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Deterministic dynamical bounds on moments of nonstationary stochastic processes

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Abstract

In this contribution, we deal with the deterministic dominance of the probability moments of stochastic processes. More precisely, given a positive stochastic process, we propose to dominate its probability moment sequence by the trajectory of appropriate lower and upper dominating deterministic processes. The analysis of the behavior of the original stochastic process is then transferred to the stability analysis of the deterministic dominating processes. The result is applied to a nonstationary auto-regressive process that appears in the system identification literature.

Keywords: nonstationary stochastic process, probability theory, nonlinear dynamic system, stability analysis, trajectory bounding.

1 Introduction

In general, the evolution of nonstationary stochastic processes is hard to obtain from the statement of the underlying stochastic equation (see [2, 7, 9, 3] and [6, chap. 13]). As a motivation example, let us consider the positive scalar stochastic process $r_k$ defined by

$$ r_k = (1 - \mu) r_{k-1} + \mu \varphi_k^2, \quad k > 0 $$

(1)

where $r_0 \geq 0$, $\mu \in [0, 1)$ and $\varphi_k$ denotes a random variable whose distribution is subject to the following “excitation” condition

$$ \varphi_k^2 \geq \delta r_{k-1} $$

(2)

with $\delta \in [0, 1]$. Note that this process arises in time-varying system identification by use of a constrained forgetting factor recursive least square algorithm (as introduced in [4], see also [8]).

Obviously, the process $r_k$ cannot be considered anymore as an auto-regressive (AR) process [1, chap. 5]. Roughly speaking, it can be viewed as a nonstationary AR process.
As valuable characteristics of stochastic processes lie in their probability moments, it is natural to ask for the trajectory of the probability moments of \( r_k \) along \( k \). But this is a hard problem to solve. The reason for this is that the excitation condition (2) is such that its probability distribution of \( \varphi_k \) depends on all its past samples (due to \( r_{k-1} \)). Thus, the distribution of the sample \( r_k \) depends in a very intricate way on its past values, so does the evaluation of its moments.

Then, instead of asking for the exact value of these moments, one may be less ambitious and desire to only characterize the evolution of their trajectory (along \( k \)). This is the purpose of the paper.

Here, we propose to lower and upper dominate the trajectory of the probability moments of nonstationary stochastic processes by the solutions of deterministic dynamical equations. Our contribution is as follows.

Given a positive scalar stochastic process \( x_k \), we show that under functional assumptions on its conditional (onto past samples) probability moments, it is possible to trace the evolution of its probability moments on that of the output of appropriately defined lower and upper bounding deterministic dynamic systems, i.e. \( w_k \leq E(x_k) \leq z_k \) with \( w_k = g(w_{k-1}) \) and \( z_k = f(z_{k-1}) \). Hence, valuable properties of these moments can be obtained from the stability analysis of these bounding dynamic systems, e.g. equilibrium points, convergence rates.

For an illustration purpose, our results will be applied to the stochastic process \( r_k \) in order to derive bounds upon its probability moments, i.e. \( E(r^n_k) \), and on those of its inverse process \( p_k = 1/r_k \), i.e. \( E(p^n_k) \).

The structure of the paper is as follows. Our main result is stated in Section 2. It deals with the convex and concave functional boundedness of the trajectory of the conditional expectation of a positive scalar stochastic process. Consequences of this functional property on the evolution of the process expectation are provided. In Section 3, we develop a simple algorithm for practically evaluating convex (lower) and concave (upper) functional bounds on a given function representing conditional expectation dynamics. In Section 4, we apply our results to the stochastic equation (1) under the excitation condition (2). More precisely, we derive deterministic dynamics dominating that of the moments of the stochastic processes \( r_k \) and \( p_k \). Finally, simulations are provided for a particular distribution of the sequence \( \varphi_k \).

2 Deterministic dominance of stochastic processes

In this section, we are interested in evaluating convergence bounds on the expectation of a positive scalar stochastic process \( x_k \). Therefore, we propose to dominate this expectation by the trajectories of appropriate lower and upper bounding deterministic processes, i.e \( w_k \leq E(x_k) \leq z_k \) for \( k \geq 0 \).

The convergence analysis of the original expectation is then transferred to that of the deterministic dominating dynamics.
The following theorem presents our main result.

