Descriptor Systems and Control Theory

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Abstract
This report is a brief survey over different concepts and methods concerning singular systems. Singular system is also called differential algebraic equations (DAE) since a DAE model may involve both differential and algebraic equations. There is a growing interest in DAE systems much depending on the increased usage of object-oriented modeling languages, like MODELICA, in the modeling of dynamical systems. These programs often give DAE models as result. Another advantage of DAE models is that they sometimes can keep the natural structure of the dynamical model.

Keywords: Singular systems, implicit systems, DAE systems
1 Introduction

Many processes can be described, at least approximately, by a system of ordinary linear differential equations

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n} \) and \( D \in \mathbb{R}^{p \times m} \). For this class of systems an extensive amount of books are written. However, sometimes this representation does not describe the behavior of the process sufficiently well and it may be necessary to use nonlinear differential equations. Then (1) becomes

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)) \\
y(t) &= h(x(t), u(t))
\end{align*}
\]

where \( f \) and \( h \) are arbitrary functions. Theory for this class of systems can be found in, e.g., (Khalil, 2002). Most physical plants are covered by this system class but there exist systems which do not fit into (2). At least they are easier to model in a more general framework in order to keep the natural structure of the problem. Instead we want to represent the system as a first order differential equation

\[
F(\dot{z}(t), z(t), u(t)) = 0
\]

where \( F \) may be any vector valued function. For example, \( F \) may be a function with singular \( F_z \) where \( F_z \) denotes \( \partial F(\dot{z}, z, u)/\partial \dot{z} \). A system with \( F_z \) singular is called a differential algebraic equation (DAE) because it can include both differential and algebraic equations of the variables. The vector \( z(t) \) is called the generalized state vector or descriptor vector (Bender and Laub, 1987) and the components are called generalized states or descriptor variables. All generalized states will in general not describe the dynamics in the system as for state space models, but can also be variables included due only to the algebraic equations. A small example of a DAE is a pendulum modeled using the Newton-Euler equations.

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**Example 1: Planar pendulum**

Consider a pendulum consisting of sphere with mass, \( m \), connected to the ground via a stiff, massless rod with constant length, \( l \), see Figure 1. The
position of the center of the sphere is \( x \) and \( y \) and the angle between the \( x \)-axis and the massless rod is given by \( \alpha \). At the connection point a time varying torque, \( T(t) \), is applied. This torque is assumed to be the control signal. In the same point, two contact forces appear, the force \( F_x \) directed along the \( x \)-axis and the force \( F_y \) pointing along the \( y \)-axis. The moment of inertia for the pendulum with respect to the connection point is \( J \). The generalized state vector \( z(t) \) is

\[
z(t) = \begin{bmatrix} x \\ y \\ \alpha \end{bmatrix}
\]

The governing equations consist of the Newton-Euler equations plus the kinematic constraints as

\[
\begin{bmatrix}
m & 0 & 0 \\
0 & m & 0 \\
0 & 0 & J
\end{bmatrix} \dddot{z}(t) = \begin{bmatrix}
0 \\
-mg \\
\frac{T}{J}
\end{bmatrix} + \begin{bmatrix} F_x \\ F_y \\ -F_x l \cos \alpha - F_y l \sin \alpha \end{bmatrix}
\]

\[
0 = \begin{bmatrix} x - l \sin \alpha \\ y + l \cos \alpha \end{bmatrix}
\]

Equation (4) form a DAE and it is often quite easy to obtain this form for general mechanical systems. Other mechanical systems formulated as DAEs can be found in (Hahn, 2003).

Systems on DAE form occur in many different areas, and have received a lot of attention since the 70’s. One reason for the attention is the increased usage of object-oriented modeling languages, e.g., Modelica (Fritzson, 2004; Tiller, 2001), when modeling systems in computers. This type of modeling tools often yield DAEs as result.

Other terms used for DAEs are implicit systems, singular systems, generalized state space systems etc. (Brenan et al., 1989). The system description studied in this report is the linear special case of the nonlinear DAE (3)

\[
\begin{align*}
Ex(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{align*}
\]

where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \) and \( E \) is a square matrix of rank \( r < n \), i.e., a singular matrix.

--- Example 2: Linear DAE ---

A linear state space system (1) can be formulated on DAE form (5) as

\[
\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} A & B \\ 0 & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ u(t) \end{bmatrix}
\]

\[
y(t) = \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}
\]

The matrix \( E \) is singular and has the same rank as the number of ordinary states, \( x(t) \), and it has some rows equal to zero, i.e., the system is a DAE. Note that a state space system with direct term \( Du(t) \) can be included in the DAE form (5), i.e., without a direct term.
Another example of a linear DAE is an electrical circuit. This example will be used throughout this report to show certain properties of DAEs. It is a modified version of an example in (Zhang et al., 2003).

--- Example 3: Electrical circuit ---

![Figure 2: Electronic circuit.](image)

The electric circuit in Figure 2 contains some typical components in electrical systems. The behavior of the capacitor and the inductor described by the differential equations

\[
CV_C(t) = i_C(t) \\
L\dot{i}_L(t) = V_L(t)
\]  

(6)

where \( V_C(t), V_L(t) \) are the voltages over the capacitor and inductance respectively and \( i_C(t), i_L(t) \) are the corresponding currents. The resistances, \( R_1 \) and \( R_2 \), are described by

\[
V_{R,j}(t) = R_j i_{R,j}(t), \quad j = 1, 2
\]  

(7)

The current source is assumed to be ideal, that is, it can provide an arbitrary current, \( i(t) \), independent of the voltage over it. Using Kirchhoff’s laws the following differential equations are obtained

\[
L\dot{i}(t) = -R_1 i(t) - V_{R,2}(t) + V_S(t) \\
CV_{R,2}(t) = -\frac{1}{R_2} V_{R,2}(t) + i(t)
\]  

(8)

where \( V_S(t) \) is the voltage over the current source. The current \( i(t) \) is the control signal which gives the equation

\[
u(t) = i(t)
\]

Using the generalized state vector

\[
z(t) = \begin{bmatrix} i(t) \\ V_{R,2}(t) \\ V_S(t) \end{bmatrix}
\]
and assuming that the output of the system, \( y(t) \), is the voltage over \( R_2 \), the matrix form of the complete circuit becomes

\[
\begin{bmatrix}
L & 0 & 0 \\
0 & C & 0 \\
0 & 0 & 0
\end{bmatrix} \dot{z}(t) = \begin{bmatrix}
-R_1 & -1 & 1 \\
1 & \frac{1}{R_2} & 0 \\
1 & 0 & 0
\end{bmatrix} z(t) + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} u(t) \quad (9a)
\]

\[y(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} z(t)\quad (9b)\]

The matrix \( E \) is singular and the result is thus a DAE.

In this example the equation was manually worked up while modeling. For example the the equations for the resistors (7) were inserted in (6) yielding (8). However, the modeling could also have been done more automatically by just model each component and then adding constraints connecting them. This method is commonly used when modeling mechanical system consisting of multiple bodies connected in different ways, see (Hahn, 2003).

The transfer function for the system (5) is an ordinary rational function in \( s \), where \( E \) has replaced an identity matrix.

\[G(s) = C(sE - A)^{-1}B + D \quad (10)\]

The difference between (10) and a transfer function for a state space model is that (10) may be non-proper. That is, the degree of the numerator can exceed the degree of the denominator. The non-properness results in that derivatives of the system’s input signal, \( u(t) \), may occur in the output signal, \( y(t) \). This sort of behavior is often unwanted and will in Section 11 be eliminated by using state feedback.

## 2 Impulsive Modes

Ordinary systems (1) only have finite modes. However, a system on DAE form (5) may have both finite and infinite modes. The latter modes correspond to infinite eigenvalues of \( sE - A \) or equivalently eigenvalues \( \lambda = 0 \) to \( E - \lambda A \). These infinite eigenvalues correspond to dynamic or nondynamic modes. The nondynamic modes are associated with directions of the generalized state vector in which purely algebraic equations describe the relationships between the generalized states, the input signal and the output signal. The number of nondynamic modes of (5) is equal to \( n - r \), i.e., the total number of generalized states minus the rank of the \( E \) matrix. The associated directions are given by the generalized eigenvectors

\[Ev_1^i = 0, \quad i = 1, \ldots, n - r \quad (11)\]

Superscript 1 indicates that this is a grade 1 generalized eigenvector (Bender and Laub, 1987). Note that the matrix \( A \) has disappeared since \( \lambda = 0 \) in \( E - \lambda A \), and that \( v_1^i \) span the nullspace of \( E \). The dynamic modes, also called the poles of the system, incorporate both infinite modes and finite modes of \( sE - A \). All modes that are not nondynamic are dynamic and therefore the number of dynamic modes is \( r \). The finite dynamic modes are obtained by calculating

\[AW = EWA \quad (12)\]
and pick the eigenvectors corresponding to finite eigenvalues. Since these modes are obtained in the ordinary way, the number of them can be computed as

\[ n_1 = \deg \det (sE - A) \]

This implies that it is always fewer finite poles than generalized states for a DAE. The infinite dynamic modes, also called impulsive modes, are associated with directions of the generalized state vector in which impulsive behavior of the solution may occur only because of the initial conditions, \( i.e., \) despite an input signal \( u(t) \) identically equal to zero. The associated directions, corresponding to the \( i \)th grade \( 1 \) infinite generalized eigenvectors, are computed as

\[ Ev_i^{k+1} = Av_i^k \]  \hspace{1cm} (13)

where \( k \geq 2 \). These higher grade generalized eigenvectors span the impulsive solution subspace of (5) and the number of infinite dynamic modes are \( r - n_1 \). If \( r = n_1 \) the number of impulsive modes equal zero. In this case the system is said to be impulse free or index one.

