Connections between L_2 -Model Reduction and Balanced Truncation

Wolfgang Scherrer* and Fredrik Tjärnström**

*Institut für Ökonometrie,
Operations Research und Systemtheorie,
Technische Universität Wien,
Argentinierstr. 8/119, A-1040, Vienna, Austria,
WWW: <http://www.eos.tuwien.ac.at/Oeko/>
E-mail: W.Scherrer@tuwien.ac.at

** Division of Automatic Control
Department of Electrical Engineering
Linköpings universitet, SE-581 83 Linköping, Sweden
WWW: <http://www.control.isy.liu.se>
E-mail: fredrikt@isy.liu.se

May 9, 2003

Report no.: LiTH-ISY-R-2521

Submitted to the 13th Symposium on System Identification in
Rotterdam 2003

Technical reports from the Control & Communication group in Linköping are
available at <http://www.control.isy.liu.se/publications>.

Connections between L_2 -Model Reduction and Balanced Truncation

Wolfgang Scherrer* and Fredrik Tjärnström**

*Institut für Ökonometrie,
Operations Research und Systemtheorie,
Technische Universität Wien,
Argentinierstr. 8/119, A-1040, Vienna, Austria,
E-mail: W.Scherrer@tuwien.ac.at

**Automatic Control,
Department of Electrical Engineering,
Linköpings universitet, SE-58183 Linköping, Sweden,
E-mail: fredrikt@isy.liu.se

May 9, 2003

Abstract

In this paper we investigate the connection between model reduction by balanced truncation and by L_2 reduction. We show that locally, i.e., close to the set of lower order systems, balanced truncation and (un-weighted) L_2 model reduction produce models that are almost identical. This implies that high order estimated models can be reduced by either L_2 reduction or balanced truncation, both methods giving a low order model with the same asymptotic variance, if the true data generating model is in the class of low order models.

Keywords: Model reduction, Balanced truncation, L_2 reduction, System identification

1 Introduction

Model reduction has been a widely studied problem since the 70's. The purpose of model reduction is to find a good low order approximation of a high order model. The reason for this is typically that the simpler low order model is more attractive for simulation and controller design. The L_2 reduction problem has been studied by several authors, e.g., [10, 7, 1]. All L_2 reduction algorithms generally suffer from the need of numerical optimization routines which cannot guarantee convergence. On the other hand model reduction by balanced truncation [4] has achieved a widespread use due to its ease of implementation with good numerical properties. See also [5, 2]

In this paper we present some results connecting L_2 model reduction and balanced truncation. We show that for a sequence of n -th order models converging to an m -th order model, the L_2 reduced model and the balanced truncated

model are “close” to each other. This paper takes a similar approach to that in [6], where the performance of minimum phase balanced truncation is analyzed with respect to the Kullback-Leibler distance rate.

The closeness of the two reduction methods can be exemplified from a system identification perspective where both reduced models will have asymptotically the same covariance. This is in accordance with simulation results in [8].

The outline of the paper is as follows. In Section 2 the problem formulation and the necessary notation is given. Section 3 presents the L_2 model reduction problem and a linear approximation of the L_2 problem for models “close” to the set of lower order models. By comparing the balanced truncated model in Section 4 with the linear approximation of the L_2 reduced model, it follows that balanced truncated models are “close” to optimal with respect to the L_2 norm, if the model is “close” to the set of lower order models. In Section 5 we apply these results to the system identification framework and in Section 6 we give some conclusions.

2 Problem formulation and notation

A stable state space model with n states, q inputs and p outputs is a quadruple $\theta = \{A, B, C, D\}$, where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times q}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times q}$ and A is asymptotically stable. The set of all such stable state space models will be denoted by $\mathbb{S}_{n,p,q} \subseteq \mathbb{R}^{(n+p)(n+q)}$. For $T \in \mathbb{R}^{n \times n}$ nonsingular and $\theta = \{A, B, C, D\}$ we will use $T \cdot \theta = \{TAT^{-1}, TB, CT^{-1}, D\}$ to denote the state space model obtained by the state space transformation T .

A model $\theta = \{A, B, C, D\} \in \mathbb{S}_{n,p,q}$ is a realization of the stable $p \times q$ transfer function $k(z) = C(z^{-1}I - A)^{-1}B + D$, where z denotes the backward shift. Let $\mathbb{M}_{n,p,q}$ ($\mathbb{M}_{n,p,q}$) denote the set of all stable $p \times q$ transfer functions of McMillan degree n ($\leq n$). Furthermore $\Pi : \mathbb{S}_{n,p,q} \rightarrow \mathbb{M}_{n,p,q}$ denotes the mapping attaching a transfer function $k = \Pi(\theta)$ to a state space model θ . Of course $\Pi(T \cdot \theta) = \Pi(\theta)$.

