Projection method in closed loop system identification

P. Carrette

Department of Electrical Engineering
Linkping University, S-581 83 Linkping, Sweden
WWW: http://www.control.isy.liu.se
Email: carrette@isy.liu.se

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Projection method in closed loop system identification*

P. Carrette - carrette@isy.liu.se

Department of Electrical Engineering, Linköping University
S-58183 Linköping, Sweden

Abstract

This paper deals with a new method for closed loop identification, i.e. the projection method. This method generalizes the already known two-stage method by allowing the estimation of non-causal closed loop sensitivity models in its first step. Such models have been shown to lead to robust identification (as the second step) of the open-loop system dynamics in the presence of marginal nonlinear closed loop behaviors.

This robustness property is investigated. It results a simple “rule of thumb” under which the influence of these closed loop nonlinearities on the open-loop model estimate remains small.

Keywords: Closed loop identification, Two-stage method, Volterra expansion.

1 Introduction

Closed loop identification methods have arisen for constructing models of systems evolving under feedback. That is to say that the open-loop models (i.e. of $G_0(q)$) are derived on the basis of data measurements collected during closed loop operations, i.e. the triplets $(y(t), u(t), r(t))$ out of Figure 1.

Depending on the feedback assumptions, three sets of closed loop identification methods can be distinguished: namely, direct, indirect and joint input-output methods [2, 4, 9, 10]. Furthermore, under the prediction identification framework (see [5] as a reference), we detail that:

- In the direct approach, the model is evaluated on the basis of input and output open-loop data, i.e. $(y(t), u(t))$, without taking care of the feedback mechanism $F(q)$. The consistency of the model estimate essentially relies on the correct modeling of the system disturbance dynamics, i.e. related to $v(t)$.

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In the *indirect* approach, a model of the closed loop system is first estimated. As it is an open-loop identification problem, the consistency of the resulting model can be achieved without putting too much attention on the system disturbance [5]. Then, under the knowledge of the feedback dynamics, it is possible to compute the open-loop model corresponding to the closed loop estimate. Its consistency relies on that of this latter estimate.

In the *joint input-output* approach, the system input $u(t)$ and output $y(t)$ are jointly considered as the output of a system driven by the reference $r(t)$ (and the noise). A model of this latter system is estimated on the basis of measurements of these three signals. No explicit knowledge of the feedback dynamics is needed except that it belongs to certain class (e.g. linear within Figure 1). By use of this 2-output model estimate, an open-loop model of the underlying system can be computed. Its consistency is readily derived from that of the primer estimate.

Among this joint input-output approach has emerged a simple and robust method called the *two-stage method* [8]. This is a two-step method that is schematically as follows:

1. A *causal* model of the sensitivity function of the closed loop system, i.e. $S(q)$, is estimated on the basis of reference and input measurements, i.e. $(u(t), r(t))$.

2. By use of the simulation of this model, i.e. $\dot{u}(t) = \hat{S}(q)r(t)$, and output data, i.e. $y(t)$, a model of the open-loop dynamics is identified.

The statistical properties of the resulting open-loop model estimate have been reviewed in [3]. It has been emphasized that the frequency distribution of its bias error highly depends on the accuracy of the sensitivity model obtained in the first step of the method. More precisely, the deviation of the sensitivity model appears as a complicate weighting of the frequency distribution of the open-loop model error to be minimized.

In order to circumvent this drawback of the two-stage method, Forssell [2] has proposed to slightly modified its first step. This results in the *projection method* where the closed
loop sensitivity model takes the form of a non-causal, doubly infinite FIR-filter. The idea underlying this method goes back to Akaike [1] while considering that the system input is separated into two orthogonal parts: the simulated input \( \hat{u}(t) \) is linearly related to the reference and the associated error signal, i.e. \( u(t) - \hat{u}(t) \), is uncorrelated with it. Furthermore, this new method has been shown to lead to good model bias (i.e. systematic error) properties even in situations where the sensitivity function exhibits marginal nonlinear behaviors. This actually is where the focus of the present paper is put: namely, the study of the non-causality aspect of the sensitivity model structure. More precisely, we want to answer the following question:

*Why and when the non-causality of the sensitivity model results in better model of the open-loop system dynamics \( G_0(q) \)?*

At first sight, this non-causality question may appear quite strange. Indeed, it is easy to see that causal open-loop and feedback dynamics can not produce non-causal closed loop sensitivity functions. In fact, it will be seen that it is really the fact that nonlinearities are present in the loop that makes the non-causality of this model structure relevant.

The structure of the paper is as follows. In Section 2, we give a detailed presentation of the projection method and fix the notations that will be used in the sequel of the paper. In Section 3, we introduce the structure of the closed loop sensitivity model and derive the related parameter estimate based on appropriate data measurements. In Section 4, we present the open-loop model estimate and end up with an expression of its deviation with respect to the corresponding system dynamics. In Section 5, we analyze the influence of the simulated signal error on the elements of a matrix that plays an important role in the open-loop model error. In Section 6, we end up with an upper-bound on the 2-norm of the open-loop model parameter deviation and show how non-causal taps in the closed loop sensitivity model may improve the open-loop model estimate. In Section 7, we derive a “rule of thumb” for choosing the closed loop sensitivity model structure in order to make the open-loop model estimate robust against nonlinearities within the closed loop behaviors. Finally, Section 8 is devoted to the illustration of the achieved results.

## 2 Notations for the projection method

Let us immediately fix notations for the open and closed loop system dynamics. As we focus on the bias property of the projection method when nonlinearities are present in the loop, we here consider a feedback configuration of the open-loop linear system \( G_0(q) \) whose output \( y(t) \) is noise-free (see Figure 1 for \( \nu(t) = 0 \)). The closed loop system is feed by a reference signal \( r(t) \) and the input of the open-loop system is denoted \( u(t) \). Furthermore, the output feedback dynamics is denoted \( F(q) \). The relation between these different entities are easily obtained by use of classical system manipulations. Namely, the open-loop equation is \( y(t) = G_0(q)u(t) \) while the input signal originates from

\[
\hat{u}(t) = r(t) - F(q)y(t)
\]
This last relation can also be written as

\[
    u(t) = (1 + F(q)G_0(q))^{-1}r(t) = S_0(q)r(t) + v(t)
\]

where \(S_0(q)\) denotes the linear term of the Volterra expansion (see e.g. [6, chap. 6]) of the (possibly nonlinear) closed loop sensitivity function, i.e.

\[
    (1 + F(q)G_0(q))^{-1} = S_0(q) + \Delta_0(q),
\]

and \(v(t)\) stands for the contributions of the remaining part of the expansion applied to the reference signal, i.e. \(v(t) = \Delta_0(q)r(t)\). It is worth noting that the (uniquely defined) dynamics \(S_0(q)\) is causal whenever the dynamics involved in the closed loop configuration are\(^1\).