**Theorem 1** Let \( x_k \) (with \( k > 0 \)) be a positive stochastic process such that
\[
g(x_{k-1}) \leq E(x_k|\mathcal{F}_{k-1}) \leq f(x_{k-1}) \quad \text{a.e.} \tag{3}
\]
where \( \mathcal{F}_{k-1} = \sigma\{x_j, 0 \leq j < k\} \) is the \( \sigma \)-algebra generated by the past events of the process and \( \mathcal{F}_{k-1} \subseteq \mathcal{F}_k \), and where the functions \( g(x) \) and \( f(x) \) are continuous nonnegative convex and concave functions in \( \mathbb{R}^+ \), respectively. Then,
\[
w_i \leq E(x_k|\mathcal{F}_{k-i}) \leq z_i \tag{4}
\]
where \( w_i, z_i > 0 \) are the samples of particular trajectories of the following deterministic scalar processes: \( w_i = g(w_{i-1}) \) and \( z_i = f(z_{i-1}) \) with \( w_0 = z_0 = x_{k-i} \).

Before going into the proof, let us note that the stochastic inequality (3) holds uniformly in \( k \). For example, in the case of a stochastic process \( x_k = h(\epsilon_k) \) with a random sequence \( \epsilon_k \) possibly dependent on the past \( x_k \) (i.e. \( x_{k-1}, \ldots, x_0 \)), we can write
\[
E(x_k|\mathcal{F}_{k-1}) = h_k(x_{k-1})
\]
where \( h_k(x) \) is possibly non-uniform in \( k \). Then, by defining \( h_-(x) := \min_k h_k(x) \) and \( h_+(x) = \max_k h_k(x) \) over \( x > 0 \), we obtain \( g(x) \leq h_-(x) \) as well as \( h_+(x) \leq f(x) \) with the desired properties for \( g(x) \) and \( f(x) \), if possible. If not, the associated deterministic process bound does not hold.

**Proof:** By use of Jensen’s inequality [5, page 47], the concavity (resp. convexity) property of \( f(x) \) (resp. \( g(x) \)) leads to

\[
E(f(x)) \leq f(E(x)) \quad \text{(resp. } g(E(x)) \leq E(g(x))\text{)}
\]

for any positive random variable \( x \). Now, the quantity \( E(x_k|\mathcal{F}_{k-i}) \) is recursively defined by
\[
E(x_k|\mathcal{F}_{k-i}) = E(E(x_k|\mathcal{F}_{k-i+1})|\mathcal{F}_{k-i})
\]
for \( i < k \). So that
\[
E(x_k|\mathcal{F}_{k-i}) = E(\cdots E(E(x_k|\mathcal{F}_{k-1})|\mathcal{F}_{k-2})\cdots|\mathcal{F}_{k-i}) \\
\leq E(\cdots f(x_{k-1})|\mathcal{F}_{k-2})\cdots|\mathcal{F}_{k-i}) \\
\leq E(\cdots f(x_{k-2})\cdots|\mathcal{F}_{k-i}) \\
\leq f(\cdots f(x_{k-i})\cdots)
\]

with \( i \) compositions of the concave function \( f(x) \). For the lower bound (i.e. in term of the convex function \( g(x) \)), we similarly have
\[
E(x_k|\mathcal{F}_{k-i}) \geq g(\cdots g(x_{k-i})\cdots)
\]

Finally, the definition of the \( w_i \) and \( z_i \) processes leads to \( w_i = g(\cdots g(z_0)\cdots) \) and \( z_i = f(\cdots f(y_0)\cdots) \) with \( i \) compositions of \( g(x) \) and \( f(x) \), respectively. Hence, the proof is completed by taking \( w_0 = z_0 = x_{k-i} \). \( \blacksquare \)
It follows from this result that the convergence properties of the expectation of the stochastic process \( x_k \) can be estimated by the analysis of particular deterministic positive processes. The two following lemmas exhibit properties of their underlying dynamics, i.e. \( f(x) \) and \( g(x) \) respectively.

**Lemma 2** Let \( z_i \) (with \( i > 0 \)) be the following positive scalar process

\[
z_i = f(z_{i-1}) \quad \text{with } z_0 > 0
\]

where \( f(z) \) is a nonnegative nondecreasing concave function in \( \mathcal{R}^+ \). If there exists \( z^* > 0 \) such that \( f(z^*) = z^* \) and \( f(z) < z \) for \( z^* < z \), then \( z^* \) is an attractive equilibrium point for \( z^* < z \), i.e.

\[
z_i \leq z^* + \gamma_z^{-1}(z_{i-1} - z^*)
\]

with \( \gamma_z > 1 \). Globally, we have that \( \lim_i z_i \leq z^* \) for \( i \to \infty \).