We will consider state space models $\theta = \{A, B, C, D\}$ of order n that are close to an m -th order model ($m < n$). These models are of the form

$$\begin{aligned} A &= A_0 + \epsilon A_1 = \begin{pmatrix} A_{0,11} + \epsilon A_{1,11} & \epsilon A_{1,12} \\ A_{0,21} + \epsilon A_{1,21} & A_{0,22} + \epsilon A_{1,22} \end{pmatrix} \\ B &= B_0 + \epsilon B_1 = \begin{pmatrix} B_{0,1} + \epsilon B_{1,1} \\ B_{0,2} + \epsilon B_{1,2} \end{pmatrix} \\ C &= C_0 + \epsilon C_1 = \begin{pmatrix} C_{0,1} + \epsilon C_{1,1} & \epsilon C_{1,2} \end{pmatrix} \\ D &= D_0 + \epsilon D_1 \end{aligned} \tag{1}$$

where all the matrices are partitioned conformingly, e.g., $A_{0,11} \in \mathbb{R}^{m \times m}$, $A_{0,21} \in \mathbb{R}^{(n-m) \times m}$. Note that by construction, the limiting model $\theta_0 = \{A_0, B_0, C_0, D_0\}$ (as $\epsilon \rightarrow 0$) is not minimal, and the truncated model $\{A_{0,11} + \epsilon A_{1,11}, B_{0,1} + \epsilon B_{1,1}, C_{0,1} + \epsilon C_{1,1}, D_0 + \epsilon D_1\}$ is a “good” approximation of the full order model for small ϵ . We impose the assumption that $A_{0,11}$ and $A_{0,22}$ are asymptotically stable matrices, and that the limiting transfer function $k_0 = \Pi(\theta_0)$ has McMillan degree m , i.e., $k_0 \in \mathbb{M}_{m,p,q}$.

The L_2 norm of the transfer function $k(z)$ is defined as

$$\|k\|_2 = \sqrt{\mathbb{E} \operatorname{tr} y(t)y(t)'} \tag{2}$$

where $y(t) = k(z)u(t)$ and $(u(t)|t \in \mathbb{Z})$ is a q -dimensional white noise process with mean zero and a unit variance.

In the sections to come we will compare the L_2 reduced model $\bar{\theta}_{L_2} = \{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ and the balanced truncated model $\tilde{\theta}_{b.t.} = \{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\}$ and show that there exist a state transformation $T(\epsilon)$ such that

$$\bar{\theta}_{L_2} = T(\epsilon) \cdot \tilde{\theta}_{b.t.} + O(\epsilon^2). \quad (3)$$

3 L_2 model reduction

The L_2 -model reduction problem may be stated as follows: Given $k \in \mathbb{M}_{n,p,q}$, find a transfer function $\bar{k} \in \mathbb{M}_{m,p,q}$, $m < n$ such that

$$J(\bar{k}, k) = \frac{1}{2} \|\bar{k} - k\|_2^2 \quad (4)$$

is minimal.

Let $\bar{\theta} = \{\bar{A}, \bar{B}, \bar{C}, \bar{D}\} \in \mathbb{S}_{m,p,q}$ and $\theta = \{A, B, C, D\} \in \mathbb{S}_{n,p,q}$ be two minimal realizations of $\bar{k} = \Pi(\bar{\theta})$ and of $k = \Pi(\theta)$, respectively. Furthermore let $e(t) = (\bar{k}(z) - k(z))u(t)$. With a slight abuse of notation, the L_2 criterion as a function of the system matrices is then given by

$$J(\bar{\theta}, \theta) := \frac{1}{2} \mathbb{E} \operatorname{tr} e(t)e(t)' \quad (5)$$

We denote an optimal L_2 reduced model by $\bar{\theta}_{L_2}$, i.e.,

$$J(\bar{\theta}_{L_2}, \theta) = \min_{\bar{\theta}} J(\bar{\theta}, \theta) \quad (6)$$

Of course one has to keep in mind that optimal models are only unique up to state transformations, i.e., $J(\bar{\theta}_{L_2}, \theta) = J(T \cdot \bar{\theta}_{L_2}, \theta)$.