Furthermore, the well-definition of these dynamics interconnection asks for a loop delay of at least one sample time.

In Figure 2, we have drawn an equivalent representation of the closed loop system in

![Figure 2: Closed loop system configuration.](image)

clearly showing up the relation between the reference signal and the open-loop system input. In the sequel of the paper, we intend to refer to this representation. Remember that the open-loop system dynamics \(G_0(q)\) has been assumed linear.

The aim of the projection method is to access a model for the open-loop system dynamics \(G_0(q)\). Therefore, the two following steps are performed:

1. Estimation of a parametric (usually FIR) model \(S(q, \beta)\) for the sensitivity function of the closed loop system. The related model structure is often chosen to lead to non-causal model dynamics.

\(^1\) In case these dynamics are linear, the dynamics \(S_0(q)\) is identical to the closed loop sensitivity function. In fact, large influences of the (nonlinear) dynamics \(\Delta_0(q)\) occur for closed loop systems being far from linear.
2. Estimation of a model \( G(q, \theta) \) for \( G_0(q) \) in terms of the linear dynamics linking the output \( y(t) \) to a modified input signal \( u(t, \beta) \) obtained by simulating the previously estimated sensitivity function model, i.e. \( u(t, \beta) = S(q, \beta)r(t) \).

The aim of the paper is to focus on the non-causality aspect of the sensitivity model structure \( S(q, \beta) \). Therefore, let us turn to the related parameter estimate.

### 3 Estimation of the sensitivity function

In this section, we give details about the first step of the projection method: namely, the estimation of an FIR (possibly non-causal) model for the sensitivity function of the closed loop system dynamics. This means that the corresponding model structure is

\[
S(q, \beta) = \sum_{i=n_1}^{n_2} b_i q^{-i}
\]

where \( n_1 \leq n_2 \) are integer related to the model order. In case \( n_1 < 0 \), the model can access non-causal dynamics. The parameter vector of the model is written as \( \beta = [b_{n_1}, \ldots, b_{n_2}]^T \in \mathbb{R}^{n_2-n_1+1} \).

Let us then make the following assumption.

**Assumption 1** The linear term of the closed loop system sensitivity function (through the Volterra expansion), i.e. \( S_0(q) \), belongs to the sensitivity model set, i.e. \( S(q, \beta_0) = S_0(q) \)

for some \( \beta_0 \in \mathbb{R}^{n_2-n_1+1} \) with \( n_1 \leq 0 \) and \( n_2 \geq l_0 \) where \( l_0 \) stands for the (effective) extent of the impulse response of \( S_0(q) \).

Note that, in case \( n_1 < 0 < l_0 < n_2 \), this “true” parameter vector \( \beta_0 \) is as follows

\[
\beta_0 = [\ldots, 0, b_{0,0}, \ldots, b_{0,l_0}, 0, \ldots]^T
\]  

(2)

with an obvious definition for the \( b_{0,i} \)'s.

By using regression matrix notation [5], the least-squares (LS) estimate of the sensitivity model parameters based on the available data measurements, i.e. \( \{(u(t), r(t))\), \( 1 \leq t \leq N\} \), takes the following form:

\[
\hat{\beta} = \Phi_r^1 u = \beta_0 + \Phi_r^1 v
\]  

(3)

where the \( t \)-th row of regressor matrix \( \Phi_r \) is \( \varphi_r(t) = [r(t-n_1), \ldots, r(t-n_2)] \) and \( M^\dagger \) denotes the Penrose pseudo-inverse of \( M \) (see [7, chap. 3]). In the right hand side expression, \( v \) stands for the column containing the samples of the “disturbance” signal \( v(t) \). From an

\[\text{2 By saying that the effective extent of the impulse response of a discrete-time dynamics, e.g. } \sum_{j \geq 0} s_j q^{-j} \text{ say, is } l, \text{ we mean that } l|s_is_{i+k}| \ll \sum_{j \geq 0} s_j^2 \text{ for any } i \text{ provided } |k| > l.\]
element point of view, this sensitivity parameter estimate is written as \( \hat{\beta} = [\hat{\beta}_1, \ldots, \hat{\beta}_{n_2}]^T \).

The expression (3) has actually been obtained by use of the regressor form of equation (1), i.e. \( u(t) = \varphi_r(t)\hat{\beta}_0 + v(t) \).

It then appears that the error of the sensitivity model parameters, i.e. \( \tilde{\beta} = \hat{\beta} - \beta_0 \) or (elementwise) \([\tilde{\beta}_1, \ldots, \tilde{\beta}_{n_2}]^T\), is due to the perturbation entering in the closed loop system description, i.e. \( v(t) \). In case of purely linear feedback dynamics, i.e. \( \Delta_0(q) = 0 \) leading to \( v(t) = 0 \), no error occurs in the closed loop sensitivity model, i.e. \( \hat{\beta} = \beta_0 \) whenever \( n_1 \leq 0 \). Thus, non-causal model dynamics do not outperform that with \( n_1 = 0 \) which is the first causal one.

Finally, it can be added that, even for marginal nonlinear behaviors of the closed loop system dynamics, the correlation between the reference regressors (i.e. \( \varphi_r(t) \)'s within \( \Phi_r \)) and its related disturbance, i.e. \( v(t) = \Delta_0(q)r(t) \), induces a systematic error (e.g. bias term) in the estimate \( \hat{\beta} \) in expression (3) with respect to the true \( \beta_0 \).

In this small perturbation situation, we shall show that non-causal sensitivity models easily outperform causal ones in the estimation of the open-loop system dynamics (performed in the second step of the method).

4 Estimation of the open-loop system

In the second step of the projection method, a model of the open-loop system dynamics is estimated on the basis of the output signal \( y(t) \) and the modified input signal \( \hat{u}(t) = u(t, \hat{\beta}) \) obtained by simulating the model estimate of the sensitivity function of the closed loop system.

