Bounds on the response time under control constraints

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May 4, 1999

Report no.: LiTH-ISY-R-2141

Submitted to 37th CDC

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Abstract

We consider the problem of transferring the output of a linear system from one equilibrium value to another under control amplitude constraint. It is possible to give a simple lower bound on the time required, expressed in the longest time constant, the gain of the system, the change in y and the control bound.

1 Introduction

Recently much interest has been paid to the various system properties that restrict performance. The references [1], [3], [4], [5] all discuss the basic performance limitations. A common practical control design problem is to find the best possible rise time or settling time for a step response when the control amplitude is limited. To use the criteria in the references mentioned above, one would scale the variables to reflect the amplitude constraints and then transform the time domain properties into bandwidth or gain crossover properties. We are looking for more direct time domain criteria. For that purpose we consider a single-input-single-output system with transfer function $G(s)$ and minimal realization

$$\dot{x} = Ax + Bu, \quad y = Cx \quad (1)$$

We assume the initial condition $x(0) = 0$, and that we want to transfer the state to a new equilibrium

$$y(t_f) = Cx(t_f) = y_0, \quad \dot{x}(t_f) = 0 \quad (2)$$

without exceeding the control constraint

$$|u(t)| \leq u_0 \quad (3)$$

The question is: what is the smallest t_f ? The answer is of course obtained by solving the minimum time optimal control problem, see e. g. [2], but we are looking for some simple lower bound. We do that by first reviewing the minimum time solution for a first order system and then extending the result to higher order systems. We also consider the case when G contains an integrator separately.

2 Minimum time control of a first order system

Consider the system

$$\dot{x} = -\frac{1}{\tau}x + \frac{k}{\tau}u, \quad |u| \leq u_0 \quad (4)$$

Suppose one wants to transfer the state from $x(0) = 0$ to $x(t_f) = y_0 > 0$ with the smallest possible value of t_f . This is a minimum time problem and it is easy to see from the maximum principle that the solution is to use the constant control $u = u_0$ until the desired state is reached. It is then necessary that $u_0 > \bar{u} = \frac{y_0}{k}$. From the equation

$$y_0 = x(t_f) = \int_0^{t_f} e^{-(t_f-\sigma)/\tau} \frac{ku_0}{\tau} d\sigma \quad (5)$$

it follows that

$$t_f = -\tau \log \left(1 - \frac{y_0}{ku_0} \right) = -\tau \log \left(1 - \frac{\bar{u}}{u_0} \right) \quad (6)$$

We see that the minimum time is proportional to the system time constant and depends critically on $\bar{u}/u_0$, i.e. the ratio between the control required to maintain the new state and the maximum control available. If the variables are scaled so that $u_0 = y_0 = 1$, then the minimum time is given by

$$t_f = -\tau \log \left(1 - \frac{1}{k} \right) \quad (7)$$

3 A lower bound on the minimum time control of a general linear system

Now consider a general linear SISO system with transfer function $G(s)$ or in state space form

$$\dot{x} = Ax + Bu, \quad |u| \leq u_0 \quad (8)$$

$$y = Cx \quad (9)$$

where the dimension of the state vector x is $n > 1$. We pose the same problem: to transfer the state from $x(0) = 0$ to an equilibrium $x(t_f)$, such that $y(t_f) = Cx(t_f) = y_0$, with the smallest possible t_f .

Suppose the system has a real negative pole $-1/\tau$. Then it is possible to perform a coordinate change so that (8) can be written in the block triangular

form

$$\frac{d}{dt} \begin{bmatrix} x_r \\ x_d \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & -\frac{1}{\tau} \end{bmatrix} \begin{bmatrix} x_r \\ x_d \end{bmatrix} + \begin{bmatrix} B_1 \\ \frac{k}{\tau} \end{bmatrix} u \quad (10)$$

$$y = \begin{bmatrix} C_1 & c_2 \end{bmatrix} \begin{bmatrix} x_r \\ x_d \end{bmatrix} \quad (11)$$

where x_d , k and c_2 are scalars, x_r is an $n-1$ -vector, and A_{11} , A_{12} and B_1 are matrices of appropriate dimensions. We assume that the mode x_d is controllable, so that $k \neq 0$. The desired final state satisfies the following equations

$$0 = A_{11} x_r(t_f) + A_{12} x_d(t_f) + B_1 \bar{u} \quad (12)$$

$$0 = -\frac{1}{\tau} x_d(t_f) + \frac{k}{\tau} \bar{u} \quad (13)$$

$$y_0 = C_1 x_r(t_f) + c_2 x_d(t_f) \quad (14)$$

where $\bar{u}$ is the control needed to maintain the steady state. The task of transferring the state from $x(0) = 0$ to $x(t_f)$ satisfying $y_0 = Cx(t_f)$ then requires the subtask of transferring x_d from $x_d = 0$ to $x_d(t_f)$ satisfying (13), i.e. $x_d(t_f) = k\bar{u}$. This subtask is of the form described in section 2 and the minimum time according to (6) is

$$t_{f,x_d} = -\tau \log \left(1 - \frac{x_d(t_f)}{ku_0} \right) = -\tau \log \left(1 - \frac{\bar{u}}{u_0} \right) \quad (15)$$