By defining $\bar{x}(t) = (z^{-1}I - \bar{A})^{-1}\bar{B}u(t)$, the derivative of the “residual” $e(t)$ with respect to the matrices $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ is given by:

$$\begin{aligned} de(t) = & d\bar{C}\bar{x}(t) - \bar{C}(z^{-1}I - \bar{A})^{-1}d\bar{A}\bar{x}(t) \\ & + \bar{C}(z^{-1}I - \bar{A})^{-1}d\bar{B}u(t) + d\bar{D}u(t) \end{aligned} \quad (7)$$

The derivative of the criterion J with respect to the matrices $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ is

$$\begin{aligned} dJ = & \mathbb{E} \operatorname{tr} de(t)e(t)' \\ = & -\operatorname{tr} \sum_{j>0} \bar{C}\bar{A}^{j-1}d\bar{A}\mathbb{E}\bar{x}(t-j)e(t)' \\ & + \operatorname{tr} \sum_{j>0} \bar{C}\bar{A}^{j-1}d\bar{B}\mathbb{E}u(t-j)e(t)' + \\ & + \operatorname{tr} d\bar{C}\mathbb{E}\bar{x}(t)e(t)' + \operatorname{tr} d\bar{D}\mathbb{E}u(t)e(t)' \end{aligned} \quad (8)$$

The quantities u , $\bar{x}$, e may be computed by the following state space model

$$\left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array} \right] = \left[\begin{array}{cc|c} A & 0 & B \\ 0 & \bar{A} & \bar{B} \\ \hline 0 & 0 & I \\ 0 & I & 0 \\ -C & \bar{C} & \bar{D} - D \end{array} \right]. \quad (9)$$

As easily can be seen, $\hat{y}(t) := (\hat{C}(z^{-1} - \hat{A})^{-1}\hat{B} + \hat{D})u(t) = (u(t)', \bar{x}(t)', e(t)')'$. Furthermore let us define

$$\hat{P} = \hat{A}\hat{P}\hat{A}' + \hat{B}\hat{B}' \quad (10)$$

$$\hat{M} = \hat{A}\hat{P}\hat{C}' + \hat{B}\hat{D}' \quad (11)$$

Then $\hat{\Omega}_0 := \mathbb{E} \hat{y}(t)\hat{y}(t)' = \hat{C}\hat{P}\hat{C}' + \hat{D}\hat{D}'$ and $\hat{\Omega}_j := \mathbb{E} \hat{y}(t-j)\hat{y}(t)' = \hat{M}'(\hat{A}')^{j-1}\hat{C}'$ for $j > 0$. Using these definitions the derivative dJ may be further simplified

$$\begin{aligned} dJ = & -\text{tr } d\bar{A}S_x\hat{M}' \sum_{j>0} (\hat{A}')^{j-1}\hat{C}'S'_e\bar{C}\bar{A}^{j-1} \\ & + \text{tr } d\bar{B}S_u\hat{M}' \sum_{j>0} (\hat{A}')^{j-1}\hat{C}'S'_e\bar{C}\bar{A}^{j-1} \\ & + \text{tr } d\bar{C}S_x\hat{\Omega}'S'_e + \text{tr } \bar{D}S_u\hat{\Omega}'S'_e \end{aligned} \quad (12)$$

Here $S_u = [I, 0, 0]$, $S_x = [0, I, 0]$ and $S_e = [0, 0, I]$ are suitably defined selection matrices. The sum $\sum_{j>0} (\hat{A}')^{j-1}\hat{C}'S'_e\bar{C}\bar{A}^{j-1}$ is the solution of the Lyapunov equation:

$$\hat{X} = \hat{A}'\hat{X}\bar{A} + \hat{C}'S'_e\bar{C} \quad (13)$$

and thus we finally obtain

$$\begin{aligned} dJ = & -\text{tr } d\bar{A}S_x\hat{M}'\hat{X} + \text{tr } d\bar{B}S_u\hat{M}'\hat{X} \\ & + \text{tr } d\bar{C}S_x\hat{\Omega}'S'_e + \text{tr } \bar{D}S_u\hat{\Omega}'S'_e. \end{aligned} \quad (14)$$

In particular note: If $\bar{\theta} = \bar{\theta}_{L_2}$ is an optimal reduced order system, then we must have

$$\begin{aligned} D_{\bar{A}} &:= S_x\hat{M}'\hat{X} = 0, \quad D_{\bar{B}} := S_u\hat{M}'\hat{X} = 0, \\ D_{\bar{C}} &:= S_x\hat{\Omega}'S'_e = 0, \quad D_{\bar{D}} := S_u\hat{\Omega}'S'_e = 0 \end{aligned} \quad (15)$$