Let us first express this simulated input signal. With the help of the sensitivity parameter estimate \( \hat{\beta} \), we can write in vector form

\[
\hat{u} = \Phi_r\hat{\beta} = \Phi_r\beta_0 + [\Phi_r\Phi_r^\dagger]v = (\Phi_r\beta_0 + v) + ([\Phi_r\Phi_r^\dagger]v - v) = u - P_r^\perp v
\]

where \( P_r^\perp \) stands for the orthogonal projector onto the kernel of \( \Phi_r^T \), i.e. \( P_r^\perp = (I - [\Phi_r\Phi_r^\dagger]) \). Its associated contribution to the modified input signal represents the part of the system input signal that can not be explained by the sensitivity model. Under the previous assumption, this part is due to the perturbation entering in the closed loop system description (1).

Now, we turn to the estimation of the open-loop system dynamics based on the modified data set, i.e. \( \{(y(t), \hat{u}(t)), 1 \leq t \leq N\} \). In order to simplify the derivations, we state the following assumption.

**Assumption 2** The open-loop system dynamics can be described by a strictly causal
FIR dynamics of order at most \( n \), i.e. \( G(q, \theta_0) = G_0(q) \) for some \( \theta_0 \) with

\[
G(q, \theta) = \sum_{i=1}^{n} g_i q^{-i}
\]

where \( \theta = [g_1, \ldots, g_n]^T \).

Then, the system output is related to delayed versions of its original input as

\[
y(t) = \varphi_u(t) \theta_0
\]

where \( \varphi_u(t) = [u(t-1), \ldots, u(t-n)] \) denotes the regressor vector of the FIR representation of the open-loop system dynamics.

Similarly to the preceding section, we can express the LS estimate of the open-loop model parameters, denoted \( \hat{\theta} \), based on the modified data set. In vector form, it is written as:

\[
\hat{\theta} = \Phi_u^T y = \Phi_u^T \Phi_u \theta_0 = \theta_0 + [\Phi_u^T \Phi_u]^{-1}[\Phi_u^T \Phi_u] \theta_0
\]

where the \( t \)-row of the matrices \( \Phi_u, \Phi_\hat{u} \) and \( \Phi_\tilde{u} \) is the regression vector corresponding to \( u(t), \hat{u}(t) \) and \( \tilde{u}(t) = u(t) - \hat{u}(t) \), respectively.

Under Assumption 2, the error in the open-loop model parameter, i.e. \( \hat{\theta} - \theta_0 \), is seen to essentially depend on the deviation between the system input signal \( u(t) \) and its simulated version \( \hat{u}(t) \) (see equation (4)). More precisely, we shall show that the interest of non-causal models of the closed loop sensitivity function, i.e. \( S(q, \beta) \), can be drawn from the analysis of the matrix \( [\Phi_u^T \Phi_u] \) expressing the “correlation” between simulated input, i.e. \( \hat{u}(t) \), and its deviation, i.e. \( \tilde{u}(t) \), with respect to the original system input.

5 Influences of the simulated input error

In the present section, we focus our attention onto the matrix \( [\Phi_u^T \Phi_u] \) appearing in the expression of the LS estimate \( \hat{\theta} \) in (5). The resulting analysis actually explains why it is relevant to add non-causal taps to the closed loop sensitivity model structure in order to achieve a smaller error of the open-loop model parameter estimate, i.e. \( \hat{\theta} \).

Therefore, let us give more details about the structure of this matrix by deriving the expression of its constitutive elements. The result lies in the following proposition.

**Proposition 1** Let Assumption 1 and 2 be satisfied. Then, up to marginal time-shift effects, the matrix \( [\Phi_u^T \Phi_u] \) (see expression (5)) is a non-symmetric Toeplitz matrix with characteristic elements, i.e. \( \hat{\omega}_{j-i} = [\Phi_u^T \Phi_u]_{i,j} \) for appropriate \( (i, j) \) pairs, graphically
represented as (only the nonzero elements are shown)

\[
\begin{bmatrix}
\hat{w}_{1-n} \\
\vdots \\
\hat{w}_{-1} \\
\hat{w}_0 \\
\hat{w}_1 \\
\vdots \\
\hat{w}_{n-1}
\end{bmatrix}
= 
\begin{bmatrix}
x_{n_2+1} & \cdots & x_{n_2+n-1} \\
\vdots & \ddots & \vdots \\
x_{n_1-1} & \cdots & x_{n_1-1}
\end{bmatrix}
\begin{bmatrix}
\ddot{b}_{n_1} \\
\vdots \\
\ddot{b}_0 \\
\dddot{b}_0 \\
\vdots \\
\dddot{b}_n
\end{bmatrix}
\]

where \(x_m = (P_r^v)^T P_r^\perp (T^m r)\) where \(T\) stands for the time-shift matrix \(T\), i.e. \(T_{k+1,l} = \delta_{k,l}\) with \(\delta_{k,l}\) denoting the Kronecker symbol.

In case \((n_2-n_1) < n-2\), the upper (resp. lower) triangular sub-matrix becomes trapezoidal with an upper-left (resp. lower-right) element identical to \(x_{n_1+n-1}\) (resp. \(x_{n_2-n+1}\)).

Before going to the proof, note that, in case the reference signal \(r(t)\) is periodic of period at most \((n_2 - n_1) + 1\), any column \(T^r r\) can be fully explained by the regression matrix \(\Phi_r\), i.e. \(T^r r = \Phi_r z\) for appropriate \(z \in \mathcal{R}^{(n_2-n_1) - 1}\), so that \(x_m = 0\) up to marginal edge effects. Thus, \(\hat{w}_{j-i} = 0\) for all \((i, j)\) pairs: the open-loop model parameter error vanishes, i.e. \(\hat{\theta} = \theta_0\).

**Proof:** First, we express the \(i\)-th and the \(j\)-th column of the regression matrices \(\Phi_{\ddot{u}}\) and \(\Phi_{\dddot{u}}\) respectively as

\[(\Phi_{\ddot{u}})_i = T^i \hat{u} = T^i \Phi_r \hat{\beta} \quad \text{and} \quad (\Phi_{\dddot{u}})_j = T^j \hat{u} = T^j P_r^\perp v\]

for which \(i\) and \(j\) belong to the interval \([1, n]\). Hence,

\[
([\Phi_{\ddot{u}} \Phi_{\dddot{u}}]_{i,j} = \begin{bmatrix} P_r^\perp (T^i \Phi_r) \end{bmatrix}^T P_r^\perp v)
\]

(6)

where we have used the fact that \(T^r T^j = T^{j-i}\) up to marginal edge effects. Note that the \(k\)-th column of the matrix \([P_r^\perp (T^i \Phi_r)]\) is such that

\[P_r^\perp (T^i \Phi_r) = T_r^i \Phi_r = 0 \quad \text{for } 0 < l + k \leq (n_2 - n_1) + 1\]

due to the fact that \((\Phi_r)_k = T^k_{n_1} r\) and \(T^m r = \Phi_r z\) for appropriate \(z\) in case \(n_1 < m \leq n_2\). This immediately implies that the diagonal elements of the presently considered matrix are identically zero, i.e. \(\hat{w}_0 = 0\).

