponding

orthogonal projector, i.e. $\hat{\theta}_i \in \mathcal{R}(\hat{P}_i)$. By use of (5), we further get

$$\hat{\theta}_i = P_i \theta_0 + \hat{\Phi}_i^+ (\varepsilon_{0,i} + [\tilde{\Phi}_i - \hat{\Phi}_i] \theta_0) \quad (16)$$

with $\hat{P}_i^\perp = [I_n - \hat{P}_i]$ so that $\hat{P}_i \hat{P}_i^\perp = 0$. So, the vector $\hat{\theta}$ can be viewed as a convex (in the matrix sense) combination of the restricted $\hat{\theta}_i$ vectors.

Now, let us elaborate on the bias error of the parameter vector $\hat{\theta}$. First, we assume that the “cleaned” regressor matrix satisfies the excitation assumption (see Assumption 1), i.e.

$$\beta \sigma_e^2 \ll \lambda_{\min}(E(\hat{\Phi}^T \hat{\Phi}))$$

In fact, a sufficient condition for this to hold is that every projected row-block satisfies a similar assumption: namely, $\beta \sigma_e^2 \ll \tilde{\lambda}_{in_i}$ that can be derived from Assumption 2 (provided N_i is large enough). Then, we can write

$$E(\hat{\theta} - \theta_0) \approx \left[\sum \hat{P}_k (\Phi_k^T \Phi_k + N_k \Delta) \hat{P}_k \right]^{-1} \sum N_i \hat{P}_i c_e$$

where we have used the fact that $\hat{P}_i$ is written in terms of the eigenvectors of $E(\tilde{\Phi}_i \tilde{\Phi}_i)$, i.e. $\hat{P}_i E(\tilde{\Phi}_i^T \tilde{\Phi}_i) \hat{P}_i^\perp = 0$, as well as the fact that the deterministic part of a regressor row-block is uncorrelated with the undermodeled system disturbance, i.e. $E(\Phi_i^T \varepsilon_{i,0}) = 0$.

Remark 3 When enlightening the contributions of each parameter vector $\hat{\theta}_i$, we can write

$$E(\hat{\theta} - \theta_0) \approx \left[E(\hat{\Phi}^T \hat{\Phi}) \right]^{-1} \sum [E(\hat{\Phi}_i^T \hat{\Phi}_i)] E(\hat{\theta}_i - \hat{P}_i \theta_0)$$

where we have assumed that each $\hat{\Phi}_i$ satisfies the excitation assumption 1, i.e. $\beta \sigma_e^2 \ll \tilde{\lambda}_{in_i}$. Furthermore, the 2-norm of the bias of $\hat{\theta}_i$ can be upper-bounded by

$$\|E(\hat{\theta}_i - \hat{P}_i \theta_0)\|_2 \leq N_i \|c_e\|_2 / \tilde{\lambda}_{in_i} \ll \|c_e\|_2 / \lambda_{\min}(\Delta)$$

because the signal-to-noise ratio in the signal subspace of the row-block is assumed large. $\square$

Thus, the 2-norm of the bias error is upper-bounded by

$$\left\| \left[\sum \hat{P}_k (\Phi_k^T \Phi_k + N_k \Delta) \hat{P}_k \right]^{-1} \sum N_i \hat{P}_i \right\|_2 \|c_e\|_2 \quad (17)$$

The first factor in the RHS is slightly different from that in the 2-norm bound of the bias error of the original $\tilde{\theta}$ in expression (11). It is now expressed in terms of the convex sum of projected matrices, i.e. $\pi_i \hat{P}_i (\Phi_i^T \Phi_i / N_i + \Delta) \hat{P}_i$, and of corresponding projectors, i.e. $\pi_i \hat{P}_i$, with $\pi_i = N_i / N$ summing to one.

Finally, let us show that, in the 2-norm sense, the vector $\hat{\theta}$ is an improved model parameter estimate compared to what $\tilde{\theta}$ is. Thus, we derive an expression of the ratio between the first factor in the RHS of (17) and that of (11).

Theorem 1 Let Ξ_i be nonnegative definite symmetric matrices having rank n_i so that $\Xi = \sum_i \pi_i \Xi_i$, with $\pi_i > 0$ summing to one, is regular. Let P_i be the orthogonal projector onto $\mathcal{R}(\Xi_i)$ and $P = \sum_i \pi_i P_i$. Let Δ denote a nonnegative definite symmetric matrix. Finally, define $\hat{P}_i$ as an orthogonal projector of rank $\hat{n}_i$ and $\hat{\Xi}_i = \hat{P}_i(\Xi_i + \Delta)\hat{P}_i$ while $\hat{P} = \sum_i \pi_i \hat{P}_i$ as well as $\hat{\Xi} = \sum_i \pi_i \hat{\Xi}_i$. Then, provided $\eta > 0$, we have