3.1 Linear approximation of the optimal reduced order model

In this section we compute a first order approximation of the optimal reduced order model. We make the assumption that the reduced order model $\bar{\theta}$ takes the following form

$$\begin{aligned} \bar{A} &= A_{0,11} + \epsilon\bar{A}_1 + O(\epsilon^2) \\ \bar{B} &= B_{0,1} + \epsilon\bar{B}_1 + O(\epsilon^2) \\ \bar{C} &= C_{0,1} + \epsilon\bar{C}_1 + O(\epsilon^2) \\ \bar{D} &= D_0 + \epsilon\bar{D}_1 + O(\epsilon^2). \end{aligned} \quad (16)$$

As mentioned above, (minimal) realizations of the optimal reduced order model are only unique up state transformations. By the above assumption w.l.o.g. we restrict ourselves to realizations which are “close” to the truncated models and thus in the limit converge to the truncation of the limiting model $\theta_0 = \{A_0, B_0, C_0, D_0\}$.

The strategy now is as follows. First a Taylor series expansion of the “derivatives” (15)

$$\begin{aligned}
D_{\bar{A}} &= 0 + \epsilon D_{1,\bar{A}} + O(\epsilon^2) \\
D_{\bar{B}} &= 0 + \epsilon D_{1,\bar{B}} + O(\epsilon^2) \\
D_{\bar{C}} &= 0 + \epsilon D_{1,\bar{C}} + O(\epsilon^2) \\
D_{\bar{D}} &= 0 + \epsilon D_{1,\bar{D}} + O(\epsilon^2)
\end{aligned} \tag{17}$$

is computed. (Note that the derivatives are zero for $\epsilon = 0$.) Setting $D_{1,\bar{A}} = 0$, $D_{1,\bar{B}} = 0$, $D_{1,\bar{C}} = 0$, and $D_{1,\bar{D}} = 0$ gives four equations, which have to be solved for the unknowns $\bar{A}_1$, $\bar{B}_1$, $\bar{C}_1$, and $\bar{D}_1$.

In the course of these evaluations Taylor series expansions for the solutions of Lyapunov equations have to be computed. Consider e.g., the controllability Grammian of the model $\theta = \{A, B, C, D\}$, which is defined as the solution of the Lyapunov equation $P = APA' + BB'$. By the asymptotic stability assumption, we can compute a Taylor series expansion of P by the following recursions:

$$\begin{aligned}
P &= P_0 + \epsilon P_1 + \epsilon^2 P_2 + \dots \\
P_0 &= A_0 P_0 A_0' + B_0 B_0' \\
P_1 &= A_0 P_1 A_0' + B_1 B_0' + B_0 B_1' + A_1 P_0 A_0' + A_0 P_0 A_1' \\
P_2 &= A_0 P_2 A_0' + B_1 B_1' + A_1 P_1 A_0' + A_0 P_1 A_1' + A_1 P_0 A_1' \\
\dots &= \dots
\end{aligned} \tag{18}$$

Furthermore due to the lower triangular block structure of A_0 , it is easy to see that P has the form:

$$P = \begin{pmatrix} P_{0,11} + O(\epsilon) & P_{0,12} + O(\epsilon) \\ P_{0,21} + O(\epsilon) & O(1) \end{pmatrix}, \tag{19}$$

where e.g., $P_{0,11} = A_{0,11} P_{0,11} A_{0,11}' + B_{0,1} B_{0,1}'$. In a similar manner, we obtain for the observability Grammian $Q = A'QA + C'C$:

$$Q = \begin{pmatrix} Q_{0,11} + O(\epsilon) & \epsilon Q_{1,12} + O(\epsilon^2) \\ \epsilon Q_{1,21} + O(\epsilon^2) & O(\epsilon^2) \end{pmatrix} \tag{20}$$

Note that by the assumption $k_0 \in \mathbb{M}_{m,p,q}$, it follows that $P_{0,11} > 0$ and $Q_{0,11} > 0$ holds.

Using the same strategy, one obtains

$$\hat{P} \doteq \begin{pmatrix} P_{0,11} + O(\epsilon) & P_{0,12} + O(\epsilon) & P_{0,11} + \epsilon \hat{P}_{1,13} \\ P_{0,21} + O(\epsilon) & O(1) & P_{0,21} + O(\epsilon) \\ P_{0,11} + \epsilon \hat{P}_{1,31} & P_{0,12} + O(\epsilon) & P_{0,11} + \epsilon \hat{P}_{1,33} \end{pmatrix} \tag{21}$$

and

$$\hat{X} \doteq \begin{pmatrix} Q_{0,11} + \epsilon \hat{X}_{1,1} \\ \epsilon Q_{1,21} \\ -Q_{0,11} + \epsilon \hat{X}_{1,3} \end{pmatrix}. \tag{22}$$

Here and in the following $\doteq$ means that only terms up to order ϵ are considered.