To follow, let us explicity evaluate the non-diagonal elements in (6) \((\hat{w}_{j-i} \text{ with } i \neq j)\) in order to enlighten the characteristics they inherit from the structure of the model of the closed loop sensitivity function used in the first identification step. It is easy to see that

\[\mathcal{R}^{(n_2-n_1)+1} \ni P_r^\perp (T^i \Phi_r) = \begin{cases} P_r^\perp [0, \ldots, 0, T_r^{n_2+l}, \ldots, T_r^{n_2+l}] & \text{for } l > 0 \\
P_r^\perp [T_r^{n_1+l}, \ldots, T_r^{n_1+l}, 0, \ldots, 0] & \text{for } l < 0
\end{cases}\]
where \( \hat{n}_{1,l} = \min(n_1, n_2 + (l + 1)) \) (resp. \( \hat{n}_{2,l} = \max(n_2, n_1 + (l - 1)) \)) is identical to \( n_1 \) (resp. \( n_2 \)) for all \( |l| < n \) in case \( (n_2 - n_1) \geq n - 2 \).

This means that we can write for \( l < 0 \)

\[
\hat{w}_{-l} = \left( [T^{n_1+l}, \ldots, T^{n_1,l-1}, 0, \ldots, 0] \hat{\beta} \right)^T P_r^\dagger v
\]

\[
= \left( \hat{n}_{1,l} - (l + 1) \right) \sum_{k=n_1}^{\hat{n}_{1,l}-(l+1)} \hat{b}_k T^{k+l} P_r^\dagger v = \sum_{k=n_1}^{\hat{n}_{1,l}-(l+1)} (b_{0,k} + \hat{b}_k) x_{k+l}
\]

Note that, in case \( (-l) > (n_2 - n_1 + 1) \), the upper summation index in the \( k \)-sum becomes \( n_2 \) (because \( \hat{n}_{1,l} = n_2 + (l + 1) \)) otherwise it is exactly \( n_1 - (l + 1) \).

A similar expression can also be derived in case of positive \( l \), i.e.

\[
\hat{w}_{-l} = \sum_{k=n_2}^{\hat{n}_{2,-l}-(l-1)} (b_{0,k} + \hat{b}_k) x_{k+l}
\]

By putting these two expressions in a matrix form, the proposed result is achieved. \( \blacksquare \)

Thus, it appears that the matrix \([\Phi_u^T \Phi_u] \) exhibit two contributions, i.e.

\[
([\Phi_u^T \Phi_u] = \begin{bmatrix} \Phi_u^T \Phi_u \end{bmatrix}_{\beta} = [\Phi_u^T \Phi_u]_{\beta_0} + [\Phi_u^T \Phi_u]_{\tilde{\beta}}
\]

One originates from parameters corresponding to the linear term in the Volterra expansion of the closed loop sensitivity function (related to \( \beta_0 \)) while the other is induced by the error in the associated LS parameter estimate, i.e. within \( \tilde{\beta} \).

Now, let us separate the contributions to the column \( \hat{\omega} \) (defined as \([\hat{w}_{1-n}, \ldots, \hat{w}_{n-1}]^T \)) conformably to those in the parameter estimate \( \tilde{\beta} \), i.e. \( \hat{\omega} = \omega_0 + \hat{\omega} \). Under assumption 2 (i.e. leading to \( n_1 \leq 0 < l_0 \leq n_2 \)), the column \( \omega_0 \) takes the following form

\[
\omega_0 = [w_{0,1-n}, \ldots, w_{0,l_2-n}, 0, \ldots, 0, w_{0,n-l_1}, \ldots, w_{0,n-1}]^T \in \mathcal{R}^{2n-1}
\]

where \( l_2 = n + (l_0 - n_2) - 1 \) (resp. \( l_1 = n - (1 - n_1) \)) denotes the number of nonzero elements at the top (resp. bottom) of this column. It is important to notice that if \( l_2 \leq 0 \) (resp. \( l_1 \leq 0 \)) for which \( n_2 \geq (l_0 + n) - 1 \) (resp. \( n_1 \leq 1 - n \)) then there is no nonzero upper (resp. lower) part in the column \( \omega_0 \).

Hence, the part of the matrix \([\Phi_u^T \Phi_u] \) due to the column \( \omega_0 \) takes the form of

\[
[\Phi_u^T \Phi_u]_{\beta_0} = \begin{bmatrix}
\begin{array}{c}
w_{0,n-l_1} & \cdots & w_{0,n-1} \\
\vdots & & \vdots \\
w_{0,1-n} & \cdots & w_{0,l_2-n}
\end{array}
\end{bmatrix}
\]
where only the nonzero elements are shown. While stating that \( l_2 \leq 0 \), i.e. \( n_2 \geq (n+l_0)-1 \), for which the lower triangular part of this matrix is trivially zero (up to marginal edge effects), it appears that the introduction of an increasing number of non-causal taps in the closed loop sensitivity model \( S(q, \beta) \) (i.e. larger \((-n_1)\)) makes the number of its nonzero upper sub-diagonals decrease. More precisely, more upper sub-diagonals (starting from the lower most one) are identical to zero. At the end, for \( n_1 \leq 1 - n \), i.e. \( l_1 \leq 0 \) (for which \( \omega_0 = 0 \)), this matrix vanishes.

In order to be complete, we can add that the number of terms \( x_m \) contributing to the element \( w_{0,i} \) out of \( \omega_0 \) (originating from the result in Proposition 1) is

\[
\hat{z}(w_{0,i}) = \begin{cases} 
\min((l_2 - n) - i, l_0) + 1 & \text{for } l_2 - n \geq i \geq 1 - n \\
\min(i - (n - l_1), l_0) + 1 & \text{for } n - l_1 \leq i \leq n - 1
\end{cases}
\] (8)

Of course, no contributions are found, i.e. \( \hat{z}(w_{0,i}) = 0 \), for \( i \in (l_2 - n, n - l_1) \).