**Proof:** First, we derive some properties of the function \( f(z) \).

It is nonnegative: \( f(z) \geq 0 \) for \( z \geq 0 \). It is nondecreasing and concave: \( 0 \leq f'_+(z_2) \leq f'_+(z_1) \) for \( 0 \leq z_1 \leq z_2 \), with \( f'_+(z) \), the right derivative of \( f(z) \). By assumption, \( 0 \leq f'_+(z^*) < 1 \) and \( f(z^*) > 0 \), so that \( f(z) > 0 \) for all \( z > 0 \) and either \( z < f(z) \leq z^* \) or \( f(z) = z \) for \( 0 < z < z^* \). We also have \( z^* \leq f(z) < z \) for \( z > z^* \), by the nondecreasing property.

Now, we show that \( [z^*, z_M] \) (with \( z_M < \infty \)) is a positively invariant compact set and we derive the result in (5). From above, if \( z_{i-1} \in [z^*, z_M] \) then \( z^* \leq z_i = f(z_{i-1}) \leq z_{i-1} \), so that \( z_i \in [z^*, z_M] \). Moreover, as \( f(z) < z \) for \( z^* < z \), we have that \( z_i < z_i-1 \) for \( z^* < z_i-1 \).

This means that the equilibrium point \( z^* \) (i.e. \( f(z^*) = z^* \)) is attractive from above. And simple calculations give:

\[
\frac{f(z) - z^*}{z - z^*} \leq \frac{f(z^*) + f'_+(z^*)(z - z^*) - z^*}{z - z^*} = f'_+(z^*)
\]

for \( z^* < z \). Hence, \( \gamma_z^{-1} = f'_+(z^*) < 1 \) in (5).

Finally, the positive invariance of \([0, z^*]\) (i.e. if \( z_{i-1} \in [0, z^*] \) then \( 0 \leq z_i = f(z_{i-1}) \leq z^* \)) completes the proof of the lemma. \( \blacksquare \)

Similarly, we have for the process \( w_i \) in Theorem 1.

**Lemma 3** Let \( w_i \) (with \( i > 0 \)) be the following positive scalar process

\[
w_i = g(w_{i-1}) \quad \text{with } w_0 > 0
\]

where \( g(w) \) is a positive nondecreasing convex function in \( \mathcal{R}^+ \). If there exists \( w^* > 0 \) such that \( g(w^*) = w^* \) and \( g(w) > w \) for \( w < w^* \), then \( w^* \) is an attractive equilibrium point for \( w < w^* \), i.e.

\[
w_i \geq w^* - \gamma_w^{-1}(w^* - w_{i-1})
\]

with \( \gamma_w > 1 \). Globally, we have that \( \lim_i w_i \geq w^* \) for \( i \to \infty \).
Proof: it is similar to the one of Lemma 2. In this case, \( \gamma_w \) can be linked with the left derivative of \( g(w) \) evaluated at \( w = w^* : \gamma_w^{-1} = g'_L(w^*) < 1 \) with \( g'_L(w) \), the left derivative of \( g(w) \).

Hence, provided that the stochastic process \( x_k \) satisfies the condition (3) in Theorem 1 and that the corresponding functions \( f(x) \) and \( g(x) \) exhibit the characteristics described in Lemma 2 and Lemma 3, respectively, we have that

\[
u^* - \gamma^{-k}(w^* - x_0)I_{x_0 < w^*} \leq E(x_k) \leq z^* + \gamma^{-k}(x_0 - z^*)I_{x_0 > z^*}
\]

with \( E(x_k) = E(x_k | F_0) \) and \( I_X \) denotes the indicator function on to the condition \( X \). Of course, if either \( g(x) \) or \( f(x) \) cannot be found then the corresponding convergence property does not hold anymore. Actually, either \( w^* = 0 \) or \( z^* = \infty \).

3 Algorithm for practical dominating functions

For a constructive use of the result presented in Section 2, we now propose to evaluate, at least numerically, dominating convex and concave functions (i.e. \( g(x) \) and \( f(x) \), respectively) of a particular function \( h(x) \) for \( x \geq 0 \).

