On a connection between minimax MPC and risk-sensitive control

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Abstract

A connection between robust synthesis of MPC using a particular minimax formulation and a probabilistic risk-sensitive approach is established. It is shown that the minimax controller basically solves a risk problem, but with the crucial property that the risk parameter is chosen automatically in the optimization in order to obtain a trade-off between performance and risk-sensitivity.

1 Introduction

The aim of this note is to show a structural similarity between two, at first glance very different, approaches to MPC. To be more precise, to show a connection between a probabilistic risk approach, and the worst-case solution introduced in [Löf01].

1.1 System setup

The class of systems we address are linear discrete-time time-invariant systems with measurement errors and external system disturbances

\[ x(k+1) = Ax(k) + Bu(k) + B_\xi \xi(k) \]  
\[ y(k) = Cx(k) + \eta(k) \]

Furthermore, there are constraints on inputs and outputs. To keep the notation simple, we will however not address these explicitly until later.

The two approaches that we will compare use different ways to model the external disturbance and the measurement error. Basically, we are comparing a stochastic and an unknown-but-bounded framework

\[ \xi(k) = P^\frac{1}{2} w(k) \]  
\[ w(k) \in W_1 = \mathcal{N}(0, I) \]  
\[ w(k) \in W_2 = \{ w : ||w|| \leq 1 \} \]

Of course, the measurement error model will be of the corresponding type. However, we will not explicitly work with the estimation part so we do not address this issue. All we have to know is that in both approaches, we write

\[ x(k) = \hat{x}(k) + P^\frac{1}{2} z(k) \]
The expression $P^{1/2}z(k)$ hence denote the state estimation error. The two different ways too specify the state estimate error are

$$z(k) \in Z_1 = \mathcal{N}(0, I) \quad (4a)$$

$$z(k) \in Z_2 = \{z : ||z|| \leq 1\} \quad (4b)$$

This corresponds to a normal distributed estimation error with covariance matrix $P$ (obtained typically using a Kalman filter), or an estimation error known to lie inside the ellipsoid $(x - \hat{x})^TP^{-1}(x - \hat{x}) \leq 1$ (obtained using an unknown-but-bounded framework, see e.g. [GC99]).

### 1.2 MPC

The underlying performance measure that is used in MPC is typically a finite horizon quadratic measure ($Q = Q^T > 0, R = R^T > 0$)

$$J = \sum_{j=0}^{N-1} ||x(k+j+1|k)||_Q^2 + ||u(k+j|k)||_R^2$$

(5)

Since $x(\cdot|k)$ is uncertain due to the estimation error and the disturbances, this has to be addressed in some way. A standard method is to minimize the expectation of the performance measure, given a normal distribution of the state estimation error and the external disturbances

$$\min_{u(\cdot)} \mathbb{E}_{z(k), w(\cdot)} J, \quad z(k) \in Z_1, \ w(\cdot) \in W_1$$

(6)

It is well known and easily shown that this is equivalent to a problem without estimation error and external disturbances, i.e. the control law will be the same as for a nominal MPC controller.

The standard approach to robustify nominal MPC is to employ a minimax strategy, i.e. optimize worst-case behavior. In [Löf01], it was shown that a minimax strategy

$$\min_{u(\cdot)} \max_{z(k), w(\cdot)} J, \quad z(k) \in Z_2, \ w(\cdot) \in W_2$$

(7)

gives a problem that can be solved using semidefinite programming.

We will show that this worst-case approach has connections to a probabilistic approach, so called risk-sensitive control [Jac77, Whi81]. The idea in risk-sensitive control is to introduce a risk-factor $\theta > 0$ and minimize

$$\min_{u(\cdot)} \theta \log \mathbb{E}_{z(k), w(\cdot)} e^{\frac{J}{\theta}}, \quad z(k) \in Z_1, \ w(\cdot) \in W_1$$

(8)

A small $\theta$ means a controller that is willing to take some risk, while a large $\theta$ corresponds to a cautious, risk-sensitive, controller. By letting $\theta$ approach zero, the nominal MPC problem is recovered. If $\theta$ is chosen too large, the problem will break down (the expectation will be infinite). This will be obvious later.