More explicitly, the number of contributions to \( w_{0,l_2-n} \) (resp. \( w_{0,l_2-n} \)) is unity and increases by unit increments while getting either to the extreme positions within that column or to the position of the element \( w_{0,d_1-n} \) (resp. \( w_{0,l_1-n} \)) with \( d_i = l_i - l_0 \) (when \( \geq 1 \)) for which this number is exactly \((l_0 + 1)\) and stays the same until the top (resp. bottom) is reached. In case \( n_2 \geq n - 2 \) (resp. \( n_1 \leq l_0 - (n - 2) \)) for which \( d_2 \leq 1 \) (resp. \( d_1 \leq 1 \)), the number of contributions to \( w_{0,1-n} \) (resp. \( w_{0,n-1} \)) is identical to \( l_2 \) (resp. \( l_1 \)).

We can also derive an upper-bound on the absolute value of the nonzero elements of the column \( \omega_0 \). It simply is

\[
|w_{0,i}| \leq \hat{z}(w_{0,i}) x_c b_{0,c}
\] (9)

where \( x_c = \max_{0 < k < n}(|x_{n_1-k}|, |x_{n_2+k}|) \) while \( b_{0,c} = \max_{n_1 \leq k \leq n_2} |b_{0,k}| \).

For what concerns the “error” column \( \hat{\omega} \), all its elements are generally nonzero. Furthermore, the lower (resp. upper) the position of an element in that column, the larger the number of its contributions (due to \( x_m \) from Proposition 1) for, respectively, the lower (resp. upper) part of that column. To be more precise, the number of terms contributing to \( \hat{\omega}_i \) is exactly \(|i|\) (in case \((n_2 - n_1) \geq n - 2\)) so that the following upper-bound holds for its absolute value:

\[
|\hat{\omega}_i| \leq \min(|i|, (n_2 - n_1) + 1) x_c \hat{b}_c
\] (10)

where \( \hat{b}_c = \max_{n_1 \leq k \leq n_2} |\hat{b}_k| \).

Finally, it is important to note the dependence of these two terms, i.e. \( w_{0,i} \) and \( \hat{\omega}_i \) for appropriate \( i \), upon the closed loop system “disturbance” expressed in terms of the nonlinear part of the Volterra expansion of the closed loop sensitivity function, i.e. \( v(t) = \Delta_0(q)r(t) \). Actually, the elements of \( \omega_0 \) depend upon \( v(t) \) via factors similar to \( x_j \) while the elements of \( \hat{\omega} \) are expressed in terms of products like \( \hat{b}_k x_j \). Roughly speaking, this means that the \( w_{0,i} \)’s (resp. \( \hat{\omega}_i \)’s) have the characteristics of variables like \( v(t) \) (resp. \( v(t) v(t - m) \)).

As a consequence, marginal nonlinear behaviors of the closed loop sensitivity function, i.e.
\[ \hat{b}_c^2 \ll \|b_0\|_2^2/(l_0 + 1), \] will imply that \( \hat{\omega} \approx \omega_0 \) (in the sense of \( \|\hat{\omega}\|_2/\|\omega_0\|_2 \ll 1 \)) so that the elements of \( \omega_0 \) form the dominant contribution to the open-loop model parameter error (through the matrix \( [\Phi^T_\omega \Phi_\omega]_{\beta_0} \)).

6 Upper-bound on the open-loop parameter error

By use of the analysis performed in the preceding section, we are ready to give expressions for the upper-bound of the open-loop model parameter error that is relevant to draw conclusions about the role of the non-causal taps within the closed loop sensitivity model \( S(q, \beta) \). In fact, we shall use a brute-force upper-bound of the 2-norm of non-symmetric Toeplitz matrices.

Here, we derive a typical bound on the 2-norm [7] of the open-loop model parameter error, i.e. \( \hat{\theta} - \theta_0 \) from equation (5). It is as follows

\[
\|\hat{\theta} - \theta_0\|_2/\|\theta_0\|_2 \leq \sigma_{\min}^{-2}(\Phi_\alpha) \left\| \Phi^T_\alpha \Phi_\alpha \right\|_2
\]

(11)

where \( \sigma_{\min}(X) \) is the smallest singular value of \( X \), i.e. \( \sigma_{\min}^2(X) = 1/\|X^TX|^{-1}\|_2 \).

In order to give expressions for the right hand side of this inequality, we analyze each of its factors.

First, we focus on the non-symmetric Toeplitz matrix \( [\Phi^T_\alpha \Phi_\alpha] \). Its squared 2-norm is upper-bounded as

\[
\|\Phi^T_\alpha \Phi_\alpha\|_2^2 \leq \text{tr} \left( [\Phi^T_\alpha \Phi_\alpha]^T[\Phi^T_\alpha \Phi_\alpha] \right) = \sum_{i=1-n}^{n-1} (n - |i|) \hat{\omega}_i^2
\]

(12)

By use of the upperbounds derived on the absolute value of the elements \( \omega_{0,i} \) and \( \hat{\omega}_i \) (see inequality (9) and (10), respectively), we end up with the following result.

**Proposition 2** Let Assumption 1 and 2 be satisfied. Then,

\[
\left\| \Phi^T_\alpha \Phi_\alpha \right\|_2^2 \leq x_c^2 \left\{ \hat{b}_c^2 \sum_{i=1,2} \left( 1 + 2 \left( \hat{b}_c/|b_0,c| \right) \mu_i \right) f(l, l_0) + \hat{b}_c^2 n^2(n^2 - 1)/6 \right\}
\]

(13)

where \( \mu_i^2 = e(l, n, \tilde{n})/f(l, l_0) \) with

\[
f(l, l_0) = \left\{ I_{l>0} \ l (l + 1)(l + 2) - I_{d>1} \ d (d^2 - 1)[d + 4(l_0 + 1)] \right\}/12
\]

\[
e(l, n, \tilde{n}) = \left\{ I_{l>0} \ l (l + 1) [n(n-1) + (n-l)(2n-3) + 3(n-l)^2]
\]

\[
- I_{n>0} \ \tilde{n} (\tilde{n} + 1)(\tilde{n} + 2)[4n - 3(\tilde{n} + 1)] \right\}/12
\]

for \( d = l - l_0 \) and \( \tilde{n} = (n - 2) - (n_2 - n_1) \). The function \( I \) stands for the indicator function, i.e. \( I_{z>a} = 1 \) if \( z > a \) and vanishes otherwise.
Proof: we can easily divide the range of \( i \) in expression (12) into three parts