The number of different modes can then be summarized as

- \( n_1 \) finite dynamic modes
- \( r - n_1 \) infinite dynamic modes, also called impulsive modes
- \( n - r \) nondynamic modes

\section{Regularity}

The property that the matrix pencil \((sE - A)\) is nonsingular in all but a countable number of points is called regularity. This is fundamental as it is equivalent to existence and uniqueness of a solution to (14). It is therefore formalized as a definition.

\textbf{Definition 3.1 (Regularity)} A linear continuous-time descriptor system

\begin{align*}
E\dot{x}(t) &= Ax(t) + Bu(t) \hspace{1cm} (14a) \\
y(t) &= Cx(t) + Du(t) \hspace{1cm} (14b)
\end{align*}

is called regular if

\[ \det(sE - A) \neq 0, \hspace{1cm} (15) \]

that is the determinant is nonzero for some \( s \).

Regularity will be assumed in the remainder of this report. This is not a very restrictive assumption and most systems occurring in practice are regular. Note that for state space systems regularity is trivially fulfilled since

\[ \det(sI - A) = 0 \]  \hspace{1cm} (16)

only for the poles of the system.
4 Transformations

4.1 Canonical form

A regular DAE on the form (10) can be transformed into something called the canonical form. This means that (14) is decomposed into two subsystems, where one of the subsystems is a state space system.

Theorem 4.1 (Canonical form) Consider a linear continuous-time, regular DAE

\[ E\dot{x}(t) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) \] (17)

By using a coordinate transformation

\[
\begin{bmatrix}
  x_1(t) \\
  x_2(t)
\end{bmatrix} = Qx(t), \quad x_1(t) \in \mathbb{R}^{n_1}, x_2(t) \in \mathbb{R}^{n_2}
\] (18)

the system (17) can always be written as a partitioned system as

\[
\begin{align*}
  \dot{x}_1(t) &= A_1x_1(t) + B_1u(t) \\
  N\dot{x}_2(t) &= x_2(t) + B_2u(t) \\
  y(t) &= C_1x_1(t) + C_2x_2(t)
\end{align*}
\] (19a-b-c)

The system matrices are

\[
\begin{bmatrix}
  I & 0 \\
  0 & N
\end{bmatrix} = \bar{E} = PEQ \quad \begin{bmatrix}
  A_1 & 0 \\
  0 & I
\end{bmatrix} = \bar{A} = PAQ
\] (20a)

\[
\begin{bmatrix}
  B_1 \\
  B_2
\end{bmatrix} = \bar{B} = PB \quad \begin{bmatrix}
  C_1 & C_2
\end{bmatrix} = \bar{C} = CQ
\] (20b)

where \( N \) is an \( n_2 \times n_2 \) nilpotent matrix, i.e., \( N^k = 0 \) for some \( k \). The matrices \( P, Q \) are constant and nonsingular, but not necessarily unitary.

Proof See (Dai, 1989; Gerdin, 2004).

Equation (19a) is a differential equation usually referred to as the slow subsystem (Cobb, 1981) or the finite subsystem (Lewis, 1986). This part of the DAE determines for example stability. Equation (19b) is a pure algebraic equation in the generalized state \( x_2(t) \), the input signal, \( u(t) \), and derivatives of \( u(t) \). This is more obvious if (19b) is reformulated to its equivalent form

\[
x_2(t) = -\sum_{i=1}^{k-1} \delta^{(i-1)}(t)N^i x_2(0-) - \sum_{i=0}^{k-1} N^i B_2 u^{(i)}(t)
\] (21)

where \( \delta^{(i)} \) is the \( i \)'th derivative of a Dirac impulse and \( u^{(i)}(t) \) is the \( i \)'th derivative of the \( u(t) \). This equation is referred to as the fast subsystem, since no dynamics are involved.

From (21) we see that if \( k \) is larger than 1, the solution may contain impulses, generated both by unappropriately chosen initial conditions and by derivatives.
of for example step inputs. Furthermore, it can be noted that a system with $k$ equal to 1 is nothing but a state space system with direct term.

Other terms for the canonical form (19) is First Equivalent Form (EF1) (Dai, 1989).

The numerical issues computing the canonical form are solved in (Gerdin, 2004), but it is important to note that the transformation is not unique.

4.2 SVD Coordinate Form

In order to verify certain properties of the system, e.g., controllability, it will prove useful to have the system in a certain coordinate system based on a SVD-decomposition of $E$.

**Theorem 4.2 (SVD Coordinate System)** Consider a linear DAE on the form (17). By performing a singular value decomposition of the $E$ matrix, it can always be rewritten as

\[
\begin{align*}
\tilde{E}\tilde{x}(t) &= \tilde{A}\tilde{x}(t) + \tilde{B}u(t) \\
g(t) &= \tilde{C}\tilde{x}(t)
\end{align*}
\]

The matrix $\tilde{E}$ is a diagonal matrix

\[
\tilde{E} = U^*EV = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}
\]

where $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, 0, \ldots, 0)$, $\sigma_i$ is positive and $U^*U = V^*V = I$.

The other system matrices in (22) are

\[
\begin{align*}
\tilde{A} &= U^*AV = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \\
\tilde{B} &= U^*B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\
\tilde{C} &= CV = \begin{bmatrix} C_1 & C_2 \end{bmatrix}
\end{align*}
\]

**Proof** See (Bender and Laub, 1987).

The input-output relation is unchanged and the transfer function (10) in the SVD coordinate system will be

\[
G(s) = \tilde{C}(s\tilde{E} - \tilde{A})^{-1}\tilde{B} = \tilde{C}\begin{bmatrix} s\Sigma - A_{11} & -A_{12} \\ -A_{21} & -A_{22} \end{bmatrix}^{-1}\tilde{B}
\]

where only the upper left corner of the inverted matrix is dependent of $s$.

4.3 Connections between the Canonical form and the SVD coordinate form

The difference between the Canonical form and the SVD Coordinate form is that in the Canonical form the descriptor states are partition as finite/infinite, while in the SVD Coordinate form the descriptor states are divided as dynamic/nondynamic. The difference between the number of finite modes in (19), $n_1$, and the number of dynamic modes in (23a), $r$, is therefore the infinite dynamic modes as can be seen in Section 2. This implies that for a system with
only finite poles (dynamic modes) we have \( n_1 \) \( = \) \( r = \text{rank} \ E \), i.e., the number of finite modes equals the rank of \( E \). The connection between \( \bar{E} \) and \( \tilde{E} \) is found by using (20a) and (23a) as
\[
\begin{bmatrix}
I & 0 \\
0 & N
\end{bmatrix} = \begin{bmatrix}
\Sigma & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
V^* & 0
\end{bmatrix}
\]
where \( PU \) and \( V^*Q \) are nonsingular matrices. Equation (25) can be used to determine the number of dynamic infinite modes. The rank of \( \tilde{E} \) is \( r \) and the rank of the identity matrix in \( \bar{E} \) is \( n_1 \). This yields that the rank of the \( N \) matrix must be \( r - n_1 \), i.e., the number of infinite dynamic modes.