$$\|\hat{\Xi}^{-1}\hat{P}\|_2 / \|[\Xi + \Delta]^{-1}\|_2 \leq \eta \quad (18)$$

where

$$\eta = \frac{1}{\gamma} (\gamma_M \lambda_{\min}(P) + \delta_M) \left(1 - \frac{\delta}{1 - \delta} \sqrt{\kappa_2} \right)^{-1} \quad (19)$$

with $\gamma_M = \max_i(\lambda_{\max}(\Xi_i))$, $\gamma = [(\gamma_M + \delta_M) + (\gamma_m + \delta_m)]/2$ and $\delta = [1 - (\gamma_m + \delta_m)/(\gamma_M + \delta_M)]/2$ while $\gamma_m = \min_i(\lambda_{\hat{n}_i}(\hat{P}_i \Xi_i \hat{P}_i))$ and $\kappa_2 = \lambda_{\max}(\hat{P})/\lambda_{\min}(\hat{P})$. ■

It is easily seen that we can use this theorem while taking $\Xi_i = \Phi_i^T \Phi_i / N_i$ for each i . Let us then give practical interpretations about the improvement factor η :

- the quantity δ almost measures the disparity of the system input excitation within the eigensubspaces, i.e. γ_m far from γ_M , in view of Assumption 2 applied to each Φ_i .
- in case $\delta_M \ll \gamma_m \approx \gamma_M$, we have that $\delta \ll 1$ and $\hat{\Xi}_i \approx \gamma \hat{P}_i$ for each i . Thus,

$$\eta \approx (\lambda_{\min}(P) + \delta_M / \gamma_M) / (1 - \delta \sqrt{\kappa_2})$$

- the condition number κ_2 measures the *non-isotropy* of the $\hat{\Xi}_i$ contributions to $\hat{\Xi}$. Two opposite situations can occur: either the different projectors $\hat{P}_i$ together manage to describe the whole space or they miss to cover a particular subspace, i.e. $\lambda_{\min}(\sum_i \hat{P}_i) \geq 1$ or $\ll 1$ respectively.
- in case the $\hat{P}_i$'s randomly cover $\hat{n}_i$ -dimensional subspaces, we are left with $\lambda_k(\hat{P}) \approx \sum \pi_i \hat{n}_i / n$ for $k \in [1, n]$ by the fact that each $\hat{P}_i$ approximately participate for $\hat{n}_i / n$ to the relative coverage of the n dimensional space. Thus, we end up with $\kappa_2 \approx 1$.

Finally, from the numerator in the upper-bound of η , it is seen that smaller $\lambda_{\min}(P)$ leads to larger LS improvement. In particular, in case the P_i 's are randomly distributed with $\delta_M \ll \gamma_m \approx \gamma_M$, we end up with $\eta \approx \lambda_{\min}(P) \approx \sum_i \pi_i n_i / n$ that can stand for an asymptotic (in the number of row-blocks) estimate of η .

6 Simulation

In this section, we illustrate the concepts and results worked out in the preceding sections while considering the identification of the following ARMAX system

$$\tilde{y}(t) = [0.3q^{-1} / (1 - 0.9q^{-1} + 0.1q^{-2})]u(t) + e(t)$$

A particular realization of the identification data set is shown in Figure 1 in the case of a Gaussian white-noise $e(t)$ with $\sigma_e^2 = 0.01$. The system input is made up of four distinct

Figure 1: Input and output data measurements.

Figure 2: Rank of the successive row-blocks of Φ .

subsignals having sinusoidal shapes within the $[21, 90]$, $[91, 160]$, $[161, 230]$ and $[231, 300]$ time intervals. Their normalized frequencies are 0.1, 0.2, 0.3 and 0.4, respectively.

In the regressor formulation, the system is characterized by $\tilde{\varphi}(t) = [-\tilde{y}(t-1), -\tilde{y}(t-2), u(t-1)]^T$ and $\theta_0 = [-0.9, 0.1, 0.3]^T$ while $C_0(q) = (1 - 0.9q^{-1} + 0.1q^{-2})$. Thus, the parameter vector θ of the ARX model is three-dimensional and the regressor matrix $\tilde{\Phi}$ is made up of all the regressor vectors, i.e. $\tilde{\varphi}(t)$ for $t \leq t < 300$.

In Figure 2, we present the rank of the different row-blocks of the noise-free matrix Φ as a function of the row position. Nine row-blocks are globally found. The first row-block has rank zero because of no system input. Note that the row-blocks having full column rank correspond to the transient periods in the identification data set: occurrence of a new sinusoid in the system input. As expected, most of the time period of a particular sinusoid leads to a regressor row-block having rank two: it allows one to estimate two linear combination of the model parameter vector.