Next these expressions are plugged into the “derivatives” (15), and then the first order terms are set equal to zero. This leads to the following four equations:

$$\begin{aligned} & Q_{1,12}A_{0,21}P_{0,11} + (\hat{X}'_{1,1} + \hat{X}_{1,3})A_{0,11}P_{0,11} + \\ & Q_{0,11}A_{0,11}(\hat{P}_{1,13} - \hat{P}_{1,33}) + Q_{0,11}A_{1,12}P_{0,21} + \\ & Q_{1,12}A_{0,22}P_{0,21} + Q_{0,11}(A_{1,11} - \bar{A}_1)P_{0,11} = 0 \end{aligned} \quad (23)$$

$$Q_{1,12}B_{0,2} + (\hat{X}'_{1,1} + \hat{X}_{1,3})B_{0,1} + Q_{0,11}(B_{1,1} - \bar{B}_1) = 0 \quad (24)$$

$$C_{0,1}(\hat{P}_{1,13} - \hat{P}_{1,33}) + C_{1,2}P_{0,21} + (C_{1,1} - \bar{C}_1)P_{0,11} = 0 \quad (25)$$

$$(D_1 - \bar{D}_1) = 0 \quad (26)$$

Finally, by some simple but tedious algebraic manipulations, we may eliminate $\hat{P}_{1,13}$, $\hat{P}_{1,33}$, $\hat{X}_{1,1}$ and $\hat{X}_{1,3}$, and we obtain the solutions as:

$$\begin{aligned} \bar{A}_1 &= A_{1,11} + X_1A_{0,11} - A_{0,11}X_1 \\ &\quad + Q_{0,11}^{-1}Q_{1,12}A_{0,21} + A_{1,12}P_{0,21}P_{0,11}^{-1} \\ &\quad + Q_{0,11}^{-1}Q_{1,12}A_{0,22}P_{0,21}P_{0,11}^{-1} \\ &\quad - A_{0,11}Q_{0,11}^{-1}Q_{1,12}P_{0,21}P_{0,11}^{-1} \\ \bar{B}_1 &= B_{1,1} + X_1B_{0,1} + Q_{0,11}^{-1}Q_{1,12}B_{0,2} \\ \bar{C}_1 &= C_{1,1} - C_{0,1}X_1 + C_{1,2}P_{0,21}P_{0,11}^{-1} \\ &\quad - C_{0,1}Q_{0,11}^{-1}Q_{1,12}P_{0,21}P_{0,11}^{-1} \\ \bar{D}_1 &= D_1 \end{aligned} \quad (27)$$

where $X_1 \in \mathbb{R}^{m \times m}$ is arbitrary. All terms on the right hand side of the above equations are linear functions of $\{A_1, B_1, C_1, D_1\}$ except for the terms involving X_1 . These terms correspond to a state space transformation $(I + \epsilon X_1 + O(\epsilon^2))$ of the reduced order model and reflect the non uniqueness of the realizations. (See also the discussion above and the next section.) Apart from this, the above relations correspond to the linearization of the mapping, attaching the optimal reduced order model to θ , in the point θ_0 .

4 Balanced truncation

Balanced truncation is a simple model reduction algorithm, which works as follows. Let T be a state space transformation, which renders a state space model $\theta = \{A, B, C, D\}$ into balanced form, i.e., such that the transformed Grammians are equal and diagonal: $TPT' = T^{-T}QT^{-1} = \text{diag}(\sigma_1, \dots, \sigma_n)$. Then simply take the truncation of the transformed system matrices as the reduced order model. To be more precise let $S = (I, 0) \in \mathbb{R}^{m \times n}$ be a selection matrix which picks the first m rows of a matrix. Then the reduced order model is obtained as $\tilde{\theta} = \{STAT^{-1}S', STB, CT^{-1}S', D\}$.

For our purpose, it suffices to consider “block” balanced truncations, where the transformation T is chosen such that the transformed Grammians are “block” diagonal and not necessarily equal. It is easy to see, that the corresponding

truncation is equivalent to the “full” balanced truncation, in the sense that both represent the same transfer function. In other words they are related via a state transformation to each other.