The two corresponding procedures read as:

- For the convex functional lower-bound on \( h(x) \) with a positive and nondecreasing function \( g(x) \), we first make \( g(0) = \min_{x \geq 0} h(x) \). Then, we estimate \( g(x) \) for \( x > 0 \) by successive Euler integration steps (with an increment \( \Delta x \) for \( x \), i.e.

\[
g(x_i) = g(x_{i-1}) + g'_L(x_i) \Delta x \quad \text{for} \quad x_i = x_{i-1} + \Delta x
\]

where \( g'_L(x_i) = \max(g'_L(x_{i-1}), \alpha_i) \) and \( \alpha_i \) is such that the line \( g(x_{i-1}) + \alpha_i(x - x_{i-1}) \) is tangent to \( h(x) \), from below, for \( x > x_{i-1} \) while \( g'_L(0) = 0 \).

- For the concave functional upper-bound on \( h(x) \) with a nonnegative and nondecreasing function \( f(x) \), we first state \( f(0) = h(0) \). Then, we successively perform

\[
f(x_i) = f(x_{i-1}) + f'_L(x_i) \Delta x \quad \text{for} \quad x_i = x_{i-1} + \Delta x
\]

where \( f'_L(x_i) = \min(f'_L(x_{i-1}), \beta_i) \) and \( \beta_i \) is such that the line \( f(x_{i-1}) + \beta_i(x - x_{i-1}) \) is tangent to \( h(x) \), from above, for \( x > x_{i-1} \) while \( f'_L(0) \) is as large as possible in order to have \( f'_L(x_1) = \beta_1 \).

In Figure 1, we present the result obtained by these two procedures for a given function \( h(x) \). We have also displayed the attractive equilibrium point \( z^* \) (resp. \( w^* \)) associated to the \( z \) (resp. \( w \)) process in Lemma 2 (resp. 3).

It appears that the growth rate of these two dominating function is asymptotically identical, i.e. \( h'(x) \) for \( x \gg 1 \). Note also that the equilibrium point interval, i.e. \( [w^*, z^*] \), is rather extended over the abscissas where the original function \( h(x) \) varies.
Figure 1: Practical concave (---) and convex (—) dominating functions of the original $h(x)$ (——). Corresponding equilibrium points: $z^*$ ('*') and $w^*$ ('o').

4 Application to the stochastic process example

In this section, we analyze in some details the stochastic process $r_k$ introduced in (1). More precisely, we provide lower and upper dynamical bounds on the trajectory of its probability moments, i.e. $E(r^n_k)$, and those of its inverse, i.e. $E(p^n_k)$.

In view of Theorem 1, such dynamical bounding is achieved in deriving lower convex and upper concave functional bounds on the conditional (onto the past events) expectation on the corresponding power of these processes, i.e. $E(x_k | \mathcal{F}_{k-1})$ with $x_k = r^n_k$ for the $n$-th probability moments of $r_k$. Let us then evaluate bounding functionals for the conditional expectation of the $n$-th power of the two processes successively.

The $n$-th power of the stochastic equation governing the process $r_k$ is written as

\[
 r^n_k = [(1 - \mu) r_{k-1} + \mu \varphi^2_k]^n \\
 = r^n_{k-1} [(1 - \mu)^n + P_n(\xi_k|k-1)]
\]

with $\xi_k|k-1 = \varphi_k^2/r_{k-1}$ and $P_n(x) = ((1 - \mu) + \mu x)^n - (1 - \mu)^n > 0$ for $x > 0$. Note that $P_n(x)$ is monotonically increasing in $x$ with $P_n(0) = 0$.

The evaluation of the conditional expectation of $r^n_k$ with respect to the past events gives

\[
 E(r^n_k | \mathcal{F}_{k-1}) = r^n_{k-1} [(1 - \mu)^n + E(P_n(\xi_k|k-1)|\mathcal{F}_{k-1})]
\]

Then, we derive the following functional bounds that are uniform in $k$

\[
 r^n_{k-1} [(1 - \mu)^n + Q^n_n(r^n_{k-1})] \leq E(r^n_k | \mathcal{F}_{k-1}) \leq r^n_{k-1} [(1 - \mu)^n + Q^n_n(r^n_{k-1})] \quad (7)
\]

6
for appropriate functions $Q_n^-(x)$ (resp. $Q_n^+(x)$) defined similarly to $g(x)$ (resp. $f(x)$) in Theorem 1 for $P_n(\xi_k|k-1)$.