We then evaluate the bias of the estimated parameter vectors $\tilde{\theta}$ and $\hat{\theta}$ with respect to the true θ_0 by use of 250 Monte-Carlo simulations of the identification data set. For each of these simulations, the “cleaned” version of the regressor matrix, i.e. $\hat{\Phi}$ presented in Section 4, is evaluated by taking into account the row-block structure of the matrix Φ . The 2-norm of these experimental bias is shown in Figure 3. As expected, the parameter vector $\hat{\theta}$ has a smaller bias error than that of the original estimate $\tilde{\theta}$. In more details, we can comment on these bias errors while following the excitation properties of the system input as time evolves:

- for $t \leq 20$, no system input occurs so that the bias of $\tilde{\theta}$ is large from system disturbance undermodeling while the vector $\hat{\theta}$ is not evaluated, i.e. $\hat{\Phi}_1 = 0$.
- the first full column rank regressor row-block enters after $t = 20$. Thus, a large system input energy is added to the already available system disturbance energy (within $\tilde{\Phi}_1$). Unfortunately, the duration of this input excitation is too short so

Figure 3: Bias 2-norm of the estimated parameter vectors: $\tilde{\theta}$ in ‘—’ and $\hat{\theta}$ in ‘---’.

Figure 4: Improvement measures of the estimated model parameter bias (see text).

that its effects on the decrease of the bias of $\tilde{\theta}$ is rather small. As a result, the associated bias is still large. Contrarily, the actual excitation episode is the only one that serves for evaluating the vector $\hat{\theta}$ at present time. The corresponding bias error is then small because of large signal-to-noise ratio.

- after that, a rank two system excitation is found. This means that a one-dimensional subspace of the corresponding regressor row-block is still excited by the system disturbance only. As this subspace is found in the actual regressor matrix, the bias of $\tilde{\theta}$ remains large. On the opposite, the third “cleaned” regressor row-block does not consider this poorly excited subspace so that the bias of $\hat{\theta}$ is only influenced by the signal-to-noise ratio associated to the excited subspaces (due to the sinusoid).
- then, the second full column rank row-block appears. As its duration is quite large, it allows the bias of the two parameter vectors to decrease. Note that the bias 2-norm of $\tilde{\theta}$ decreases really slowly because the actual input excitation has to take over the already integrated disturbance energy .
- as the input excited subspace associated to the second rank two row-block, i.e. $t \in [104, 160]$, intersects this poorly excited subspace, the bias error of $\tilde{\theta}$ goes on decreasing at a similar rate while that of $\hat{\theta}$ has already achieved its value for this row-block because of almost invariant signal-to-noise ratio in the various input excited subspaces.

Then, the bias of the two parameter vectors goes on responding to the different row-block excitations. As time evolves, the improvement in the bias error becomes smaller but is always present.

In Figure 4, we present improvement measures of the bias of $\hat{\theta}$ over that of $\tilde{\theta}$ as functions of the rows in the original regressor matrix. The ratio between the experimental bias in Figure 3 appears in ‘—’. The LHS of the expression (18) in Theorem 1 (e.g. the

Figure 5: Standard deviation of the estimated parameter vectors: $\tilde{\theta}$ in ‘—’ and $\hat{\theta}$ in ‘--’.

theoretical improvement factor) is shown in ‘--’ while a particular realization of this 2-norm ratio appears in ‘-.’. Finally, the effective subspace coverage performed by the input excitation, i.e. $\sum_i \pi_i \hat{n}_i / n$, is depicted in ‘...’.

First, it is seen that the theoretical improvement factor well reflects the ratio between the two experimental bias 2-norms. This tends to make one consider the 2-norm upperbound in either (11) or (17) for the bias 2-norm itself. Furthermore, particular realizations of this theoretical factor are relevant for estimating the achieved improvement. It is also worth noticing that the effective subspace coverage performed by the input excitation gives a good approximation of the improvement factor for $t \geq 160$, i.e. after the first five regressor row-blocks. As proposed above, this coverage measure may serve as an asymptotic (in the number of row-blocks) estimate of the improvement factor.

Finally, we present in Figure 5 the standard deviation of the two estimated parameter vectors, i.e. $[E(\|\theta - E(\theta)\|_2^2)]^{1/2}$ for $\theta = \tilde{\theta}$ and $\hat{\theta}$. The important thing to see is that the variability of these two estimates are almost identical for $t \geq 90$. That is to say that the performed regressor projections do not at all make the variance of $\hat{\theta}$ become larger than that of the original $\tilde{\theta}$. So, the whole accuracy of the parameter vector $\hat{\theta}$, e.g. its total mean square error, is improved essentially because its bias is.

References

- [1] Carrette P., *Discarding data to perform more accurate identification of input/output system dynamics*, PhD Thesis, Louvain-la-neuve (Belgium), March 1997.
- [2] Ljung L., *System Identification : theory for the user*, Prentice Hall, 1987.

- [3] Stewart G.W. and J.G. Sun, *Matrix Perturbation Theory*, Computer science and Scientific computing, Academic Press, Inc. San Diego, 1990.
- [4] Stewart G.W., *An updating algorithm for subspace tracking*, IEEE Trans. SP, Vol. 40, pp. 1535-1541, 1992.
- [5] Yang B. and F. Gersemsky, *Adaptive direction-of-arrival estimation based on rank and subspace tracking*, in *SVD and signal processing III*, M. Moonen and B. De Moor (Ed.), pp. 287-294, 1995.