Let

$$\begin{aligned} T &= T_0 + \epsilon T_1 + O(\epsilon^2) \\ &\doteq \begin{pmatrix} I + \epsilon T_{1,11} & \epsilon T_{1,12} \\ -T_{0,21} + \epsilon T_{1,21} & I + \epsilon T_{1,22} \end{pmatrix} \end{aligned} \quad (28)$$

be a state space transformation, which renders $\{A, B, C, D\}$ into a block balanced form. By easy calculations one may see that the transformed system matrices are of the form:

$$\begin{aligned} TAT^{-1} &= \begin{pmatrix} \tilde{A} & O(\epsilon) \\ O(1) & O(1) \end{pmatrix}, \quad TB = \begin{pmatrix} \tilde{B} \\ O(1) \end{pmatrix} \\ CT^{-1} &= (\tilde{C} \quad O(\epsilon)), \quad D. \end{aligned} \quad (29)$$

The (block) balanced truncation is therefore given by:

$$\begin{aligned} \tilde{A} &\doteq A_{0,11} + \epsilon(A_{1,11} + T_{1,11}A_{0,11}) \\ &\quad - \epsilon(A_{0,11}(T_{1,12}T_{0,21} + T_{1,11})) \\ &\quad + \epsilon(T_{1,12}A_{0,21} + A_{1,12}T_{0,21}) \\ &\quad + \epsilon(T_{1,12}A_{0,22}T_{0,21}) \\ \tilde{B} &\doteq B_{0,1} + \epsilon(B_{1,1} + T_{1,11}B_{0,1} + T_{1,12}B_{0,2}) \\ \tilde{C} &\doteq C_{0,1} + \epsilon(C_{1,1} - C_{0,1}T_{1,11} + C_{1,2}T_{0,21}) \\ &\quad - \epsilon C_{0,1}T_{1,12}T_{0,21} \end{aligned} \quad (30)$$

The transformed Grammians $\tilde{P} = TPT'$, $\tilde{Q} = T^{-T}QT^{-1}$ are of the form

$$\tilde{P} = \begin{pmatrix} P_{0,11} + O(\epsilon) & \tilde{P}_{12} \\ \tilde{P}_{21} & O(1) \end{pmatrix} \quad (31)$$

$$\tilde{Q} = \begin{pmatrix} Q_{0,11} + O(\epsilon) & \tilde{Q}_{12} \\ \tilde{Q}_{21} & O(\epsilon^2) \end{pmatrix} \quad (32)$$

where

$$\tilde{P}_{21} \doteq (P_{0,21} - T_{0,21}P_{0,11}) + O(\epsilon) \quad (33)$$

$$\tilde{Q}_{21} \doteq \epsilon(Q_{0,21} - T'_{1,12}Q_{0,11}) \quad (34)$$

If T is a block balancing transformation, then the off diagonal blocks of $\tilde{P}$ and $\tilde{Q}$ must be zero and thus $T_{1,12} = Q_{0,11}^{-1}Q_{1,12}$ and $T_{0,21} = P_{0,21}P_{0,11}^{-1}$ must hold. (Note that both P and Q are symmetric by construction.) Comparing (30) with (27) reveals that the (block) balanced truncation of $\{A, B, C, D\}$ is up to terms of $O(\epsilon^2)$ equal to the optimal reduced order model given in (27), i.e.,

$$\begin{aligned} \bar{A} &= \tilde{A} + O(\epsilon^2), \quad \bar{B} = \tilde{B} + O(\epsilon^2) \\ \bar{C} &= \tilde{C} + O(\epsilon^2), \quad \bar{D} = \tilde{D} + O(\epsilon^2), \end{aligned} \quad (35)$$

if $T_{1,11} = X_1$ is chosen.

In order to make the above results independent of the choice of a particular realization, we need a (local) parametrization of the set $\mathbb{M}_{m,p,q}$ as follows: Let two smooth maps

$$\begin{aligned} \Phi : \mathbb{S}_{m,p,q} &\rightarrow \mathbb{R}^{n(p+q)} \\ \bar{\theta} &\mapsto \mu \end{aligned} \quad (36)$$

and

$$\begin{aligned} \Phi^{-1} : \mathbb{R}^{n(p+q)} &\rightarrow \mathbb{S}_{m,p,q} \\ \mu &\mapsto \bar{\theta} \end{aligned} \quad (37)$$

be given such that $\Phi(\theta_1) = \Phi(\theta_2)$ iff $\Pi(\theta_1) = \Pi(\theta_2)$ and $\Phi(\Phi^{-1}(\mu)) = \mu$. Thus μ may be interpreted as a vector of (free) parameters describing the transfer function $k = \Pi(\theta)$. Note that Φ and Φ^{-1} only need to be defined in suitably chosen neighborhoods of $\bar{\theta}_0 = \{A_{0,11}, B_{0,1}, C_{0,1}, D_0\}$ and of $\mu_0 = \Phi(\bar{\theta}_0)$.

An example of such a parametrization for the SISO case $p = q = 1$ is given in the next section. For the general case canonical forms may be used to define Φ and Φ^{-1} .