\[
\|\Phi^T \Phi\|_2^2 \leq \sum_{i=1-n}^{l_2-n} (n - |i|) \bar{w}_i^2 + \sum_{i=-1}^{i_+} (n - |i|) \bar{w}_i^2 + \sum_{i=n-l_1}^{n-1} (n - |i|) \bar{w}_i^2
\]

\[
\leq (1 + \alpha) \left[ \sum_{i=1-n}^{l_2-n} (n - |i|) \left( \frac{w_{0,i}}{x_c b_{0,c}} \right)^2 + \sum_{i=n-l_1}^{n-1} (n - |i|) \left( \frac{w_{0,i}}{x_c b_{0,c}} \right)^2 \right] + \sum_{i=-1}^{i_+} (n - |i|) \bar{w}_i^2
\]

where \( i_- = \max((-l_2 - n) + 1, 1 - n) \) and \( i_+ = \min((n - l_1) - 1, n - 1) \). The last inequality holds because of \((z_1 + z_2)^2 \leq (1 + \alpha)(z_1^2 + z_2^2/\alpha)\) for any \( \alpha > 0 \).

First, let us consider the contributions due to the elements in the upper part of the column \( \omega_0 \), i.e. \( w_{0,i} \) for \( i \in (-n, 0) \). From the counting (8) leading to the expression (9), we straightforwardly obtain

\[
\sum_{i=1-n}^{l_2-n} (n - |i|) \left( \frac{w_{0,i}}{x_c b_{0,c}} \right)^2 \leq \sum_{i=1-n}^{(d_2-1)-n} (n + i)(l_0 + 1)^2 + \sum_{i=d_2-n}^{l_2-n} (n + i)((l_2 - n) - i + 1)^2
\]

\[= f(l_2, l_0)\]

Similarly, we get

\[
\sum_{i=n-l_1}^{n-1} (n - |i|) w_{0,i}^2 \leq x_c^2 b_{0,c}^2 f(l_1, l_0),
\]

Then, we turn to the influence of the elements in the column \( \bar{\omega} \). We also work it out in first considering the upper part of this column. From the expression (10), we write

\[
\left( (1 + 1/\alpha) \sum_{i=1-n}^{l_2-n} + \sum_{i=-1}^{0} \right) (n - |i|) \left( \frac{\bar{w}_i}{x_c b_c} \right)^2
\]

\[
\leq \alpha^{-1} \sum_{i=-[n_2-n_1]+2}^{(n_2 - n_1) + 1} (n + i) \left( (n_2 - n_1) + 1 \right)^2 + \alpha^{-1} \sum_{i=-[n_2-n_1]+1}^{l_2-n} (n + i) i^2
\]

\[= (1 + 1/\alpha) (e(l_2, n, n)/\alpha + n^2(n^2 - 1)/12)\]

Similarly,

\[
\left( (1 + 1/\alpha) \sum_{i=n-l_1}^{n-1} + \sum_{i=0}^{i_+} \right) (n - |i|) \bar{w}_i^2 \leq x_c^2 \bar{b}_c^2 [e(l_1, n, \bar{n})/\alpha + n^2(n^2 - 1)/12]
\]

Finally, by making use of two different \( \alpha 's \), we globally have

\[
\|\Phi^T \Phi\|_2^2 \leq x_c^2 \left\{ b_{0,c}^2 \sum_{i=1,2} [(1 + \alpha_i) f(l_i, l_0) + e(l_i, n, \bar{n})/\alpha_i] + \bar{b}_c^2 n^2(n^2 - 1)/6 \right\}
\]

The smallest upperbound is achieved for \( \alpha_i = (\bar{b}_c/b_{0,c}) [e(l_i, n, \bar{n})/f(l_i, l_0)]^{1/2} \) leading to the proposed value of \( \mu_i \).
In Figure 3, we have presented the function \( f(l, l_0) \) as a function of \( l \) in a semilogarithmic plot for different \( l_0 \).

The symbol “*” indicates values of \( f(l, l_0) \) in case its second term vanishes, i.e. \( d \leq 1 \) for sufficiently large \( l_0 \geq l - 1 \). Each dashed curve corresponds to values of \( f(l, l_0) \) for a particular value of \( l_0 \), i.e. \( l_0 = l^* - 1 \) where \( l^* \) stands for the value of \( l \) associated to the symbol “*” within that curve. It appears that \( f(l, l_0) \) is a nondecreasing function of \( l_0 \).

Hence, depending on the balance between \( l_1 \) and \( l_2 \) (i.e. \( l_1 \leq l_2 \)), only one term of the sum in the right hand side of expression (13) virtually remains.

Now, we can draw the following remarks:

- In case \( l_2 \leq 0 \), i.e. \( n_2 \geq (n + l_0) - 1 \), only the summation over the non-causal taps of the closed loop sensitivity model remains. So, the upper-bound on the considered non-symmetric Toeplitz matrix becomes

\[
\| \Phi_u^T \Phi_u \|_2^2 \leq x_c^2 \left\{ \frac{b_{0,c}^2}{\bar{b}_c} \left( 1 + 2 \left( \frac{\bar{b}_c}{b_{0,c}} \right) \mu_1 \right) f(l_1, l_0) + \frac{\bar{b}_c^2}{c} n^2 (n^2 - 1) / 6 \right\}
\approx x_c^2 \frac{b_{0,c}^2}{\bar{b}_c} f(l_1, l_0)
\]

for \( b_{0,c} \gg \bar{b}_c \) and \( l_1 > 0 \) (close to \( n - 1 \)). Obviously, this bound decreases with the argument \( l_1 \) that represents the number of non-causal taps that are lacking in order to achieve the extent of the impulse response of the open-loop system, i.e. \((n - (1 - n_1))\) such taps.

- In the situation where \((l_1, l_2) \leq 0\), the remaining contributions to the 2-norm bound originate from the nonlinear behavior of the system sensitivity function, i.e. inducing
the vector \( \hat{\omega} \). It takes the following form

\[
\| \Phi^T u \|_2^2 \leq \frac{1}{6} x_c^2 \beta_0^2 n^2 (n^2 - 1)
\]

(14)

Depending on the importance of the nonlinearity, these contributions (based upon \( \hat{b}_u \)) can be rather small (for marginal nonlinearities).

Hence, the upper-bound derived in Proposition 2 gives us insights for the need of sufficiently large causal and non-causal parts of the closed loop sensitivity model \( S(q, \beta) \) as well as to the relevance of these bounds in case of marginally nonlinearly perturbed closed loop behaviors (in view of expression (14)).

