Model (In-)Validation from a $\mathcal{H}_\infty$ and $\mu$ perspective

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Abstract

We give a short overview on methods of Model (In-)Validation, that fit to the robust control framework. The idea is that the mismatch between a measured datum and an expected datum is explained by a disturbance signal $w$ and an error model $\Delta$, representing unmodelled dynamics. The key question is if there exists a pair $(w, \Delta)$, sufficiently small, that can produce the measured datum. In particular, we view the different approaches by Smith et.al. and Poolla et.al., their numerical solution and given examples.

1 Problem Setup

The general setup for robust control is depicted in figure 1(a): a generalized plant $P$ with the inputs control signal $u$ and disturbance $w$ and an error $\Delta$, representing unmodelled dynamics. The plant $P$ is given (modelling, identification) and we have a measured datum $(u_{\text{meas}}, y_{\text{meas}})$. The question is: Does the datum fit to the model? The main idea is that the (possible) mismatch between a measured datum and the expected (output-)datum is explained by a disturbance signal $w$ and an error model $\Delta$. The relation between inputs and output is given by ULFT:

$$(z) = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} =: P \cdot \begin{pmatrix} v \\ w \end{pmatrix}$$

(1)

$$\Rightarrow y = (P_{21} \Delta (I - P_{11} \Delta)^{-1} [P_{12}, P_{13}] + [P_{22}, P_{23}]) \cdot \begin{pmatrix} w \\ u \end{pmatrix} =: F_U(P, \Delta) \cdot \begin{pmatrix} w \\ u \end{pmatrix}$$

(2)

Figure 1(b) shows a simplification of the general case: a weighted additive error. As a special case, it "disables" the feedback ($P_{11} = 0$) and fixes the location of the disturbance ($P_{12} = 0$). These two simplifications make the optimization problem (presented in the next section) convex. We state the Problem, treated in the following sections:

**Problem-Definition:** Suppose the setup in figure 1(b). The (scaled) model $P$ (plant and weights) is given, also given a measured datum $(u_{\text{meas}}, y_{\text{meas}})$. Do there exist $||w|| \leq 1$ and $||\Delta|| \leq 1$ so that eqn.(2) holds?

Scaling is possible by exploiting $F_U(\gamma P, \Delta) = \gamma F_U(P, \gamma \Delta)$.

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2 Three Different Frameworks

We have a certain plant input $u_{\text{meas}}$ with a measured output $y_{\text{meas}}$, a nominal output $y_{\text{nom}}$ (without errors) and a modelled output $y_{\text{mod}}$ (including the error model). The last two signals are calculated by the (error-)model with input $u_{\text{meas}}$. All signals are of finite length $N$. Ideally, the residuum $r = y_{\text{meas}} - y_{\text{nom}}$ should be zero\(^1\).

To check the quality of the error model $(w, \Delta)$, we compare the measured output with the modelled output, i.e. we compare the residuum with $y_{\text{mod}} - y_{\text{nom}}$. Considering the above simplified plant structure, we get the following

$$
r = y_{\text{meas}} - y_{\text{nom}} \overset{?}{=} y_{\text{mod}} - y_{\text{nom}} = P_w w + P_{\text{nom}} u_{\text{meas}} + P_v \Delta P_{\text{meas}} - P_{\text{nom}} u_{\text{meas}} = P_v v + P_w w \quad (3)
$$

Notable that there is no need for $P_{\text{nom}}$ to be linear. The lhs of eqn.(3) is known by measurement of $y_{\text{meas}}$ resp. calculation of $y_{\text{nom}} = P_{\text{nom}} u_{\text{meas}}$. In the rhs, $w$ varies by $||w|| \leq 1$ and $v$ is given by

$$
v = \Delta z \quad (\ast)$$

$$
z = P_z u_{\text{meas}} \quad (5)
$$

The rhs of eqn.(5) is known, eqn.(\ast) contains the dynamics-error $\Delta$, which will be removed in the next step. As eqn.(\ast) must hold for all $\Delta \leq 1$, the question is, if there exists a relation between input $z$ and output $v$? This question is answered by the so-called

**Extension Theorem:** Eqn.(\ast) holds for a $\Delta \leq 1$, iff the input signal $z$ is larger than the output signal $v$.

Using this, eqn.(\ast) degenerates to

$$
v \text{ "smaller than" } z \quad (4)
$$

---

\(^1\)that’s why the computational complexity of the problem depends in operator theoretical sense on the dimension of the kernel of the mapping $(v, w) \rightarrow r$, compare with eqn.(3).
and the two degree freedom optimization is reduced to a convex optimization problem: minimize $||w||$ with respect to $(3,4,5)$. The result of the optimization is the minimum-norm disturbance $w$, responsible for the given datum $(u_{meas}, y_{meas})$. Finally, we get sufficient invalidation theorems of the following kind (dropping technical statements):

**Invalidation Theorem:** The model is invalid, if $||w|| > 1$.

The extension theorems for the three frameworks are responsible for the computational complexity, because they increase the "size" of eqn.(4) in different amounts.