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**Example 4: State transformations**

Consider the electric circuit described in Figure 2. Assume that \( L = 3 \), \( R_1 = 2 \), \( R_2 = 1 \) and \( C = 2 \) which gives the system description
\[
\begin{bmatrix}
3 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 0
\end{bmatrix} \dot{z}(t) = \begin{bmatrix}
-2 & -1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix} z(t) + \begin{bmatrix}
0 \\
0 \\
-1
\end{bmatrix} u(t) \\
y(t) = \begin{bmatrix}
0 & 1 & 0
\end{bmatrix} z(t)
\]
\( (26) \)

The canonical form of (26) is
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} \\
0 & 0 & 0
\end{bmatrix} \dot{w}(t) = \begin{bmatrix}
-\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix} w(t) + \begin{bmatrix}
-\frac{1}{\sqrt{2}} \\
\sqrt{2} \\
1
\end{bmatrix} u(t) \\
y(t) = \begin{bmatrix}
-\frac{1}{\sqrt{2}} & -\sqrt{2} & 0
\end{bmatrix} w(t)
\]
\( (27) \)

where \( n_1 = 1 \), \( n_2 = 2 \) and
\[
x(t) = \begin{bmatrix}
0 & 0 & -1 \\
-\frac{1}{\sqrt{2}} & 0 & 0 \\
-\frac{1}{\sqrt{2}} & -\sqrt{2} & 0
\end{bmatrix} w(t)
\]

The number of finite and infinite modes for this system is thus \( n_1 = 1 \) and \( n_2 = 2 \) respectively and the infinite modes are both dynamic and nondynamic. The electric system (26) is already expressed in SVD coordinate form, with
\[
\Sigma = \begin{bmatrix}
3 & 0 \\
0 & 2
\end{bmatrix}
\]

The number of dynamic modes is \( r = \text{rank} \Sigma = 2 \). This shows that for some systems is \( n_1 \neq r \).

---

5 Index

For a system on DAE form the \( E \) matrix is singular. Then, perhaps after some Gauss elimination, the system can always be written as
\[
\begin{bmatrix}
\hat{E} \\
0
\end{bmatrix} \dot{x}(t) = \begin{bmatrix}
\hat{A}_1 \\
\hat{A}_2
\end{bmatrix} x(t) + \begin{bmatrix}
\hat{B}_1 \\
\hat{B}_2
\end{bmatrix} u(t)
\]
\( (28) \)
where $\tilde{E}$ has full row rank. If the last block row in (28) is differentiated, (28) becomes
\[
\begin{bmatrix}
\tilde{E} \\
\tilde{A}_2 
\end{bmatrix} \dot{x}(t) = \begin{bmatrix}
\tilde{A}_1 \\
0 
\end{bmatrix} x(t) + \begin{bmatrix}
\tilde{B}_1 \\
0 
\end{bmatrix} u(t) + \begin{bmatrix}
0 \\
\tilde{B}_2 
\end{bmatrix} \dot{u}(t) 
\]
and the zero in the $E$ matrix is replaced by $\tilde{A}_2$. The new $E$ matrix may or may not be nonsingular. If nonsingularity is obtained it is possible to multiply with $E^{-1}$ from the left and get a state space system. If the new $E$ is also singular the procedure can be repeated until nonsingularity is reached. This method is called the Shuffle algorithm and was invented by (Luenberger, 1978). The number of differentiations needed before the matrix $E$ becomes nonsingular will be defined as the index of the DAE, and is an important indicator of how hard the system is to solve. An example of how to use the Shuffle algorithm in order to calculate the index is given below.

--- Example 5: Index ---

Consider the electric circuit in Example 3. The system is given by (9) as
\[
\begin{bmatrix}
L & 0 & 0 \\
0 & C & 0 \\
0 & 0 & 0 
\end{bmatrix} \dot{z}(t) = \begin{bmatrix}
-R_1 & -1 & 1 \\
1 & \frac{1}{R_2} & 0 \\
1 & 0 & 0 
\end{bmatrix} z(t) + \begin{bmatrix}
0 \\
0 \\
-1 
\end{bmatrix} u(t) 
\]
\[y(t) = \begin{bmatrix}
0 \\
1 \\
0 
\end{bmatrix} z(t) \] \hspace{1cm} (30)

In (30) the matrix $E$ is already on the form (28). Differentiation of the last row gives
\[
\begin{bmatrix}
L & 0 & 0 \\
0 & C & 0 \\
1 & 0 & 0 
\end{bmatrix} \dot{z}(t) = \begin{bmatrix}
-R_1 & -1 & 1 \\
1 & \frac{1}{R_2} & 0 \\
0 & 0 & 0 
\end{bmatrix} z(t) + \begin{bmatrix}
0 \\
0 \\
-1 
\end{bmatrix} \dot{u}(t) 
\]
and this is the shuffle step. As was said in Section 1 derivatives of the input signal might occur and that happens in this case. After Gauss elimination of the new $E$ matrix and differentiation of the last row the system becomes
\[
\begin{bmatrix}
L & 0 & 0 \\
0 & C & 0 \\
R_1 & 1 & -1 
\end{bmatrix} \dot{z}(t) = \begin{bmatrix}
-R_1 & -1 & 1 \\
1 & \frac{1}{R_2} & 0 \\
0 & 0 & 0 
\end{bmatrix} z(t) + \begin{bmatrix}
0 \\
0 \\
-1 
\end{bmatrix} \ddot{u}(t) 
\]
After two differentiations the $E$ matrix has become nonsingular and hence the index of this system is 2.

The result above is formalized in the definition below.

**Definition 5.1** The number of differentiations needed in order to get a nonsingular matrix $E$ is called the index.

The definition above is not unique. Another equivalent definition of the index is obtained by using the $N$ matrix in the canonical transformation of the system, see Section 4.1. The index is then given by the degree of nilpotency of this matrix, i.e., the integer $k$.

One can ask when the iterations in the Shuffle algorithm terminate. In (Stengel et al., 1979; Dai, 1989) it is shown that it happens in either of two ways. One way is that the $E$ matrix becomes nonsingular and the other way
is that the block-row in the $A$ matrix, corresponding to the zero-row in the $E$ matrix, becomes all zero. Then it is impossible to get a nonsingular matrix $E$. Which case that happens depends on the system regularity. The first case corresponds to regular system and the latter case to a nonregular one.

The state space model acquired after the differentiation is only apparently a state space model, because the state variables must still fulfill the algebraic constraints given in the original DAE. One method to solve the DAE would then be to find some consistent initial variables and then use an ordinary ODE-solver on the state space model. However, this would probably after some iterations result in a solution not fulfilling the algebraic constraints due to numerical noise. In fact, the higher the index, the more difficult the system is to solve numerically (Brenan et al., 1989).

A very useful result is Lemma 3 in (Bender and Laub, 1987), here formulated as a theorem.

**Theorem 5.1** Assume that the linear DAE system (14) is on SVD coordinate form. Then if $A_{22}$ is invertible it can be concluded that

1. the pencil $sE - A$ is regular
2. the system has no impulsive modes or equivalently is index one (Dai, 1989).

**Proof** See (Bender and Laub, 1987) or Section 11.

This theorem is used when deriving state feedback gains making the closed loop system regular and of index one.

### 6 Solvability

For a linear system to be solvable, i.e., to have a unique solution, it is necessary that the system is regular (Brenan et al., 1989). This is formalized in the following theorem.

**Theorem 6.1 (Solvability)** Consider a linear continuous-time DAE (14). A unique solution exists if and only if the DAE is regular, i.e., $\det(sE - A) \neq 0$.

**Proof** See (Brenan et al., 1989).

For linear state space systems, regularity is trivially fulfilled and the solution, for any given initial condition $x(0-)$, is given by

$$x(t) = e^{At}x(0-) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

(31)

A motivation for the theorem is that a regular DAE always can be transformed to canonical form (19). The solution is then given by the solution to (19a), which is an expression like (31), together with (21), i.e.,

$$x_1(t) = e^{A_1t}x_1(0-) + \int_0^t e^{A_1(t-\tau)}B_1u(\tau)d\tau$$

(32a)

$$x_2(t) = -\sum_{i=1}^{k-1} \delta(t-i)N^ix_2(0-) - \sum_{i=0}^{k-1} N^iB_2u^{(i)}(t)$$

(32b)
Regularity guarantees the existence of a unique solution. This solution might however contain impulses as mentioned in Section 4.1. In order to get a solution \( x(t) \) without impulsive behaviour further requirements on the input signals and the initial conditions are needed. The control signals must fulfill \( u(t) \in C_{p}^{k-1} \). That is, the input signal must be \( k-1 \) times piecewise differentiable. The requirements for \( x(0-) \) are more intricate. In (32b) it can be seen that if \( Nx_2(0-) = 0 \) no impulses occur. In some articles \( Ex(0-) = 0 \) is required instead. The connection between these two requirements can be seen in (33).

\[
Ex(0-) = 0 \iff PEQQ^{-1}x(0-) = 0 \iff \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} x_1(0-) \\ x_2(0-) \end{bmatrix} = 0 \tag{33}
\]

This means that if \( x(0-) \) fulfills \( Ex(0-) \) it is also known that \( Nx_2(0-) = 0 \). In this case no impulsive behaviour will occur, but the solution will not necessarily be continuous. A jump in some generalized states at \( t = 0 \) might still be needed in order for the generalized states to fulfill the algebraic equations. Continuity of the solution requires the initial values to be consistent. Consistency is in (Dai, 1989) defined as

\[
x(0^+) = P \begin{bmatrix} I \\ 0 \end{bmatrix} x_1(0) - P \begin{bmatrix} 0 \\ I \end{bmatrix} \sum_{i=0}^{h-1} N^i B_2 u^{(i)}(0^+)
\]

where \( x(0^+) = \lim_{t \to 0^+} x(t) \).