Next let $\bar{\mu}_{L_2}(\theta) = \Phi(\bar{\theta}_{L_2})$ and $\tilde{\mu}_{b.t.}(\theta) = \Phi(\tilde{\theta}_{b.t.})$.

Using these definitions the main result now may be stated as follows:

Theorem 1 *Let $\theta = \theta_0 + \epsilon\theta_1 \in \mathbb{S}_{n,p,q}$ be a state space model of the form (1). Then*

$$\bar{\mu}_{L_2}(\theta) = \tilde{\mu}_{b.t.}(\theta) + O(\epsilon^2) \quad (38)$$

and

$$\left. \frac{\partial \bar{\mu}_{L_2}}{\partial \theta} \right|_{\theta_0} = \left. \frac{\partial \tilde{\mu}_{b.t.}}{\partial \theta} \right|_{\theta_0}. \quad (39)$$

Sloppy speaking this means that balanced truncation is “close” to the optimal reduced order model if the high order model is “close” to the class of low order models.

5 Applications to system identification

The result presented above can be applied to a system identification problem. Assume that a high order model has been estimated and we want to reduce this model to a good low order approximant. When doing this, it is of great interest to minimize the mean square error of the low order model. What is presented below is basically that the mean square error for L_2 model reduction and balanced truncation are asymptotically the same, when the high order model is reduced to a “correct” lower order. This result confirms the simulation results and ideas in [8]. See also [9]. For simplicity we describe the results in the SISO case, i.e., $p = q = 1$.

Assume that data are generated from

$$y(t) = \bar{k}_0(z)u(t) + e(t), \quad (40)$$

where u is the input signal and e is white noise with variance λ . We assume that

$$\bar{k}_0(z) = \frac{b_1^0 z^{n_k} + \dots + b_{n_b}^0 z^{n_k+n_b-1}}{1 + f_1^0 z + \dots + f_{n_f}^0 z^{n_f}}, \quad (41)$$

and for simplicity we also assume that $n_k = 1, n_b = n_f = m$. (More general cases can be treated in a similar fashion.) Let $\mu_0 = (b_1^0, \dots, b_m^0, f_1^0, \dots, f_m^0)'$ be the stacked vector of the coefficients of the denominator and nominator polynomials.

The considered high order models ($n > m$) are of the form

$$k(z, \eta) = \frac{b_1 z^1 + \dots + b_n z^n}{1 + f_1 z^1 + \dots + f_n z^n} \quad (42)$$

where $\eta = (b_1, \dots, b_n, f_1, \dots, f_n)'$ again is the stacked coefficient vector.

Moreover, for the model $y(t) = k(z, \eta)u(t) + e(t)$ we define the prediction error as

$$\xi(t, \eta) = y(t) - \hat{y}(t|\eta) = y(t) - k(z, \eta)u(t). \quad (43)$$

Due to the overmodeling there exist a set $\mathcal{D} = \{\eta : k(e^{i\omega}, \eta) = \bar{k}_0(e^{i\omega})\}$ and in particular

$$\eta_0 = (b_1^0 \dots b_m^0 \ 0 \dots 0 \ f_1^0 \dots f_m^0 \ 0 \dots 0)' \in \mathcal{D}. \quad (44)$$

Since the model uses too many parameters we usually lack from identifiability. To avoid this we estimate η from

$$\hat{\eta} = \arg \min_{\eta} V_N(\eta) + \frac{\delta}{2} \|\eta - \eta_0\|_2^2, \quad (45)$$

where δ is a positive scalar (usually called a regularization parameter), N is the number of data, and

$$V_N(\eta) = \frac{1}{N} \sum_{t=1}^N \xi^2(t, \eta). \quad (46)$$

We know that [3] the asymptotic covariance of $\hat{\eta}$ equals

$$\text{Cov } \hat{\eta} \approx \frac{\lambda}{N} (\mathbb{E} \Psi(t, \eta_0) \Psi'(t, \eta_0) + \delta I)^{-1}, \quad (47)$$

where

$$\Psi(t, \eta_0) = \left. \frac{\partial}{\partial \eta} \xi(t, \eta) \right|_{\eta_0}. \quad (48)$$

Writing $k(z, \eta_0)$ in controller form gives

$$\begin{aligned} x(t+1) &= A_0 x(t) + B_0 u(t) \\ y(t) &= C_0 x(t) + D_0 u(t), \end{aligned} \quad (49)$$

where

$$\begin{aligned} A_0 &= \left(\begin{array}{cccc|cccc} -f_1^0 & -f_2^0 & \dots & -f_m^0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0 \end{array} \right) \\ B_0' &= (1 \ 0 \ \dots \ 0 \mid 0 \ \dots \ 0) \\ C_0' &= (b_1^0 \ b_2^0 \ \dots \ b_m^0 \mid 0 \ \dots \ 0) \\ D_0 &= 0 \end{aligned} \quad (50)$$