Finally, let us consider the first factor in the right hand side of the upperbound for the 2-norm of the open-loop model parameter error, i.e. \( \| \hat{\theta} - \theta_0 \|_2 / \| \theta_0 \|_2 \) in expression (11). It consists in the smallest singular value of the regressor matrix associated to the simulated system input, i.e. \( \hat{u}(t) \). The following result holds for it.

**Lemma 3** Under Assumption 1, we have that

\[
\left| \sigma_i (\Phi_{\hat{u}(n_1,n_2)}) - \sigma_i (\Phi_{\hat{u}(n_1,n_2)}) \right| \leq \delta \quad \text{for } i = 1, \ldots, n
\]

where \( \sigma_i(X) \) stands for the \( i \)-th singular value of \( X \) while \( \delta \leq n^{1/2} \| P^{-1}_r v \|_2 \) whenever \( n' \geq n_1 \) and/or \( n'' \geq n_2 \).

The notation \( \hat{u}(n_1,n_2) \) explicitly states the dependence of the simulated system input with respect to the parameters of the closed loop sensitivity model, i.e. \( n_1 \) and \( n_2 \).

**Proof:** while denoting \( A = \Phi_{\hat{u}(n_1,n_2)} \) and \( \tilde{A} = \Phi_{\hat{u}(n'_1,n'_2)} \) as well as \( E = \tilde{A} - A \), we immediately end up with (see [7, chap. 4])

\[
\left| \sigma_i (\tilde{A}) - \sigma_i (A) \right| \leq \| E \|_2
\]

Now, the simulated system input is denoted \( \hat{u} = \Phi_r \Phi_r^d u \) while that associated to \( (n'_1, n'_2) \) is written as \( \hat{u}' = \Phi_r' \Phi_r^{d'} u \) with

\[
\Phi_r' = [T^{n_1}r, \ldots, T^{n_1-1}r, \Phi_r, T^{n_2+1}r, \ldots, T^{n_2+2}r]
\]

In fact, these are two different orthogonal projections of the system input signal \( u(t) = \varphi_r(t) \beta_0 + v(t) \) (or \( = \varphi_r'(t) \beta_0' + v(t) \) with \( \beta_0' = [\ldots, 0, \beta_0^T, 0, \ldots]^T \)), i.e.

\[
\hat{u} = P_r u \quad \text{and} \quad \hat{u}' = P_r' u
\]

\[
= \Phi_r \beta_0 + P_r v \quad \text{and} \quad \hat{u}' = \Phi_r' \beta_0' + P_r' v
\]

As \( \Phi_r \) is contained in the matrix \( \Phi_r' \), we have that \( P_r = P_r P_r' \) by orthogonal projection and range arguments. This means that \( P_r' = P_r + P_r' \) with \( P_r P_r = 0 \) or \( P_r = P_r' P_r \). So, the columns of the matrix \( E \) are formed by delayed versions of the difference between these two simulated system input, i.e.

\[
E_i = T^i (\hat{u}' - \hat{u}) = T^i \hat{P}_r v \quad \text{for } i = 1, \ldots, n
\]

Finally, a brute-force bound on the 2-norm of \( E \) consists in its Frobenius norm, i.e. \( \| E \|_2^2 \leq \| E \|_F^2 = n \| P_r v \|_2^2 \). As \( \hat{P}_r = P_r P_r' \), the result immediately follows.
A consequence of this is that the addition of (causal or non-causal) taps in the closed loop sensitivity model makes the smallest eigenvalue of the information matrix associated to the simulated system input \( \hat{u}(t) \), i.e. \( \sigma^2_{\text{min}}(\Phi_{\hat{u}(n'_1,n'_2)}) \) for increasing \( n'_1 \) and/or \( n'_2 \), vary in a range given by the constant \( \delta \).

It is worth noting that this \( \delta \) increases with \((n'_1, n'_2)\). The reason for this is that the orthogonal projector \( \hat{P}_r \) (in the proof) projects onto a subspace of increasing dimensions, i.e. \( \|\hat{P}_r v\|_2 \) becomes larger but upper-bounded by \( \|P_r^\perp v\|_2 \).

As a conclusion, we can state that the benefit of non-causal taps in the closed loop sensitivity model is fully understood by their influence upon the second factor of the right hand side of expression (11) (see the preceding remarks). Furthermore, the behavior of its first factor does generally not annihilate this beneficial effect.

7 “Rule of thumb” for the sensitivity model

In the preceding section, we have analyzed the deviation of the estimated parameter vector of the open-loop system model \( G_0(q, \theta) \) as a function of possible additional taps in the closed loop sensitivity model \( S(q, \beta) \).

It has appeared that, except from effects on the smallest eigenvalue of the information matrix associated to the simulated input \( \hat{u}(t) \), the extent of the impulse response of the closed loop sensitivity model is closely related to the order of the open-loop model as well as to that of the linear term of the Volterra expansion of the closed loop sensitivity of the system, i.e. \( S_0(q) \). In fact, it has been derived that a way to prevent us from parameter errors coming from this linear dynamics is to choose the characteristics of its related model as follows

\[
\begin{align*}
n_1 &\leq -(n - 1) \quad \text{as well as} \quad n_2 \geq l_0 + (n - 1) \tag{15}
\end{align*}
\]

so that \((l_1, l_2) \leq 0 \) in equation (7), i.e. \( \omega_0 = 0 \). Thus, we see that a certain number of taps, i.e. at least \((n - 1)\), must be added to the left and the right of the extent of the impulse response of the linear dynamics \( S_0(q) \).

These two conditions actually lead to the following “rule of thumb” that is depicted in Figure 4.

**Proposition 4 (rule of thumb)** Add taps in \( S(q, \beta) \) until the absolute value of the first and the last \((n - 1)\) elements of \( \beta \) are below a certain threshold, i.e.

\[
|\hat{b}_{n_1+i}|, |\hat{b}_{n_2-i}| < \gamma \quad \text{for} \quad 0 \leq i < n - 1
\]

where \( \gamma \) denotes the threshold, e.g. \( \gamma = \alpha \|\hat{\beta}\|_2/(n - 1) \) for small \( \alpha > 0 \).

Such a rule forces the extreme elements of the parameter estimate of the closed loop sensitivity model to be small so that they lie almost surely outside the extent of \( S_0(q) \) and therefore only induce a marginal error in the open-loop model parameter error (from
It is worth adding some more remarks.