\[- \]

**Example 6: Initial conditions**

Assume that the system is

\[
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} x \tag{34}
\]

which means that \( x_1(t) = x_2(t) \) for all \( t \geq 0 \). The initial condition is assumed to be \( x(0-) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T \) which fulfills \( Ex(0-) = 0 \). The canonical form of (34) is given by

\[
\begin{align*}
\dot{w}_1 &= 3w_1 \\
\dot{w}_2 &= 0
\end{align*} \tag{35a, 35b}
\]

and the transformation is

\[
\begin{align*}
w_1 &= -2x_1 \\
w_2 &= x_1 - x_2
\end{align*} \tag{36a, 36b}
\]

The initial conditions expressed in the new variables is

\[
w(0-) = \begin{bmatrix} 0 \cdot 2 \\ 0 - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}
\]

Here it can be seen that for \( t = 0 \), \( w_2 \) jumps from \(-1\) to \(0\), i.e., there is a jump in the solution but no impulse. However, if the initial condition is chosen as \( x(0-) = [1 \ 1]^T \) which does not fulfill \( Ex(0-) = 0 \), we obtain

\[
w(0-) = \begin{bmatrix} -2 \cdot 1 \\ 1 - 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}
\]

which will give a continuous solution since \( w_2(0-) = w_2(0) \).
7 Controllability Concepts

For ordinary linear systems only one concept of controllability is used. However, for singular systems it is necessary to separate the controllability of the slow part (19a) and the controllability of the fast part (19b). This depends on the fact that systems on DAE form may include algebraic constraints implicitly or explicitly which most often restricts the reachable region to a subspace of $\mathbb{R}^n$. This implies that singular systems seldom are controllable in the ordinary sense. The theory in this section comes from (Dai, 1989), but another article that thoroughly sort out the different concepts of controllability and observability is (Cobb, 1984).

7.1 Controllability

Controllability means that for any initial condition $x(0-)$ it should be possible to choose a “nice” input signal such that the state of the system can reach an arbitrary point in $\mathbb{R}^n$ arbitrarily fast. This can be formalized as

**Definition 7.1 (Controllability)** A linear singular system is controllable if, for any $t_1 > 0$, $x(0-) \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$, there exists a $u(t) \in C^{k-1}$ such that $x(t_1) = w$.

The following theorem shows how to check the controllability of a DAE.

**Theorem 7.1 (Controllability)** Given a singular system on canonical form (19), controllability can be concluded if and only if

$$\text{rank} \begin{bmatrix} B_1 & A_1 B_1 & \ldots & A_1^{n_1-1} B_1 \end{bmatrix} = n_1 \text{ and } \text{rank} \begin{bmatrix} B_2 & NB_2 & \ldots & N^{k-1} B_1 \end{bmatrix} = n_2$$  \hspace{1cm} (37a)

or

$$\text{rank} \begin{bmatrix} sE - A & B \end{bmatrix} = n, \forall s \in \mathbb{C}, s \text{ finite and } \text{rank} \begin{bmatrix} E & B \end{bmatrix} = n$$  \hspace{1cm} (37b)

**Proof** See (Dai, 1989).

It can be seen, especially in (37b), that the test the controllability of a singular system is very similar to the ordinary case, except for an extra condition, e.g., $\text{rank} \begin{bmatrix} E & B \end{bmatrix} = n$, corresponding to the nondynamic modes. The advantage of (37a) is that transformation of the system is unnecessary. In both (37a) and (37b) the left rank test corresponds to controllability of the slow subsystem while the right rank test correspond to controllability of the fast subsystem.

Note that for singular systems reachability is not equivalent to controllability as for state space systems (5) (Lewis, 1986). Controllability of the fast subsystem is also called reachability at $\infty$ (Lewis, 1986).

7.2 R-controllability

R-controllability is controllability within the consistent reachable set of the generalized states. This can also be formulated as controllability for the slow subsystem. Since the slow subsystem is a state space system this property can be
verified as usual, see e.g., (37),

$$\text{rank } \begin{bmatrix} B_1 & A_1 B_1 & \ldots & A_1^{n-1} B_1 \end{bmatrix} = n_1$$

(38)

In similarity with state space systems, it is often not necessary to require the system to be R-controllable, but to be stabilizable.

### 7.3 Impulse Controllability

In equation (21) it was seen that a DAE may have impulsive modes due to the initial value, resulting in a solution containing impulses. This behavior is, except for in a few cases, unwanted for the closed loop system since it may saturate the system or even destroy it (Dai, 1989). It is therefore necessary to be able to eliminate this sort of behavior using only an input signal, \( u(t) \in C_{p-1} \). This property is called impulse controllability and can be concluded if

$$\text{rank } \begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} = n + \text{rank } E$$

(39)

In (Dai, 1989) it can be seen that the test for impulse observability and observability of the fast subsystem are similar. However, the nature of the concepts are different. Controllability of the fast subsystem corresponds to the idea of reachability, i.e., bringing \( x_2(t) \) to a specific position. Impulse controllability, on the other hand, guarantees that all possible impulses, generated by inconsistent initial conditions, should be possible to eliminate using an admissible control signal. In one of the tests for impulse controllability, i.e.,

$$\text{rank } \begin{bmatrix} A_{22} & B_2 \end{bmatrix} = n - \text{rank } E$$

(40)

it can be seen that the kernel of the matrix \( N \) is excluded. The interpretation of this is that \( x_2(0-) \in \text{Ker } N \) does not generate any impulses and is therefore not necessary to cancel, see Section 6.

An important property for impulse controllable systems is that it is possible to choose a feedback gain, \( L \), in

$$u(t) = -Lx(t) + r(t)$$

such that the closed loop system

$$E\dot{x}(t) = (A - BL)x(t) + Br(t)$$

$$y(t) = Cx(t)$$

has an index not exceeding one. Other designations for impulse controllability are impulse or modal controllability at \( \infty \) (Verghease et al., 1981; Bender and Laub, 1987).

It should be noted that impulse controllability is not very common for physical systems. If for example a mechanical system should have this property it requires full control of all constraints (Müller, 2000).

---

Example 7: Controllability
This is an example showing the concepts R-controllability and impulse controllability for electric circuit in Example 3. Using the system (27), the R-controllability test becomes

\[
\text{rank } [B_1] = \text{rank } \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -1 \end{bmatrix} = 1 = n_1
\]

Since condition (38) is fulfilled the conclusion is that the system is R-controllable. One way to check if the system is impulse controllable is

\[
\text{rank } \begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} = \text{rank } \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ -2 & -1 & 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 5 = \text{rank } E + n
\]

Another way is to use the SVD coordinate form of the electric circuit, which in this case is equal to the original description (26). The test (40) then becomes

\[
\text{rank } [A_{22} \quad B_2] = \text{rank } [0 \quad -1] = 1 = n - \text{rank } E
\]

and it is of course fulfilled. The system is hence impulse controllable.

8 Stability and stabilizability

8.1 Stability

A system is stable if

\[
||x(t)||_2 \leq \alpha e^{-\beta t}||x(0^-)||_2, \quad t > 0
\]  

(42)

for any \(\alpha > 0, \beta > 0\) when \(u(t) \equiv 0\). Assuming the DAE to be regular and inserting \(u(t) \equiv 0\) in (19a) and (21) gives the solution

\[
\begin{align*}
x_1(t) &= e^{A_1 t} x_1(0^-) \\
x_2(t) &= 0
\end{align*}
\]  

(43)

This implies that the DAE is stable if and only if the slow subsystem is stable. This feels quite natural since some of the generalized states is algebraically connected to the other generalized states.

8.2 Stabilizability

Stabilizability means that it is possible to obtain a stable closed loop system using state feedback

\[
u(t) = -Lx(t) + r(t)
\]  

(44)

Since stability was determined by the slow part only, stabilizability will be the same as to require the unstable finite dynamic modes to be controllable. Assume the system to be on canonical form (19). Stabilizability can then be concluded by decomposing the slow subsystem into Kronecker canonical form (Dai, 1989; van Dooren, 1981)

\[
\dot{x}_1(t) = \begin{bmatrix} A_{c,11} & 0 \\ A_{c,21} & A_{c,22} \end{bmatrix} x_1(t) + \begin{bmatrix} 0 \\ B_{c,2} \end{bmatrix} u(t)
\]  

(45)
Here the pair \((A_{c,2}, B_{c,2})\) is controllable and the upper row of (45) is not affected by the input. The singular system (17) is therefore stabilizable only if all eigenvalues of \(A_{c,11}\) have negative real parts. For methods like LQ and \(H_2\) stabilizability, rather than R-controllability, is necessary (Bender and Laub, 1987; Takaba et al., 1998).

--- Example 8: Stabilizability ---
Consider the electric circuit in Example 3. In this case the slow subsystem has only one state and it is R-controllable, see Example 7. This implies that the system is stabilizable.