\footnote{For notational convenience we have changed the sign on the risk factor compared to standard notation [Whi81]. Furthermore, in the general case, $\theta$ can be negative, leading to an ‘optimistic’ controller}
1.3 Related results

Finding connections between deterministic worst-case and stochastic risk-sensitive approaches has been done before for various problems. In [HSK96], they show connections between LMS, $H_1$, and risk-sensitive filters. Connections between $H_1$ control and risk-sensitive control are established in [GD88].

2 Initial results for the case $\xi \equiv 0$

To begin with, we look at the special case $\xi \equiv 0$, i.e. only an estimation error. The calculations are most easily done in a vectorized form, so we introduce the predicted future states, and the control sequence that we wish to find.

$$X = \begin{bmatrix} x(k+1|k) \\ x(k+2|k) \\ \vdots \\ x(k+N|k) \end{bmatrix}, \quad U = \begin{bmatrix} u(k|k) \\ u(k+1|k) \\ \vdots \\ u(k+N-1|k) \end{bmatrix}$$

(9)

By introducing the matrices $H$ and $S$

$$H = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \quad S = \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}, \quad K = HP^{1/2}$$

(10)

we can write

$$X = H\hat{z}(k) + Kz(k) + SU$$

(11)

The performance measure (5) can, after redefining $Q := \text{diag}(Q, \ldots, Q)$ and $R := \text{diag}(R, \ldots, R)$, be written as

$$J = X^TQX + U^TRU$$

(12)

Let us now compare the three performance measures for this special case.

2.1 Nominal MPC

It is well known that

$$U = \arg\min_U E \mathbf{J}, \quad z(k) \in Z$$

$$= \arg\min_U (H\hat{z}(k) + SU)Q(H\hat{z}(k) + SU) + U^TRU$$

(13)

so we will not dwell on this issue. Furthermore, when the constraints on $U$ and $X$ are linear, the solution can be found with quadratic programming. The only reason for stating this optimization problem is that it might be interesting for the reader to compare with the optimization problems that will be derived for the risk and worst-case approach.
2.2 Risk-sensitive MPC

Inserting the performance in the risk expression yields

\[
\frac{2}{\theta} \log \mathbb{E}_z e^{\frac{1}{2}((H\hat{x}(k)+Kz+SU)^T Q(H\hat{x}(k)+Kz+SU) + U^T RU)}
\]  

(14)

In the expression above and in the derivation that follows, we use the shorthand notation \( z \) to save space. The performance measure can be rewritten by pulling out the deterministic part and we obtain

\[
(H\hat{x}(k) + SU)^T Q(H\hat{x}(k) + SU) + U^T RU
\]

\[
+ \frac{2}{\theta} \log \mathbb{E}_z e^{\frac{1}{2}((z^T K^T Q K z + z^T K^T Q (H\hat{x}(k) + SU))}
\]

(15)

The normalized state estimation error has a normal distribution, i.e. a pdf (with \( n \) denoting the dimension of the state vector)

\[
\frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}z^T z}
\]

(16)

Using this, we can calculate the expectation

\[
\mathbb{E}_z e^{\frac{1}{2}((z^T K^T Q K z + z^T K^T Q (H\hat{x}(k) + SU))}
\]

\[
= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{1}{2}((z^T K^T Q K z + z^T K^T Q (H\hat{x}(k) + SU))} e^{-\frac{1}{2}z^T z} \, dz
\]

\[
= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}z^T (I-\theta K^T Q K)z + \theta z^T K^T Q (H\hat{x}(k) + SU)} \, dz
\]

(17)

To simplify notation, we introduce the matrices

\[
M = I - \theta K^T Q K, \quad S = \theta K^T Q (H\hat{x}(k) + SU)
\]