### 2.1 Discrete Frequency Domain (DFD)

*Initial work:* Smith [8], overview [1, sec 3], example [6].

The DFD approach transforms the time domain data, given in $(3,4,5)$ into frequency domain data by DFT [1, eqns.(5-7)] for all $N$ frequencies. The extension theorem for replacing the uncertainty $\Delta$ replaces eqn.(4) equivalently by

$$V_n^*V_n \leq Z_n^*Z_n, \quad \forall n$$  \hspace{2cm} (DFD 4)

The conditions for the optimization are eqns.(3,5) transformed into the frequency domain and eqn.(DFD 4). Exact formulation [1, Lemma 2 + Theorem 3].

**Properties/Comments**

- Quadratic objective + linear constraints $\Rightarrow$ no local minima.
- Even full sized LFT problems can be solved, applying $\mu$ techniques [8]. The problem remains convex as long as the SSV can be calculated by its upper bound (depends on the number of blocks).
- A computational example exists [6]: two liquids of different temperature are mixed in a tank (MIMO, $2 \times 2$), two different models are validated.
- Application of DFT: signals $v, w$ have to be zero for negative times. No problem for $v$ (depends on $u$), but for $w$, therefore restriction to static $P_w$, see [6, sec 2.3].
- Computational complexity $\sim N$ tractable.
- Another type of problem is posed in [8]: minimize the size of $\Delta$ and $w$. The problem is similar to the computation of $\mu$, but not solved.

To avoid problems with DFT, we jump back into the time domain:

### 2.2 Discrete Time Domain (DTD)

*Initial work:* Poolla [7], overview [1, sec 4].

The DTD approach transforms the pulse response coefficients of $P_v, P_u, P_z$, given in $(3+5)$ into their associated lower block Toepliz matrices (this a $N \times N$ matrix) and the signals into appropriate ones ($N$ vector); see [1, eqns.(10+11)]. The extension theorem for replacing the uncertainty $\Delta$ replaces eqn.(4) equivalently by

$$V^*V \leq Z^*Z$$  \hspace{2cm} (DTD 4)
where $V$ and $Z$ are the associated lower block Toeplitz matrices of the signals (i.e. eqn.(DTD 4) is a matrix inequality). Exact formulation [1, Theorem 4+5].

Properties/Comments

- Problem convex as long as $Z^* Z$ constant ($\iff P_{11} = P_{12} = 0$). General case?
- No restrictions on $v$ and $w$ for negative times as in the DFD (appearing from DFT): nonzero $v$ can be handled by residuum, nonzero $w$ be initial conditions of $P_w$.
- Also LTV perturbations possible in the framework.
- This theory works also for multidimensional signals.
- Computational complexity $\sim N^5$, not feasible for reasonable data-length, even in LMI formulation [2, sec 4].
- Therefore no examples given.

2.3 Sampled Data Domain (SDD)

Initial work: Smith and Dullerud [3] (technical details [5]), equivalent results independently derived by Poolla [4], overview [1, sec 5], example [2].

DTD regards the plant as a purely discrete-time system, with a "built-in" sampling time $T$. SDD is based on DTD, but interprets the plant as a sampled continuous system. The separation of plant and sample/hold unit enables us to subsample, i.e. we use the DTD-machinery for a subset of our data to get a feasible problem. The theoretical results are derived using the lifting operation. After this transformation, size and appearance of the extension theorem for replacing the uncertainty $\Delta$ are similar to the DTD case (including transformation to lower block Toeplitz matrices):

$$\hat{V}^* \hat{V} \leq \hat{Z}^* \hat{Z} \quad \text{(SDD 4)}$$

Exact formulation [1, Theorem 7+8].

Properties/Comments

- The invalidation theorem gets necessary and sufficient for $T \to 0$ (which is only of theoretical interest).
- Same comments as in DTD, but subsampling possible. Start with subsampling time $T_{\text{sub}} \gg T$ and decrease until the model is invalid or $T_{\text{sub}} = T$.
- Example: heating system, SISO [2].
- Computational time within the example: model invalid for data-length of $N = 64$, this iteration-step needed $72 \cdot 10^3$ Mflops ($2h40mins$ CPU-time on Ultra1). The final step was the 6th. [2, table 1].

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3 Questions

1. DTD and SDD in case of full LFT: convexity is lost. Other solutions?
2. DFD solvable for full LFT problems (µ): implementation? examples?
3. Exploiting sparse structure in SDD to get a faster implementation [2]?
4. MIMO problems in SDD
5. "iff" invalidation theorems?
6. Suppose model is invalid because of min ||w||₂ = 1.36, how to adjust the model knowing this value 1.36? Scaling and bounds in general?
7. All approaches compute the minimum size of w for all ||Δ||ₘₐₓ ≤ 1. What about the question: (uₖ, yₖ) given, minimum size of Δ and w [8, Problem 4.1]?

References


(references in reversed chronological order)