9 Observability Concepts

9.1 Observability

Observability of a system is the dual property to controllability, i.e., observability can be shown by showing controllability for the system

\[ E^T \dot{x}(t) = A^T x(t) + C^T y(t) \]

as can be seen in (Bender and Laub, 1987). Therefore, most of the tests are very similar to the tests in Section 7. Observability reflects the ability to reconstruct the whole generalized state vector \(x(t)\) of a singular system

\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) \]

(46)

including the impulse terms, using the measurements and the input signals. The following definition formalizes this (Dai, 1989)

**Definition 9.1 (Observability)** A linear continuous-time DAE (5) is observable if the initial condition \(x(0-)\) may be uniquely determined by \(u(t), y(t)\), for \(0 \leq t < \infty\).

Controllability can be verified by the following theorem

**Theorem 9.1** Consider a singular system on the form (5). Observability can be concluded if and only if

\[ \text{rank} \begin{bmatrix} C_1 \\ C_1 A_1 \\ \vdots \\ C_1 A_1^{n_1-1} \end{bmatrix} = n_1 \text{ and } \text{rank} \begin{bmatrix} C_2 \\ C_2 N \\ \vdots \\ C_2 N^{k-1} \end{bmatrix} = n_2 \]  

(47a)

or

\[ \text{rank} \left[ sE - A \right] = n, \ \forall s \in \mathbb{C}, s \text{ finite and } \text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n \]  

(47b)

**Proof** See (Dai, 1989).
The rank conditions to the left in (47) is for testing if the slow subsystem of (46) is observable, while the rank conditions to the right checks if the fast subsystems is observable. More on observability for systems with rectangular $E$ matrices can be found in (Özçaldiran and Lewis, 1990).

9.2 R-observability

R-observability characterizes the ability to reconstruct the states of the slow subsystem (19) using the input signal, $u(t)$, and the output signal, $y_1(t)$. Hence, for an R-observable system the slow subsystem is observable if

$$\text{rank} \begin{bmatrix} C_1 \\ C_1A_1 \\ \vdots \\ C_1A_1^{n-1} \end{bmatrix} = n_1$$

(48)

Another test for R-observability is the PBH test, see e.g., (Glad and Ljung, 2000). In many cases R-observability will not be a necessary requirement. Instead it will be enough to require the singular system to be detectable. This means that it should be possible to choose a $K$ such that

$$Ex(t) = (A - KC)x(t)$$

(49)

is a stable system. This means that of the finite modes, only those corresponding to positive eigenvalues have to be observable (compare with stabilizability).

9.3 Impulse Observability

Impulse observability guarantees the ability to uniquely determine the impulse terms in $x(t)$ from information of the impulses in the output signal and the jump behavior in the input signal (Dai, 1989). This property will be very important when estimating the state of the system, see Section 10. A system is impulse observable if and only if

$$\text{rank} \begin{bmatrix} E & A \\ 0 & E \\ 0 & C \end{bmatrix} = n + \text{rank} E$$

(50)

If the system is in SVD coordinate form impulse observability can, in similarity to the test for impulse controllability, be concluded if

$$\text{rank} \begin{bmatrix} A_{22} \\ C_2 \end{bmatrix} = n - \text{rank} E$$

(51)

In Section 7.3 it was seen that impulse controllability was equivalent to the existence of a state feedback such that the closed loop system was of index one or lower. In the same manner, impulse observability guarantees that an observer gain can be chosen such that

$$Ex(t) = (A - KC)x(t) + Bu(t) + Ky(t)$$

can be written as a state space system, possibly with direct term.

--- Example 9: Observability ---
Study the electric circuit in Figure 2 again. With the numerical values given in Example 3, the system is (27). Checking for R-observability is done as

\[
\text{rank } \begin{bmatrix} C_1 \end{bmatrix} = \text{rank } \begin{bmatrix} -\frac{1}{\sqrt{2}} \end{bmatrix} = 1 = n_1
\]

Criterion (48) is fulfilled and the system is R-observable. Using the test for impulse observability based on the SVD coordinate form (51) we obtain

\[
\text{rank } \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \neq n - \text{rank } E
\]

As can be seen, this electric circuit is not impulse controllable with this output signal.

10 State Estimation

10.1 Observer

For state feedback \( u(t) = -Lx(t) + r \) it is necessary that all generalized states are available for feedback. Often in physical systems this is not the case, and some of the states have to be estimated. The estimation is done by an observer using the input signal \( u(t) \) and the output signal \( y(t) \). A singular observer for

\[
\begin{align*}
E\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{align*}
\]  

(52)

can be written as

\[
E\hat{x}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))
\]  

(53)

where \( \hat{x}(t) \in \mathbb{R}^n \) is the estimate of the generalized states and \( K \) is the observer gain (Dai, 1989). If the observer should be useful, the estimated generalized state has to converge to the true generalized state, i.e.,

\[
\lim_{t \to \infty} (\hat{x}(t) - x(t)) = 0, \ \forall x(0-), \hat{x}(0-)
\]  

(54)

The dynamics of the estimation error, \( \tilde{x}(t) = x(t) - \hat{x}(t) \), are described by

\[
E\tilde{x}(t) = (A - KC)\tilde{x}(t), \quad \tilde{x}(0-) = x(0-) - \hat{x}(0-)
\]  

(55)

If the singular system (52) is detectable, the observer gain, \( K \), can be chosen such that the eigenvalues of \( A - KC \) have negative real part. This will result in \( \tilde{x}(t) \to 0, \ t \to \infty \) (compare to stability).

The problem with (53) is that derivatives of the inputs, in this case both \( u(t) \) and \( y(t) \), may appear in the state estimation (compare with (21)). These derivatives are often not available. Furthermore, these derivatives make the estimation sensitive to high frequency noise. It is therefore desirable to estimate the states using an observer on the form

\[
\begin{align*}
\dot{x}_e(t) &= Ax_e(t) + Bu(t) + Ky(t) \\
\dot{x}(t) &= Fx_e(t) + Hu(t) + Fy(t)
\end{align*}
\]  

(56a)

(56b)
where the expressions for \( A_c, B_c, F_c, H, F \) and \( K \) can be found in (Dai, 1989). This is possible if the system (52) is detectable and impulse observable. It might seem odd to be able to estimate a possibly impulsive solution of a singular system using a state space system that can not generate impulses. The idea is to let any impulses in the solution, \( x(t) \), affect the state estimate, \( \hat{x}(t) \), through the output signal, \( y(t) \), in (56b). From Section 9.3 it is known that if the system is impulse observable all impulsive behavior in the generalized states is visible through the output signal. The drawback with (56) is that the estimation also in this case is noise sensitive due to the direct involvement of the \( u(t) \) and \( y(t) \) in the measurement equation (Dai, 1989).

10.2 Kalman Filtering

For state space systems the observer usually chosen is the Kalman filter, see e.g., (Glad and Ljung, 2000; Kailath et al., 2000). Then it is assumed that a process noise, \( w(t) \), and a measurement noise, \( v(t) \), affect the state space system as

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + N_ww(t) \\
y(t) &= Cx(t) + v(t)
\end{align*}
\] (57)

The process noise and the measurement noise are also assumed to be stationary and to have zero-mean, i.e.,

\[
\left\langle \begin{bmatrix} w(t) \\ v(t) \end{bmatrix}, \begin{bmatrix} w(t + \tau) \\ v(t + \tau) \end{bmatrix} \right\rangle = \begin{bmatrix} Q\delta(\tau) & S\delta(\tau) & 0 \\ S^*\delta(\tau) & R\delta(\tau) & 0 \end{bmatrix}
\] (58)

where

\[
\langle y(t), s(\tau) \rangle = Ey(t)s(\tau)^T, \ -\infty < t, \tau < \infty
\]

The Kalman gain gives the optimal balance between how the process noise and measurement affect the state estimate. A small process noise compared to the measurement noise, i.e., a small ratio \( Q \), gives a small \( K \) and vice versa. Generalization of this result to descriptor systems is not trivial. First noises fulfilling (58) are introduced in the DAE (5). The descriptor system will then look like

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + N_ww(t) \\
y(t) &= Cx(t) + v(t)
\end{align*}
\] (59)

Earlier it has been seen that derivatives of the inputs, in this case both \( u(t) \) and \( w(t) \), may occur. The problem is that the derivative of white noise is not well defined, see (Åström, 1970). However, some methods to handle this problem exist. The first method is to lowpass filter the process noise. The derivative of the filtered noise then exist and the generalized states can be estimated using an ordinary Kalman filter(Kalman and Bucy, 1961). Another method, described in (Schön et al., 2003), requires \( N_w \) to be zero in certain positions so that no derivatives of \( w(t) \) occur in the solution, \( x(t) \). This is equivalent to require the covariance matrix \( Q \) to have a certain structure since

\[
\langle N_ww(t), N_ww(t + \tau) \rangle = N_w\langle w(t), w(t + \tau) \rangle N_w^T = N_wQN_w^T\delta(\tau)
\]
Equation (59) can then be transformed to canonical form (19) and an ordinary Kalman filter can be used. However, note that the transformation may result in a direct term of \( w(t) \) in the measurement equation. The new noise in the measurement equation \([w(t) \ v(t)]^T\) is then correlated with the process noise, \( w(t) \). For discrete time DAEs many references exist, e.g., (Nikoukhah et al., 1992).