(18)

and obtain

\[
\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}z^T M z + z^T S} \, dz
\]

\[
= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}(z - M^{-1} S)^T M (z - M^{-1} S) + \frac{1}{2}S^T M^{-1} S} \, dz
\]

\[
= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}(z - M^{-1} S)^T M (z - M^{-1} S) + \frac{1}{2}S^T M^{-1} S} \, dz
\]

\[
= \frac{1}{(2\pi)^{n/2}(\det M^{-1})^{1/2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}(z - M^{-1} S)^T M (z - M^{-1} S)} \, dz
\]

\[
= \frac{1}{(2\pi)^{n/2}(\det M^{-1})^{1/2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}(z - M^{-1} S)^T M (z - M^{-1} S)} \, dz
\]

(19)

The last simplification followed from the definition of a normal distribution and the fact that a pdf integrates to 1. Note that the expectation is finite (and defined) only if the matrix \( M \) is positive definite.

Our original performance measure simplifies to

\[
(H\hat{x}(k) + SU)^T Q(H\hat{x}(k) + SU) + U^T RU
\]

\[
+ \frac{2}{\theta} (\log(e^{\frac{1}{2}S^T M^{-1} S (\det M^{-1})^{1/2}}))
\]

(20)
After some simplifications, we obtain

\[(H\hat{x}(k) + SU)Q(H\hat{x}(k) + SU) + U^T RU + \frac{1}{2}S^T M^{-1}S + \frac{1}{2} \log \det M^{-1}\]

(21)

By inserting the definition of \(S\) and removing all constant terms, our optimization problem will be

\[U = \arg \min_U (H\hat{x}(k) + SU)^T Q(H\hat{x}(k) + SU) + U^T RU + \theta(H\hat{x}(k) + SU)^T QK(I - \theta K^T K)^{-1} K^T Q(H\hat{x}(k) + SU)\]

(22)

Note that we still have a quadratic program (if the constraints on \(U\) and \(X\) are linear).

### 2.3 Minimax MPC

Let us now derive the optimization problem for the minimax approach.

We first write the problem as

\[
\min_{t,U} \quad t \\
\text{subject to} \quad \max_z X^T QX + U^T RU \leq t
\]

Rewrite the constraint using a Schur complement

\[
\begin{bmatrix}
    t & H\hat{x}(k) + Kz + SU \\
    H\hat{x}(k) + Kz + SU & U^T
\end{bmatrix}
\begin{bmatrix}
    \hat{x}^T(k)H^T + z^T K^T + U^T S^T & 0 \\
    0 & R^{-1}
\end{bmatrix} \geq 0
\]

(23)

We extract the uncertainty by factorizing

\[
\begin{bmatrix}
    t & H\hat{x}(k) + SU & U^T \\
    U & \hat{x}^T(k)H^T + U^T S^T & 0 \\
    0 & 0 & R^{-1}
\end{bmatrix}
\begin{bmatrix}
    I \\
    z^T [0 \ 0] \\
    0
\end{bmatrix} + \begin{bmatrix}
    0 \\
    0 \\
    K
\end{bmatrix} \begin{bmatrix}
    I \\
    0 \\
    0
\end{bmatrix} \geq 0
\]

(24)

The above matrix inequality should hold for all admissible normalized estimation errors \(z\). To proceed, we use the following theorem [GL97]

**Theorem 1 (Robust LMI)** Robust satisfaction of the uncertain matrix inequality

\[F + LR + R^T \Delta R \geq 0 \quad \forall \|\Delta\| \leq 1\]

is equivalent to the matrix inequality

\[
\begin{bmatrix}
    F & L \\
    L^T & 0
\end{bmatrix} \succeq \begin{bmatrix}
    R & 0 \\
    0 & I
\end{bmatrix} \begin{bmatrix}
    \tau I & 0 \\
    0 & -\tau I
\end{bmatrix} \begin{bmatrix}
    R & 0 \\
    0 & I
\end{bmatrix}
\]