The time for the transfer of the complete state is then bounded below by this value:

$$t_f \geq -\tau \log \left(1 - \frac{\bar{u}}{u_0} \right) \quad (16)$$

If the static gain is finite, then $\bar{u} = y_0/G(0)$, so it is also possible to write

$$t_f \geq -\tau \log \left(1 - \frac{y_0}{G(0)u_0} \right) \quad (17)$$

If the variables are scaled so that $u_0 = 1$, $y_0 = 1$, then

$$t_f \geq -\tau \log \left(1 - \frac{1}{G(0)} \right) \quad (18)$$

4 The case of infinite static gain

The formulas of the preceding section are not useful if G has infinite static gain. Suppose G has a single pole at the origin. We can then modify (10) and (11) to

$$\frac{d}{dt} \begin{bmatrix} x_r \\ x_d \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_r \\ x_d \end{bmatrix} + \begin{bmatrix} B_1 \\ k \end{bmatrix} u \quad (19)$$

$$y = \begin{bmatrix} C_1 & c_2 \end{bmatrix} \begin{bmatrix} x_r \\ x_d \end{bmatrix} \quad (20)$$

We assume controllability of the pole at the origin so that $k \neq 0$. The final state then satisfies the following relations

$$0 = A_{11}x_r(t_f) + A_{12}x_d(t_f) + B_1\bar{u} \quad (21)$$

$$0 = k\bar{u} \quad (22)$$

$$y_0 = C_1x_r(t_f) + c_2x_d(t_f) \quad (23)$$

Since there is no other pole at the origin, A_{11} is nonsingular and

$$y_0 = (-C_1A_{11}^{-1}A_{12} + c_2)x_d(t_f)$$

The minimum time to transfer the integrator

$$\dot{x}_d = ku$$

from the origin to the final state

$$\frac{y_0}{-C_1A_{11}^{-1}A_{12} + c_2}$$

is then

$$t_{f,x_d} = \frac{y_0}{(-C_1A_{11}^{-1}A_{12} + c_2)ku}$$

This is then a lower bound on the minimum time t_f . Defining

$$g_1 = \lim_{s \rightarrow 0} sG(s)$$

and noting that

$$g_1 = (-C_1A_{11}^{-1}A_{12} + c_2)k$$

we get the expression

$$t_f \geq \frac{y_0}{g_1 u_0} \quad (24)$$

5 Tightness of the lower bound

We have given the lower bounds (17), (24) on the time of transfer from $y = 0$ to $y = y_0$. To investigate how tight this bound is we investigate a system with two time constants

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1/\tau_1 & 0 \\ 0 & -1/\tau_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u \quad (25)$$

$$y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (26)$$

The minimum time problem is solved by a control of the form

$$u = u_0, \quad 0 \leq t \leq t_1 \quad (27)$$

$$u = -u_0, \quad t_1 < t \leq t_f \quad (28)$$

Substituting this control into the system equations gives the following condition on the final time t_f

$$2^{1-\nu} \left(1 + k^{-1} + e^{-t_f/\tau_2} \right)^\nu - e^{-t_f/\tau_1} - 1 - k^{-1} = 0 \quad (29)$$

where $\nu = \tau_2/\tau_1$ and $k = G(0)u_0/y_0$. This equation can be solved numerically to give the true value of t_f , which can then be compared with the lower bound (17). The ratio of the true value of t_f and the estimated lower bound is given in figure 1 as a function of k for the case $\tau_1 = 1$, $\tau_2 = 0.1$. The corresponding

Figure 1: Ratio of true to estimated value of t_f as a function of k for $\tau_1 = 1$, $\tau_2 = 0.1$.

figure for $\tau_1 = 1$, $\tau_2 = 0.01$ is given in figure 2. In figure 3 the minimum time

Figure 2: Ratio of true to estimated value of t_f as a function of k for $\tau_1 = 1$, $\tau_2 = 0.01$.

response of the system

$$G(s) = \frac{9}{(s+1)(0.1s+1)} \quad (30)$$

with the constraint $|u| \leq 1$ is given. The lower bound according to (17), is 0.12 in this case. If the system gain is lowered so that

Figure 3: Minimum time step response. $G(0) = 9$, $u_0 = y_0 = 1$, time constants 1 and 0.1.

$$G(s) = \frac{1.5}{(s+1)(0.1s+1)} \quad (31)$$

the response is as shown in figure 4. Equation (17) gives the lower bound 1.1 in this case.

Figure 4: Minimum time step response. $G(0) = 1.5$, $u_0 = y_0 = 1$, time constants 1 and 0.1.

6 Comparison with standard controllers

It is also of interest to compare the performance with that obtained with standard industrial controllers of PI type, with anti-windup to take the input saturation into account. The responses of such a controller for the systems (30) and (31) are given in figures 5 and 6 respectively. In this case there is no definite

Figure 5: Step response of (30) under PI control with $u_0 = y_0 = 1$.

final time t_f , since the new equilibrium is reached only asymptotically. We see that our computed lower bounds for t_f (0.12 and 1.1) respectively provide a good estimate of the step response rise time.

7 Conclusions

With the formula

$$t_f \geq -\tau \log \left(1 - \frac{y_0}{G(0)u_0} \right)$$

where u_0 is the control amplitude, y_0 is the step amplitude, and τ is the largest time constant, we have a practical lower bound of the duration of the step response. A similar formula gives a lower bound when G has a pole at the origin. There are many extensions and refinements possible. The formula above does not apply if there is no real pole. Other extensions concern for instance the influence of unstable poles, asymmetries when the initial state is nonzero, and estimates of the influence of the shorter time constants.

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Figure 6: Step response of (31) under PI control with $u_0 = y_0 = 1$.

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