11 Feedback

In this report, the focus will be on generalized state feedback

\[
u(t) = -Lx(t) + r(t) \quad (60)\]

which gives a closed loop system

\[
\dot{x}(t) = (A - BL)x(t) + Br(t) \\
y(t) = Cx(t) \quad (61)
\]

Other controllers investigated for systems on descriptor form are, e.g., PID-controllers (Rao et al., 2003).

Three important objectives with the state feedback will be to

- make the closed loop stable
- remove impulse behavior in the output response
- guarantee regularity of the closed loop system

Stability is, as mentioned in Section 8, only dependent of the finite modes of the closed loop system. Therefore stabilizability will be necessary and sufficient to fulfill the first objective.

The second objective, i.e., elimination of impulses in the solution is, as can be seen in (21), equivalent to \( N = 0 \). That is, the closed loop system is only allowed to have an index not exceeding 1. Two equivalent formulations of this requirement is that the closed loop system should have

- no infinite dynamic modes (see Section 2)
- the number of finite modes equal to \( \text{rank } E \)

Since the finite modes, or the finite poles, are given by the characteristic equation of (61) the requirements can also be formulated as

\[
\deg(\det(sE - A + BL)) = \deg(\det(sI - A_1)) = \text{rank } E \quad (62)
\]

If (62) is fulfilled, regularity of the closed loop is also guaranteed.

If the system is on SVD coordinate form (22), Theorem 5.1 gives that the requirements 2 and 3 is met if \( A_{22} - B_2L_2 \) is nonsingular (Bender and Laub, 1987). This can be seen in the following expression

\[
\det \{(sE - A + BL)\} = \det \left\{ \begin{bmatrix} s\Sigma - H_{11} & -H_{12} \\ -H_{21} & -H_{22} \end{bmatrix} \right\} = \det \{-H_{22}\} \cdot \det \left\{ s\Sigma - H_{11} - H_{12}H_{22}^{-1}H_{21} \right\} \quad (63)
\]
where $\tilde{H}_{ij} = A_{ij} - B_i L_j$. If $H_{22}$ is nonsingular, its determinant is nonzero. The latter part of (63) is a polynomial of degree rank $E$ since $\Sigma$ is a nonsingular $r \times r$ matrix. Hence the closed loop system will have $r$ finite and no infinite poles. It is also obvious that $\det \{sE - A + BK\} \neq 0$ and the closed loop system is thus regular. The last requirement, i.e., no infinite poles or $N = 0$ can also be seen from the following calculation. The closed loop system on SVD coordinate form is

$$\begin{bmatrix} \Sigma & 0 \\ 0 & H_{22} \end{bmatrix} \dot{x}(t) = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t)$$

(64)

Apply the Shuffle algorithm (Luenberger, 1978) and the result is

$$\begin{bmatrix} \Sigma & 0 \\ 0 & H_{22} \end{bmatrix} \dot{x}(t) = \begin{bmatrix} H_{11} & H_{12} \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \dot{u}(t)$$

(65)

The new $E$ matrix is nonsingular if and only if $H_{22}$ has full rank, and the original DAE has an index equal to one. Choosing $L_2$ such that $A_{22} - B_2 L_2$ is nonsingular is possible only if $[A_{22} \ B_2]$ is nonsingular, or with other words the system is impulse controllable. This is formalized in the following theorem.

**Theorem 11.1 (Pole placement)** Given a linear singular system

$$E \dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

(66)

Then a state feedback

$$u(t) = -L x(t) + r(t)$$

(67)

exist such that the closed loop system (61) fulfill the objectives above if and only if the system (66) is stabilizable and impulse controllable.

**Proof** See Corollary 3-3.1 in (Dai, 1989).

Impulse controllability also guarantees that it is possible to place the infinite poles, or infinite dynamic modes, arbitrarily in the complex plane (Bender and Laub, 1987). However, if the finite poles should be possible to place arbitrarily R-controllability is needed (compare with pole placement for state space systems).

Numerical aspects and methods for pole placement for DAEs are described in (Varga, 2000; Fletcher et al., 1986) and the references therein. An example of pole placement for a DAE system is 10.

---

**Example 10: Pole placement**

Consider the electric circuit in Figure 2. This system is, as can be seen in Example 7, both R-controllable and impulse controllable. The characteristic equation for the closed loop system (62) using $L = [l_1 \ l_2 \ l_3]$ becomes

$$\det(sE - A + BL) = -6l_3 s^2 + (-2 - 7l_3 - 2l_1)s - 1 - 3l_3 - l_2 - l_1 = 0$$

(68)

The degree of the characteristic equation is two, and placing the corresponding two poles in $-2$ give the state feedback

$$L = \begin{bmatrix} -\frac{17}{12} & -\frac{1}{12} & \frac{1}{6} \end{bmatrix}$$
In (68) it was possible to place the two poles arbitrarily and the degree of the characteristic equation is equal to \( \text{rank } E \) and \( r = n_1 \). The system will thus have no infinite poles. However, the system will still have \( n - r \) infinite nondynamic modes.

### 12 LQ

In this section the linear quadratic controller for descriptor systems will be derived. In the Section 12.1 the general case will be handled, while Section 12.2 and Section 12.3 handle more specific cases.

#### 12.1 The General Case

In Section 11 the criteria for eliminating the impulsive behavior in the solution and making the closed loop stable using state feedback was stated. In this section the controller that minimizes

\[
J(Ex_0, u, t_f) = \frac{1}{2} x^T(t_f) P_f E x(t_f) + \frac{1}{2} \int_0^{t_f} \begin{bmatrix} x^T(t) & u^T(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt
\]

where \( R > 0 \) and \( \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \succeq 0 \) under the constraint that the system is described by (5) and the initial condition is \( E x(0^-) = E x_0 \), is sought. The controller must also fulfill the objectives stated in Section 11.

Equation (69) can be interpreted as a minimization of the energy in \( x(t) \) and \( u(t) \). To find a minimum of (69), the integral has to exist. It is assumed that the integral in (69) is defined as in (Cobb, 1983), i.e.,

\[
\int_0^{t_f} ||\delta(t)v||_2 dt < \infty \quad \text{but} \quad \int_0^{t_f} ||\delta(t)v||_2^2 dt = \infty
\]

where \( \delta(t)v \) is the impulse function along \( v \). Hence, \( x(t) \) and \( u(t) \) have to be impulse free. In (Cobb, 1983), this was done by introducing a preliminary feedback placing the infinite poles in finite positions. Thereafter, a number of transformations of the descriptor space were made in order to account both for the preliminary feedback but also for the nondynamic part of the DAE.

The method described in this report is originally from (Bender and Laub, 1987). Their approach is to minimize (69) using calculus of variations directly without any preliminary feedback. To solve the finite-horizon problem or to compute the Riccati equation solution \( P(t) \) a transformation to isolate the dynamic portion, i.e., the part of the descriptor space that is the orthogonal complement to part contained in the kernel of \( E \), is required. In the infinite-horizon problem the solution of the algebraic Riccati equation can be solved by calculating a generalized eigenvalue problem as will be seen. The weight matrix

\[
\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}
\]

(70)
is handled in the problem by including a output signal

\[ y(t) = Cx(t) + Du(t) \]  

(71)

where the matrices \( C \) and \( D \) is determined by a Cholesky factorization

\[
\begin{bmatrix}
Q & S \\
S^T & R
\end{bmatrix} = \begin{bmatrix} C^T & D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \quad R = D^T D
\]  

(72)

The factorization is possible since (70) is positive semidefinite. The concatenated system is then

\[
\begin{align*}
E \dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]  

(73)

and will be assumed to be expressed on SVD coordinate form. This assumption is not necessary in the infinite horizon case, i.e., when \( t_f = \infty \). The weight matrices will also be assumed to be on SVD coordinate form

\[
VQV^T = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \quad VS = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}
\]  

(74)

The optimal control signal \( u(t) \) is derived using the first order necessary conditions (Bender and Laub, 1987), (Jonckheere, 1987). The optimal trajectory must fulfill

\[
\begin{bmatrix}
E & 0 & 0 \\
0 & E^T & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\dot{x}(t) \\
\dot{\gamma}(t) \\
\dot{u}(t)
\end{bmatrix} = \begin{bmatrix}
A & 0 & B \\
-Q & -A^T & -S \\
S^T & B^T & R
\end{bmatrix} \begin{bmatrix}
x(t) \\
\gamma(t) \\
u(t)
\end{bmatrix}
\]  