It is now clear that writing $k(z, \hat{\eta})$ in controller form gives a state space model of the form (1) with the following matrices equal to zero

$$A_{1,21}, A_{1,22}, B_{1,1}, B_{0,2}, B_{1,2}, D_1. \quad (51)$$

Now we can construct two estimates of the low order model:

1. Compute the balanced truncation $\tilde{\theta}_{b.t.}$ of the above mentioned controller form state space model $\hat{\theta}$ of $k(z, \hat{\eta})$, and let $\tilde{\mu}_{b.t.}$ denote the stacked vector of coefficients of the polynomial representation of the corresponding transfer function.
2. Compute the optimal L_2 reduced order model $\bar{\theta}_{L_2}$ of the state space model $\hat{\theta}$, and let $\bar{\mu}_{L_2}$ denote the stacked vector of coefficients of the polynomial representation of the corresponding transfer function. (Of course the L_2 reduction can also be directly performed for the corresponding polynomial representations, without the need to transform forth and back from state space to polynomial representations.)

Note that here the mapping Φ , used in section 4, maps a state space model $\bar{\theta} \in \mathbb{S}_{m,1,1}$ to the coefficient vector of the corresponding polynomial representation, and Φ^{-1} gives the controller form state space model of a polynomial representation.

Both $\tilde{\mu}_{b.t.}$ and $\bar{\mu}_{L_2}$ are functions of the estimates $\hat{\eta}$ of the high order model and by Theorem 1, and in particular from (39), it follows for the Jacobians that

$$H_0 := \left. \frac{\partial \bar{\mu}_{L_2}}{\partial \hat{\eta}} \right|_{\eta_0} = \left. \frac{\partial \tilde{\mu}_{b.t.}}{\partial \hat{\eta}} \right|_{\eta_0} \quad (52)$$

This in turn implies that

$$\text{Cov } \bar{\mu}_{L_2} = H_0 \text{Cov } \hat{\eta} H_0' = \text{Cov } \tilde{\mu}_{b.t.} \quad (53)$$

i.e., both methods have asymptotically the same covariance.

6 Conclusions

In this paper we have discussed model reduction of an n -th order system that is “close” to an m -th order approximant. We have shown that reducing such a model by minimizing an unweighted L_2 model reduction criteria essentially gives the same result as reducing the model by balanced truncation. This result has also been applied in a system identification setting. Here we showed that estimating a high order model and then reducing it to correct order by subjecting it to L_2 reduction or balanced truncation gives low order models with the same covariance.

From the results presented here it is clear that balanced truncation can be used instead of or as an initial guess for L_2 reduction in certain situations. This will avoid problems with local minima that are present in the numerical optimization problems for computing the L_2 reduced model.

References

- [1] A. Ferrante, W. Krajewski, A. Lepschy, and U. Viaro. Convergent algorithm for l_2 model reduction. *Automatica*, 35:75–79, 1999.
- [2] K. Glover. All optimal hankel-norm approximations of linear multivariable systems and their L^∞ -error bounds. *International Journal of Control*, 39(6):1115–1193, 1984.
- [3] L. Ljung. *System Identification: Theory for the User*. Prentice-Hall, Upper Saddle River, NJ, 2nd edition, 1999.
- [4] B. Moore. Principal component analysis in linear systems: Controllability, observability and model reduction. *IEEE Transactions on Automatic Control*, 26:17–31, 1981.
- [5] L. Pernebo and L. M. Silverman. Model reduction via balanced state space representations. *IEEE Transactions on Automatic Control*, 27(2):382–387, 1982.
- [6] W. Scherrer. Local optimality of minimum phase balanced truncation. In *Proceedings of the 15th IFAC World Congress*, Barcelona, Spain, July 2002.
- [7] J. T. Spanos, M. H. Milman, and D. L. Mingori. A new algorithm for L_2 optimal model reduction. *Automatica*, 28(5):897–909, 1992.
- [8] F. Tjärnström. *Variance Expressions and Model Reduction in System Identification*. PhD thesis 730, Department of Electrical Engineering, Linköpings Universitet, Linköping, Sweden, 2002.
- [9] B. Wahlberg. *On the Identification and Approximation of Linear Systems*. PhD thesis 163, Department of Electrical Engineering, Linköping University, 1987.
- [10] D. A. Wilson. Optimum solution of model reduction problem. *Proceedings IEE*, 117(6):1161–1165, 1970.