Moreover, when appropriate (lower convex and upper concave, respectively) dominating functions are estimated for $E(p_k^n|F_{k-1})$ (as presented in Section 3), we can define $\tilde{Q}_n^-(x)$ and $\tilde{Q}_n^+(x)$ as the functions that make the bounding expressions in (7) identical to the corresponding dominating function estimates. The attractive equilibrium points of these dominating processes, i.e. $w^*_r$ and $z^*_r$ are found by solving

$$\tilde{Q}_n^-(w^*_r) = \tilde{Q}_n^+(z^*_r) = 1 - (1 - \mu)^n$$

while the lower bounds for the convergence rate to these solutions (see expression (6)) are given by

$$\gamma_{w,r}^{-1} = 1 + w^*_r(\tilde{Q}_n^-(w^*_r))'_- \quad \text{and} \quad \gamma_{z,r}^{-1} = 1 + z^*_r(\tilde{Q}_n^+(z^*_r))'_+$$

where $(h(x))'_-$ (resp. $(h(x))'_+$) stands for the first left (resp. right) derivative of $h(x)$ evaluated at $x = x^*$.

The $n$-th power of the process $p_k$, i.e. $p^n_k$, is treated similarly. We first write

$$p^n_k = p^n_{k-1} \left[ (1 - \mu)^n + P_n(\xi_k|k-1) \right]^{-1}$$

where $P_n(x)$ is the same polynomial as before and $\xi_k|k-1$ can be written as $\xi_k|k-1 = \varphi_k^2 p_{k-1}$. Then, the uniform (in $k$) functional bounds on $E(p^n_k|F_{k-1})$ takes the following form

$$\frac{p^n_{k-1}}{(1 - \mu)^n + T^+_n(p^n_{k-1})} \leq E(p^n_k|F_{k-1}) \leq \frac{p^n_{k-1}}{(1 - \mu)^n + T^-_n(p^n_{k-1})}$$

for appropriate functions $T_n^-(x)$ and $T_n^+(x)$. In fact, by Jensen’s inequality, it can be shown that $T_n^-(x) \leq Q_n^+(1/x) \leq T_n^+(x)$ with $Q_n^+(x)$ from above.

Finally, when convex and concave dominating functions are estimated for the lower and upper bounds of $E(p^n_k|F_{k-1})$, we similarly obtain the functions $\tilde{T}_n^-(x)$ and $\tilde{T}_n^+(x)$. The attractive equilibrium points of these dominating processes, i.e. $w^*_p$ and $z^*_p$ are found by solving

$$\tilde{T}_n^-(w_p) = \tilde{T}_n^+(z_p) = 1 - (1 - \mu)^n$$

while the lower bounds for the convergence rate to these solutions are given by

$$\gamma_{w,p}^{-1} = 1 - w^*_p(\tilde{T}_n^-(w^*_p))'_- \quad \text{and} \quad \gamma_{z,p}^{-1} = 1 - z^*_p(\tilde{T}_n^+(z^*_p))'_+$$

In the next section, we give simulations of the dynamical (lower convex and upper concave) bounds we have derived for the probability moments of the processes $v_k$ and $p_k$ in the case of a particular distribution of their “independent” random variable $\varphi_k$. The role of the corresponding equilibrium points, i.e. $w^*$ and $z^*$, will also be demonstrated.
5 Simulation results

Here, we illustrate the theoretical results presented in the preceding sections. More precisely, we evaluate asymptotic bounds on the trajectories of the second probability moments of the stochastic processes \( r_k \) and \( p_k = 1/r_k \).

As seen above, these bounds are made of the equilibrium points of the dominating deterministic dynamics associated to the corresponding conditional moments trajectories, i.e. \( E(r_k^2 | \mathcal{F}_{k-1}) \) and \( E(p_k^2 | \mathcal{F}_{k-1}) \). By use of Jensen’s inequality, we further have that

\[
1/z_p^* \leq E(p_k^2) \leq E(r_k^2) \leq z_r^* \quad \text{for large } k
\]

where \( z_r^* \) (resp. \( z_p^* \)) is related to the estimated function \( \tilde{Q}_2^+(r^2) \) (resp. \( \tilde{T}_2^+(p^2) \)) introduced in Section 4.

First, let us introduce the density function of the “input” random variable. Each sample \( \varphi_k \) is taken independently of the others. Its energy density function is based upon a reference density function, i.e. \( d(\varphi^2) \), whose distribution function is denoted \( D(\varphi^2) \). We then consider a modification of this reference density function in order to generate energy sequences that satisfy the excitation condition \( \varphi_k^2 \geq \delta r_{k-1} \) along \( k \geq 0 \) for a chosen value of \( \delta \in [0, 1] \). By defining the conditional (on \( \sigma^2 \)) reference density function as

\[
d(\varphi^2 | \sigma^2) = \frac{1}{1 - D(\sigma^2)} d(\varphi^2) I_{\varphi^2 \geq \sigma^2},
\]

the sample \( \varphi_k^2 \) can be seen as a random variable following a density function identical to \( d(\cdot | \delta r_{k-1}) \).