(75)

with the boundary conditions

\[
Ex(0-) = Ex_0, \quad E^T \gamma(t_f) = E^T P_f Ex(t_f)
\]  

(76)

where \( x(t) \) is the generalized states, \( \gamma(t) \) is the costates and \( u(t) \) is the control signal. In order to get a regular and impulse free solution to (75), two assumptions are done:

**Assumption 1 :**

1. The system (73) is impulse controllable, or equivalently \([A_{22} \ B_2]\) has full row rank.
2. \[
\begin{bmatrix} Q_{22} & S_2 \\ S_2^T & R \end{bmatrix} > 0
\]

The latter part is equivalent to that \([C_2 \ D]\) has full column rank, for which impulse observability is necessary but not sufficient. Define the matrices \( \bar{R}, \bar{S} \) and \( \bar{B} \) as

\[
\bar{R} = \begin{bmatrix} 0 & A_{22} & B_2 \\ A_{22}^T & Q_{22} & S_2 \\ B_2^T & S_2^T & R \end{bmatrix}, \quad \bar{S} = [A_{21}^T \ Q_{12} \ S_1], \quad \bar{B} = [0 \ A_{12} \ B_1]
\]  

(77)
and an extended control vector as

$$\bar{u}(t) = \begin{bmatrix} \gamma_2(t) \\ x_2(t) \\ u(t) \end{bmatrix}$$  \hspace{1cm} (78)$$

If Assumption 1 is fulfilled, the matrix $\bar{R}$ is invertible and the extended control signal can then be expressed in the variables $x_1(t)$ and $\gamma_1(t)$ as

$$\bar{u}(t) = -\bar{R}^{-1} \begin{bmatrix} \bar{S} & \bar{B} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \gamma_1(t) \end{bmatrix}$$  \hspace{1cm} (79)$$

Inserting (79) into (75) gives the expression

$$\begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{\gamma}_1(t) \end{bmatrix} = \begin{bmatrix} A_{11} - \bar{B}\bar{R}^{-1}\bar{S} & -\bar{B}\bar{R}^{-1}\bar{B} \\ -(Q_{11} - S\bar{R}^{-1}\bar{S}) & -(A_{11} - S\bar{R}^{-1}\bar{B}) \end{bmatrix} \begin{bmatrix} x_1(t) \\ \gamma_1(t) \end{bmatrix}$$  \hspace{1cm} (80a)$$

The differential equation (80a) together with the initial conditions (76), expressed in SVD coordinate form,

$$\begin{bmatrix} x_{1(0-)} \\ \gamma_{1(t_f)} \end{bmatrix} = \begin{bmatrix} x_{10} \\ P_{f(t)\Sigma x_{1(t_f)}(t_f)} \end{bmatrix}$$  \hspace{1cm} (80b)$$

constitutes a two-point boundary value problem of linear-quadratic optimal control with $x_1(t)$ and $\gamma_1(t)$ as state and costate instead of $x(t)$ and $\gamma(t)$. This boundary value problem can be solved by computing the corresponding state transition matrix, see (Bender and Laub, 1987). The criterion (69) will under Assumption 1 exist and be finite for the case when $t < \infty$. This depends on that $\bar{R}$ is nonsingular. If the horizon is infinite, $t_f = \infty$, stabilizability of (73) is also required. In order to get a feedback solution to the optimal control problem, it is necessary to derive a linear relationship between the control signal, $u(t)$, and the generalized state, $x(t)$, for a given initial value. This can be done by computing a matrix whose columns span the entire space in which the optimal descriptor-codescriptor-control trajectory

$$\begin{bmatrix} x(t) \\ \gamma(t) \\ u(t) \end{bmatrix}$$

can lie. Denote this $(2n + m) \times r$ matrix by

$$\begin{bmatrix} X(t) \\ \Gamma(t) \\ U(t) \end{bmatrix} = \begin{bmatrix} X_1(t) \\ X_2(t) \\ \Gamma_1(t) \\ \Gamma_2(t) \\ U(t) \end{bmatrix}$$

in which $X_1(t)$, $\Gamma_1(t)$ can be computed from the state transition matrix and $X_2(t)$, $\Gamma_2(t)$, $U(t)$ are computed using (78) together with (79) replacing lowercase letters with uppercase letters. The control law, $L(t)$, is then a solution to

$$U(t) = L(t)X(t)$$  \hspace{1cm} (81)$$

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The matrix $L(t)$ has $m(n-r)$ degrees of freedom. This freedom will in the infinite-horizon problem be used to guarantee the regularity of the closed-loop pencil, $sE - A + BL$. The optimal cost from (69) using $L(t)$ can be expressed as

$$J = \frac{1}{2}x^T \Sigma P(0) \Sigma x$$

(82)

The matrix $P(t)$ is the solution to the Riccati differential equation

$$-\Sigma \frac{dP}{dt} \Sigma = \Sigma P(t) A_{11} + A_{11}^T P(t) \Sigma + Q_{11} - (\Sigma P(t) \bar{B} + \bar{S}) \bar{R}^{-1} (\Sigma P(t) \bar{B} + \bar{S})^T$$

(83)

where $P(t_f) = P_f$. This Riccati equation can also be written in other ways, see (Bender and Laub, 1987).

12.2 The Infinite-Horizon Problem

Until now, mostly the finite-horizon problem has been studied. The thoughts in the infinite-horizon problem are however very similar. It is well known that for state space systems the optimal feedback gain and the unique positive semidefinite or positive definite solution of the associated Riccati equation of the infinite horizon LQ problem, i.e., $t_f = \infty$ and $P_f = 0$, is related to the eigenstructure of a certain Hamiltonian matrix. In this case the Riccati equation is

$$0 = \Sigma P(t) A_{11} + A_{11}^T P(t) \Sigma + Q_{11} - (\Sigma P(t) \bar{B} + \bar{S}) \bar{R}^{-1} (\Sigma P(t) \bar{B} + \bar{S})^T$$

(84)

i.e., equation (83) with $\frac{dP}{dt} = 0$ and the corresponding Hamiltonian gives the generalized eigenvalue problem

$$\begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix} \begin{bmatrix} X_1 \\ \Gamma_1 \end{bmatrix} = \begin{bmatrix} A_{11} - \bar{B} \bar{R}^{-1} \bar{S}^T & -\bar{B} \bar{R}^{-1} \bar{B}^T \\ -\bar{Q}_{11} - S \bar{R}^{-1} S^T & -(A_{11} - \bar{S} \bar{R}^{-1} \bar{B}^T) \end{bmatrix} \begin{bmatrix} X_1 \\ \Gamma_1 \end{bmatrix}$$

(85)

The equation (85) could easily be rewritten as an ordinary eigenvalue problem since the matrix $\begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix}$ is nonsingular. In the finite-horizon case some assumptions had to be made, i.e., the assumptions in Assumption 1. However in the infinite-horizon problem these assumptions have to be strengthened in order to ensure for example convergence of the integral (69).

Assumption 2 :

1. the system (73) fulfills Assumption 1, is stabilizable and detectable.

2. the matrix $\begin{bmatrix} C_1 & C_2 & D \end{bmatrix}$ is of the form

$$\begin{bmatrix} C_1 & C_2 & D \end{bmatrix} = \begin{bmatrix} 0 & C_2 & D \\ C_1 & 0 & 0 \end{bmatrix}$$

(86)

Assumption 2 guarantees different properties. For instance the first part of the assumption guarantees the existence of a unique, impulse-free solution of the necessary condition (75) and it also assures that for $\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$ along this solution the criterion (69) exist and is finite. The stabilizability together with detectability gives that

$$\frac{dP}{dt} \rightarrow 0, t \rightarrow \infty$$
The second part of the Assumption 2 yields that $C^T D = 0$. This part is in the state space case equal to assuming $C^T D = S = 0$.

Further properties that the eigenvalue problem (85) have under Assumption 2 and under the assumption of distinct eigenvalues are:

1. if $\lambda$ is an eigenvalue, then $-\lambda$ is also an eigenvalue. No eigenvalues with zero real part exist and the total number of eigenvalues is $2r$.

2. the $r$ eigenvalues with negative real part correspond to the steady-state optimal regulator, since this regulator will stabilize the system.

3. let $\Lambda$ be an $r \times r$ diagonal matrix satisfying (85) and whose elements have negative real part. Then the unique positive definite (semidefinite) symmetric solution of the algebraic Riccati equation (84) is $P = \Gamma X^{-1} \Sigma^{-2}$. Furthermore, since the finite modes are assumed at least stabilizable and detectable, $X_1$ is guaranteed to be nonsingular.