\[
\tau \geq 0
\]

\[\blacksquare\]
After introducing the multiplier $\tau \geq 0$ and applying Theorem 1, we obtain
\[
\begin{bmatrix}
t & \tilde{x}^T(k)H^T + U^T S^T & U^T & I \\
H\tilde{x}(k) + SU & Q^{-1} & 0 & 0 \\
U & 0 & R^{-1} & 0 \\
I & 0 & 0 & 0 \\
\end{bmatrix} \succeq 0
\]
(25)
Simplification yields
\[
\begin{bmatrix}
t & \tilde{x}^T(k)H^T + U^T S^T & U^T & I \\
H\tilde{x}(k) + SU & Q^{-1} - \tau KK^T & 0 & 0 \\
U & 0 & R^{-1} & 0 \\
I & 0 & 0 & \tau I \\
\end{bmatrix} \succeq 0
\]
(26)
This is the formulation that would be used to solve the optimization problem, since it is an LMI in the variables $U$, $t$ and $\tau$. However, the aim of our calculations is to show the connections to risk-sensitive control. To do this, we rewrite the LMI using a Schur complement
\[
(H\tilde{x}(k) + SU)^T(Q^{-1} - \tau KK^T)^{-1}(H\tilde{x}(k) + SU) + U^T RU + \frac{1}{\tau} \leq t
\]
(27)
Matrix-inversion lemma
\[
(Q^{-1} - \tau KK^T)^{-1} = Q + QK\left(\frac{1}{\tau} I - K^T Q K\right)^{-1} K^T Q
\]
(28)
gives us
\[
(H\tilde{x}(k) + SU)^T Q (H\tilde{x}(k) + SU) + U^T RU
+ (H\tilde{x}(k) + SU)^T Q K \left(\frac{1}{\tau} I - K^T Q K\right)^{-1} K^T Q (H\tilde{x}(k) + SU) + \frac{1}{\tau} \leq t
\]
(29)
Rearrange the $\tau$ term a bit and we find the final optimization problem for the minimax controller
\[
U = \arg\min_{U,t} \left( H\tilde{x}(k) + SU \right)^T Q (H\tilde{x}(k) + SU) + U^T RU
+ \tau( H\tilde{x}(k) + SU)^T Q K (I - \tau K^T Q K)^{-1} K^T Q (H\tilde{x}(k) + SU) + \frac{1}{\tau}
\]
(30)
After comparing this to the expression for the risk-sensitive approach (22) we are able to state our main result.

The performance measure minimized in minimax MPC corresponds to a risk-sensitive performance measure with the inverse risk parameter $\tau^{-1}$ added.

3 The general case

Let us now look at the general case when we have external disturbances. We first introduce the stacked predictions, now containing also the future disturbances
\[
X = H\tilde{x}(k) + HP^T z(k) + SU + GW
\]
(31)
\[ G = \begin{bmatrix}
B_k P^2_w & 0 & \ldots & 0 \\
AB_k P^2_w & B_k P^2_w & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
A^{N-1} B_k P^2_w & A^{N-2} B_k P^2_w & \ldots & B_k P^2_w
\end{bmatrix} \tag{32} \]

\[ W = \begin{bmatrix}
w(k) \\
w(k+1) \\
\vdots \\
w(k+N-1)
\end{bmatrix} \tag{33} \]

### 3.1 Risk-sensitive MPC

As before, we begin with the risk-sensitive MPC controller. We define a new stochastic variable containing all the uncertainties

\[ \zeta = \begin{bmatrix} z(k) \\ W \end{bmatrix} \in \mathcal{N}(0, I) \tag{34} \]

This can be done since all uncertainties are independent and normally distributed. This allows us to write

\[ X = H \hat{x}(k) + SU + K\zeta \]

\[ K = \begin{bmatrix} HP^{1/2} \\ G \end{bmatrix} \tag{36} \]