- In practice, the extent of $S_0(q)$ is usually smaller than that of the open-loop system $G_0(q)$. The reason for this is that the closed loop system is designed in order to improve the performances of the underlying system, e.g. faster response, larger frequency band, etc. This means that the largest part of the parameter vector $\beta$ will then be devoted to the additional taps imposed by the conditions (15).

- These two conditions have been derived for a general sensitivity function $S_0(q)$ corresponding to $\tilde{\beta}$ as in expression (2). In case this dynamics contains a (effective) delay, e.g. $d_0$ say, we find additional zeros in this true vector $\beta_0$: namely, $b_{0,i} = 0$ for $i < d_0$. This implies that the condition in the left hand side of (15) can be replaced by $n_1 \leq d_0 - (n - 1)$ (see the upper arrow in Figure 4). Hence, for $d_0 \geq n - 1$, no non-causal taps are needed in the closed loop sensitivity model.

- As previously pointed out, too large nonlinear behaviors of the closed loop sensitivity function make the conditions (15) useless for performing a good estimation of the open-loop model $G(q, \theta)$. The reason for this is that, in that case, the errors in the parameter estimate $\hat{\beta}$, i.e. $\hat{\beta} - \beta_0$, are of the same order of magnitude than the nonzero elements of $\beta_0$. This induces contributions of the column $\hat{\omega}$ that are no more marginal compared to those participating to the “true” column $\omega_0$. More precisely, this leads to $\tilde{b}_c \sim b_{0,c}$ so that the second term in the right hand side of expression (13) becomes relevant whatever $(l_1, l_2)$ is. Hence, there is no way that additional (causal and/or non-causal) taps in the closed loop sensitivity model can improve the open-loop model estimation.

This concludes our analysis of the influence of the extent of the impulse response of the closed loop sensitivity model (used in the first step of the projection method) on the accuracy of the open-loop system estimation (i.e. its second step).
8 Simulations

In this section, we intend to illustrate the results that have been achieved in the preceding sections. More precisely, we want to show how the addition of non-causal taps in the closed loop sensitivity model leads to a more accurate open-loop model estimate in the situation of marginally nonlinear feedback dynamics within the closed loop system.

The open-loop system dynamics is chosen as

$$G_0(q) = -2.78 q^{-1} + 3.21 q^{-2} - 1.93 q^{-3} + 0.60 q^{-4} + 0.08 q^{-5}$$

for which $\theta_0 = [-2.78, 3.21, -1.93, 0.60, 0.08]^T$ with $n = 5$. The static feedback dynamics is

$$F(x) = x - 0.35 e^{-x^2} \sin x$$

It is represented in Figure 5. Obviously, it is a nonlinear dynamics that is a slight perturbation of the linear operator $x$.

By closing the loop, we obtain a nonlinear dynamics whose sensitivity function participates to the reference-to-input relation as

$$u(t) = (1 + F(q)G_0(q))^{-1}r(t)$$

The corresponding linear dynamics $S_0(q)$, i.e. first term of the Volterra expansion of that nonlinear operator, is simply $1/(1 + G_0(q))$.

In Figure 6, we have represented the impulse response of these two dynamics, i.e. input signals $u(t)$ produced by unit impulse reference at the origin, i.e. $r(t) = \delta(t)$. It appears that the impulse response of the dynamics $S_0(q)$ does not differ too much from that of the original nonlinear sensitivity function. Furthermore, it is seen that it (not surprisingly)}
corresponds to a causal dynamics whose “effective” extent is fixed to $l_0 = 11$.

Now, let us consider the projection method (i.e. LS estimation of the open-loop system “parameters”, i.e. within $\theta_0$, through that of the linear part of the sensitivity function, i.e. $S_0(q)$) applied to data measurements originating from a Gaussian white noise reference signal $r(t)$ of unit variance. Monte-Carlo simulations have been performed over 100 data sets made of the triplets $(y(t), u(t), r(t))$ for $1 \leq t \leq 300$.

First, we turn to the LS estimation of the parameters of the sensitivity model $S(q, \beta)$, i.e. within $\hat{\beta}$. In Figure 7, we present the average of the absolute value of the deviation between the estimated parameters $\hat{b}_i$ and the corresponding “true” values $b_{0,i}$ (taken from Figure 6), i.e. $\hat{b}_i = \hat{b}_i - b_{0,i}$, as a function of the index $i \in [-8, 18]$. It appears that this parameter error is negligible compared to the nonzero $b_{0,i}$, within the vector $\hat{\beta}_i$. This actually means that the feedback nonlinearity only induces marginal perturbations of the closed loop sensitivity. More importantly, these deviation samples are rather uniform in $i$ so that the induced perturbations also appear to be non-causal.

Moreover, we have displayed in Figure 8 the average of the quantity $x_m^2$ appearing in Proposition 1. As predicted, it is negligible for $m \in [n_1, n_2]$.

From this, we are in position to illustrate the results derived through the analysis performed above: namely, how the addition of taps in the closed loop sensitivity model (leading to the “rule of thumb” in Proposition 4) can improve the accuracy of the open-loop parameter estimate $\hat{\theta}$.

Therefore, we perform the second step of the projection method (i.e. estimation of the open-loop model parameters (within $\hat{\theta}$) based on the simulated input $\hat{u}(t)$ evaluated by use of the sensitivity model estimate $S(q, \hat{\beta})$) for each of the Monte-Carlo simulations. In Figure 9, we have represented the average of the 2-norm of the deviation between the estimated open-loop parameter estimate and its “true” value, i.e. $\|\hat{\theta} - \theta_0\|_2$, for different
values of the closed loop sensitivity model variables $n_1$ and $n_2$. Furthermore, we have displayed the corresponding 2-norm upper bounds (see expression (11)) divided by 230! Apart from this factor, the behavior of these two quantities appears similar so that the interest of adding taps in the closed loop sensitivity model becomes obvious. In particular, non-causal taps are seen to be crucial for improving the open loop parameter estimate, i.e. $n_1 \leq 1 - n$.

9 Conclusions

In the present paper, we have studied a new method for closed loop identification. It is called the projection method and was first proposed by Forssell in [2]. This method generalizes the already known two-stage method [8] by making use of non-causal sensitivity models in its first step. Due to decorrelation properties, such models have been shown to lead to robust identification (as the second step) of the open-loop system dynamics in the presence of marginal nonlinear closed loop behaviors.

The influence of such nonlinear dynamics has been investigated while playing around with the non-causality property of the sensitivity models. From this investigation, we have proposed a simple “rule of thumb” for ending up with a closed loop sensitivity model structure that makes the open-loop model estimate robust against these nonlinear system behaviors.
References