The optimal feedback gain is then given as in the finite horizon case, i.e., by calculating the transition matrix corresponding to (80) and using (78) together with (79). However, it is not necessary to form the Hamiltonian in (85). Another equivalent method is to solve the generalized eigenvalue problem

$$\begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ \Gamma \\ U \end{bmatrix} = \begin{bmatrix} A & 0 & B \\ -Q & -A^T & -S \\ ST & BT & R \end{bmatrix} \begin{bmatrix} X \\ \Gamma \\ U \end{bmatrix} \tag{87}$$

where $[X \Gamma U]^T$ is partitioned as earlier. Then all variables needed to compute the optimal feedback, i.e., $X$ and $U$ are obtained at once. The optimal stabilizing control gain is

$$U_s = LX_s$$

or equivalently

$$L = U_s X_s^\dagger + Y (I - X_s X_s^\dagger) \tag{88}$$

where $U_s$ and $X_s$ are the eigenvectors corresponding to the $r$ eigenvalues with negative real part. The freedom in the solution, i.e., the free parameter $Y$, is used to chose a feedback gain making the closed loop pencil $sE - A + BL$ regular. This is always possible because the system is assumed to be impulse controllable. The closed loop will then be regular, have finite poles and be stable. In (Bender and Laub, 1987) the following conjecture is presented.

**Conjecture 1** If the minimum norm feedback gain, i.e., if the free parameter in (88) is chosen as

$$Y = \arg\min_Y ||L||_2 = \arg\min_Y ||U_s X_s^\dagger + Y (I - X_s X_s^\dagger)||_2$$

the closed loop pencil is regular and the closed loop dynamic modes are finite and stable.

Below an example of LQ control can be seen. This example also serves as a description of the algorithm.

---

**Example 11: LQ**
In this example the LQ controller for the electric circuit described in Example 5 will be derived. The system matrices for the system are given as

\[
\begin{bmatrix}
L & 0 & 0 \\
0 & C & 0 \\
0 & 0 & 0
\end{bmatrix}
\dot{z}(t) =
\begin{bmatrix}
-R_1 & -1 & 1 \\
1 & \frac{1}{R_2} & 0 \\
1 & 0 & 0
\end{bmatrix}
z(t) +
\begin{bmatrix}
0 \\
0 \\
-1
\end{bmatrix}
u(t) \quad (89a)
\]

\[y(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} z(t) \quad (89b)\]

The algorithm can be described in a number of steps.

   - Impulse controllability was for this system concluded in Example 7 so that requirement is fulfilled.
   - The matrix \( \begin{bmatrix} Q_{22} & S_2 \\ S_2^T & R \end{bmatrix} \) must be positive definite. By choosing the penalty matrices as

   \[
   Q = \begin{bmatrix} 1 & 0 & 0 \\
   0 & 1 & 0 \\
   0 & 0 & 1 \end{bmatrix}, \quad R = 1, \quad S = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}
   \]

   and noticing that the system matrices for the electric circuit (89) are on SVD coordinate form the matrix becomes

   \[
   \begin{bmatrix} Q_{22} & S_2 \\ S_2^T & R \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
   \]

   which is a positive definite matrix.

   Assumption 1 is thus fulfilled.

2. Verify Assumption 2. One part of Assumption 2 is Assumption 1. This part is already tested and therefore only the extra conditions in Assumption 2 need to be checked.
   - Stabilizability of the system was concluded in Example 8.
   - Detectability of the system consisting of the dynamics (89a) and an output given by

   \[
y(t)^T y(t) = (C x(t) + D u(t))^T (C x(t) + D u(t))
   = [x^T(t) \quad u^T(t)] \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} [x(t) \quad u(t)]
   \]

   The matrices \( C \) and \( D \) are obtained by a Cholesky factorization of the matrix \( \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \) and the result is

   \[
   C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
   \]
Since detectability is dual to stabilizability, the test is to verify if
\[ E^T \dot{x}(t) = A^T x(t) + C^T y(t) \] (91)
is stabilizable. The slow subsystem of (91) is for this example given by
\[ \dot{w}_1(t) = -\frac{1}{2}w_1(t) + \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} y(t) \] (92)
which obviously is stable since the A matrix is negative. The electric circuit (89) is therefore both stabilizable and detectable.

- To fulfill Assumption 2 it is also necessary that \( C^T [C_2 \ D] = 0 \). For this example this expression becomes
\[
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]
and the requirements is thus satisfied. Note that the matrix \([C_1 \ C_2 \ D] \) does not have the form given by (78), but the zeros in \( C_1, C_2 \) and \( D \) may change place and still fulfill \( C^T [C_2 \ D] = 0 \). The conclusion is that Assumption 2 is fulfilled.

3. Calculate the optimal feedback by first solving the generalized eigenvalue problem (87). The stable part of the solution is
\[
\begin{bmatrix} X_s \\ \Gamma_s \\ U_s \end{bmatrix} = \begin{bmatrix} 0.626 - 0.001i & 0.626 + 0.001i \\ -0.297 - 0.703i & -0.297 + 0.703i \\ -0.282 + 0.007i & -0.2821 - 0.0068i \\ 0.626 - 0.001i & 0.626 + 0.001i \end{bmatrix}
\]
\[
\Lambda_s = \begin{bmatrix} -0.659 + 0.378i & 0 \\ 0 & -0.659 - 0.378i \end{bmatrix}
\]
The optimal, stabilizing feedback gain then is obtained from (88) as
\[ L = \begin{bmatrix} -0.829 & 0.003 & 0.377 \end{bmatrix} \]
with \( Y = [0 \ 0 \ 0]^T \). The poles of the closed loop system is given by the diagonal elements in \( \Lambda \). In Figure 3 the free response of the system is shown. This means that the closed loop system is released from the the initial value, \( x_0 \),
\[ x_0 = \begin{bmatrix} 1 & 0.5 & 0.5 \end{bmatrix} \]
which fulfills the implicit algebraic constraints, with the reference values equal to zero.

Other, more recent, papers on LQ control in the general case are (Zhaolin et al., 1988; Zhu et al., 2002).
12.3 LQ for Causal Systems

For causal systems, i.e., when the output contains no derivatives of the input signal, and the system has consistent initial conditions, the procedure described in Section 12.2 can be simplified. The idea presented in (Müller, 2000) is to use the canonical form of the system. It will in the causal case be given by

\[
\dot{x}_1(t) = A_1 x_1(t) + B_1 u(t) \quad (93a)
\]

\[
x_2(t) = B_2 u(t) \quad (93b)
\]

\[
y(t) = C_1 x_1(t) + C_2 x_2(t) = C_1 x_1(t) + D u(t) \quad (93c)
\]

where \(D = C_2 B_2\) and

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Tx \quad (94)
\]

This is an state space system, which makes it possible to use the ordinary LQ method. However, it is preferable to use a cost function similar to (69), i.e., a cost function based on the generalized state variables \(x(t)\). The reason for this is that the generalized states most often have physical interpretations which is used when choosing appropriate penalties for the variables. In this section we assume the cost function be given as

\[
J = \frac{1}{2} \int_0^\infty \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix}^T \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} & \tilde{S}_1 \\ \tilde{Q}_{12}^T & \tilde{Q}_{22} & \tilde{S}_2 \\ \tilde{S}_1^T & \tilde{S}_2^T & \tilde{R} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} dt \quad (95)
\]

where the penalty matrices \(\tilde{Q}_{ii}, \tilde{S}_i\) and \(\tilde{R}\) have appropriate sizes and are given by

\[
\begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{12}^T & \tilde{Q}_{22} \end{bmatrix} = T^{-T} QT^{-1}, \quad \begin{bmatrix} \tilde{S}_1 \\ \tilde{S}_2 \end{bmatrix} = ST^{-1}, \quad \tilde{R} = R \quad (96)
\]

Using the fact that

\[w_2 = B_2 u\]
the cost function (95) can be formulated as

$$J = \frac{1}{2} \int_0^\infty \begin{bmatrix} x_1 \\ u \end{bmatrix}^T \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{S}_1 & \tilde{S}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ u \end{bmatrix} dt$$

The cost function (97) together with the slow part of the DAE, i.e. (93a) constitutes an ordinary LQ problem. The conditions for which the problem has a unique solution can be found in almost any textbook on control theory, for example (Glad and Ljung, 2000). Numerically this optimal control problem can be solved in MATLAB by using the function lqr.

Note that the obtained feedback gain, \( L \), corresponds to the states in the canonical form. However, we want to have a feedback gain for the physical variables, i.e., the original generalized state variables \( x \). Using (94) we obtain

$$u = -L \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -LT x$$

**References**


This report is a brief survey over different concepts and methods concerning singular systems. Singular system is also called differential algebraic equations (DAE) since a DAE model may involve both differential and algebraic equations. There is an growing interest in DAE systems much depending on the increased usage of object oriented modeling languages, like Modelica, in the modeling of dynamical systems. These programs often give DAE models as result. Another advantage of DAE models is that they sometimes can keep the natural structure of the dynamical model.