Performing exactly the same calculations as in the previous section yields

\[ U = \arg\min_U (H \hat{x}(k) + SU)^T Q (H \hat{x}(k) + SU) + U^T RU \]

\[ + \theta (H \hat{x}(k) + SU)^T Q K (I - \theta K^T Q K)^{-1} K^T Q (H \hat{x}(k) + SU) \tag{37} \]

### 3.2 Minimax MPC

The minimax controller in the general case can be derived in the same way as it was done in the special case when \( \xi \equiv 0 \). The only difference is that the robustness theorem now has to be employed for all uncertainties along the horizon. This will force us to introduce \( N + 1 \) variables (one for the estimation error and \( N \) for the external disturbances). The details of the derivation are omitted. However, by introducing the matrix (each block having a size equal to the dimension of the corresponding component in the vector \( \zeta \))

\[ \Gamma = \text{diag}(\tau_0 I, \tau_1 I, \ldots, \tau_N I) \succeq 0 \tag{38} \]

it can be shown that repeated application of Theorem 1, followed by a Schur complement and use of the matrix inversion lemma, gives the following optimization problem

\[ U = \arg\min_{U, \tau} (H \hat{x}(k) + SU)^T Q (H \hat{x}(k) + SU) + U^T RU \]

\[ + (H \hat{x}(k) + SU)^T Q K (I - \Gamma K^T Q K)^{-1} K^T Q (H \hat{x}(k) + SU) + \sum_{i=0}^{N} \frac{1}{\tau_i} \tag{39} \]
The formulation above is not the one that actually is used in the optimization. As for the derivation of the minimax controller in the previous section, it is only written in this form in order to simplify comparison with the risk-sensitive solution.

So how should we interpret this optimization problem. One way is to look at a conservative solution where we introduce the additional constraint $\tau_0 = \tau_1 = \ldots \tau_N$. For this special case, we see if we compare with (37) that we once again obtain a tuned risk-sensitive controller.

3.3 State and input constraints

As we mentioned in the introduction, MPC is typically applied to systems with state and inputs constraints. Let us look at how the constraint handling in the minimax approach in [Löf01] can be interpreted in a stochastic setting. Of course, any constraint on $U$ is taken care of straightforwardly, since these constraints are unaffected by the uncertainties.

Let us look at a single state constraint (M is thus a row vector)

$$MX \leq 1$$

Inserting the definition of $X$ yields

$$MH \hat{x}(k) + MSU + MK\zeta \leq 1$$

In the minimax approach we want this constraint to hold for all admissible uncertainties. This gives us the constraint

$$\max_{||\zeta|| \leq 1} MH \hat{x}(k) + MSU + \sum_{j=0}^{N} m_j\zeta_j \leq 1$$

where

$$MK = \begin{bmatrix} m_0 & m_1 & \ldots & m_N \end{bmatrix}, \quad \zeta = \begin{bmatrix} z(k) \\ W \end{bmatrix} = \begin{bmatrix} \zeta_0 \\ \vdots \\ \zeta_N \end{bmatrix}$$

Finding the maximum over the uncertainty $\zeta$ can be done analytically and yields the robustified constraint

$$MH \hat{x}(k) + MSU + \sum_{j=0}^{N} \sqrt{m_j m_j^T} \leq 1$$

In the stochastic counterpart $\zeta_j \in \mathcal{N}(0, I)$ we have $m_j \zeta_j \in \mathcal{N}(0, m_j m_j^T)$, hence $\sqrt{m_j m_j^T}$ corresponds to the standard deviation of each stochastic part that is added to the nominal part of the constrained variable.

4 Conclusion

By some algebraic manipulations, we showed that there is a clear connection between deterministic minimax MPC, and a stochastic risk-sensitive approach. In the case when the only uncertainty comes from the estimation error, the minimax performance measure is equivalent to a risk-sensitive performance measure. In the more general case with external disturbances, it was shown that a conservative variant of the minimax problem is equivalent to a risk-sensitive approach.
References