For the simulations, we consider an energy sequence \( \varphi^2 \) that has a small probability, i.e. \( \nu \) say, of being large with a density function centered at \( \sigma_0^2 \) and a complementary probability, i.e. \( 1 - \nu \), of being small. In Figure 2, we present the density function of \( |\varphi| \) corresponding to a particular example of such a reference density function \( d(\varphi^2) \) for \( \nu = 0.1 \) and \( \sigma_0^2 = 1 \). We also show the constitutive density functions.

In Figure 3, we have presented two realizations of the process \( r_k \) for this reference density function \( d(\varphi^2) \) with \( \delta \) identical either to zero or 0.3 for \( \mu = 0.05 \). Obviously, the two realizations behave very differently: for zero \( \delta \), it exhibits small \( r_k \) (leading to large \( p_k \) ) values due to irrelevant samples \( \varphi_k^2 \) while, for \( \delta = 0.3 \), its focuses on the significant samples out of that density function. Note that memory of the process (or of the initial condition \( r_0 \)) is similar to the inverse of the forgetting factor \( \mu \), i.e. \( \sim 1/\mu \).

Now, let us turn to the trajectory of the second probability moment of the processes \( r_k \) and \( p_k \). Although these two processes are not auto-regressive (AR) per se, they can roughly speaking be seen (from their realizations in Figure 3) as almost-stationary AR processes.

The independence of the samples \( \varphi_k \) over \( k \) and the fact that their density function is at most \( r_{k-1} \)-dependent imply that the conditional (onto the past) probability moments of these processes are uniform in \( k \), i.e. \( E(x_k | \mathcal{F}_{k-1}) = h(x_{k-1}) \). Therefore, the results
derived in Section 2 are easily applied.

Let us then consider the deterministic dynamics that upper dominate the second probability moment of these processes, i.e. $E(r_k^2)$ and $E(p_k^2)$. These dynamics are related to the bounding functions $Q_2^+(r^2)$ and $T_2^+(p^2)$ that are represented in Figure 4 (normalized to $[1 - (1 - \mu)^2]$). As mentionned in Section 4, the equilibrium points of the dominating trajectories are found by making these bounding functions identical to $[1 - (1 - \mu)^2]$.

i.e. leading to $z_{r_k}^*$ and $z_{p_k}^*$, respectively. Furthermore, from the expression (9), the interval
made of these two equilibria, i.e. \(1/z_p^*\) and \(z_r^*\), will asymptotically (in \(k\)) contain the expectation of the square processes. This can be seen in the figure where we have displayed the estimated values of these process 2-norms, i.e. \(\|r_k\|_2\) and \(1/\|p_k\|_2\) with \(\|x\|_2 = [E(x^2)]^{1/2}\). These estimates has been obtained by averaging a particular realization of the associated processes (for \(\delta = \mu = 0.4\)). In fact, the equilibria interval appears to provide bounds that tightly surround the estimated moments.

Finally, in Figure 5, we present the estimated (lower and upper) bounds on the asymptotic value of these process 2-norms, i.e. \(1/(z_p^*)^{1/2}\) and \((z_r^*)^{1/2}\), as functions of the value \(\delta\) for several values of \(\mu\).

It can be emphasized that these bounding intervals are not linear in the \(\delta\) value. Indeed, for small \(\delta\) (i.e. < 0.02), the \(r_k\) process exhibits small sample values that characterize the global distribution of \(\varphi^2\). For increasing \(\delta\)'s (i.e. 0.02 < \(\delta\) < 0.2), the process \(r_k\) tends to exhibit the distribution of more energetic regressors samples \(\varphi_k\). For larger \(\delta\) values (i.e. > 0.2), the samples \(r_k\) already concentrate on the significant part of this distribution, so that more “selective” \(\delta\)'s have only small effects on the realization of \(r_k\). Note also that the bounds on the second probability moments of the process \(r_k\) are rather independent of the forgetting factor \(\mu\) for \(\delta > 0.2\).

6 Conclusions

The main result of this paper is the constructive possibility of transferring the analysis of the trajectory of the probability moments of particular stochastic processes into the stability analysis of dominating deterministic dynamics.

This result has been applied to a nonstationary auto-regressive process obtained by conditioning its “input” random variable on past values of its output.

References


