Iteration varying filters in Iterative Learning Control

Mikael Norrlöf

Division of Automatic Control
Department of Electrical Engineering
Linköpings universitet, SE-581 83 Linköping, Sweden
WWW: http://www.control.isy.liu.se
E-mail: mino@isy.liu.se

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Abstract

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Keywords: Iterative learning control, measurement noise, disturbance rejection, ILC
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M. Norrlöf

Department of Electrical Engineering, Linköpings universitet, SE-581 83 Linköping, Sweden
Email: mino@isy.liu.se
http://www.control.isy.liu.se

Abstract

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1 Introduction

The aim of this paper is to show how a new iteration varying Iterative Learning Control (ILC) algorithm can be found using ideas from estimation and identification theory [5]. Classically iterative learning control has been considered to be a method for achieving trajectory tracking, see e.g., the surveys [7, 8, 1]. In this contribution ILC will be used in a different setting, where ILC instead is applied for disturbance rejection. Disturbance rejection aspects of ILC have been covered earlier in e.g., [13, 3, 2], where disturbances such as initial state disturbances and measurement disturbances are addressed. More details on the approach presented here can also be found in [9]. There it is also shown how it is possible, in the linear system case, to transform a problem from one formulation to the other.

In Figure 1 the structure of the system, used in the disturbance rejection formulation approach to ILC, is shown as a block diagram.

For ILC the goal is to, iteratively, find the input to a system such that some error is minimized. In the disturbance rejection formulation, the goal is to find an input $u_k(t)$ such that the output $z_k(t)$ is minimized, i.e., such that the disturbance acting on the output is compensated for. If the system is known and invertible, and the disturbance $d(t)$ is known, then the obvious approach would be to filter $d(t)$ through the inverse of the system and use the resulting $u_k(t)$ as a control input. This means that the optimal input looks like,

$$u_k(t) = -(G^0(q))^{-1}d(t)$$

Different aspects of the disturbance rejection approach to ILC will be considered in this contribution. The effects of measurement disturbances will be especially considered. Results from simulations using the methods are also presented.

2 An ILC algorithm using disturbance estimation

If the system $G^0$ in Figure 1 is a discrete time linear time invariant system, then the following equations give a mathematical description of the behavior of the system,

$$z_k(t) = G^0(q)u_k(t) + d(t)$$
$$y_k(t) = z_k(t) + n_k(t)$$

For simplicity it is assumed that the system disturbance $d(t)$ is $k$-independent.

Now, assume $G(q)$ to be a model of $G^0(q)$. Using the model of the system the disturbance $d(t)$ can be estimated using the measurement from the system and the model,

$$\hat{y}_k(t) = y_k(t) - G(q)u_k(t)$$

Let $\hat{d}_k(t)$ be the estimate of the disturbance in the $k$th iteration. A straightforward approach to estimate the
disturbance is to minimize the loss function

\[ V_{k,t}(\hat{d}_k(t)) = \frac{1}{2} \sum_{j=0}^{k-1} (\tilde{y}_j(t) - \hat{d}_k(t))^2 \]  

(3)

and the corresponding estimate is given by,

\[ \hat{d}_k(t) = \frac{1}{k} \sum_{j=0}^{k-1} \tilde{y}_j(t) \]  

(4)

This can also be written in a recursive form

\[ \hat{d}_{k+1}(t) = \frac{k}{k+1} \hat{d}_k(t) + \frac{1}{k+1} \tilde{y}_k(t) \]  

(5)

The corresponding ILC algorithm is an updating equation for the control signal \( u_k(t) \). In order to minimize \( y_k(t) \) the best choice for the input is

\[ u_{k+1}(t) = -\frac{1}{G(q)} \hat{d}_{k+1}(t) \]  

(6)

which means that,

\[ u_{k+1}(t) = u_k(t) - \frac{1}{(k+1)G(q)} y_k(t) \]  

(7)

where (5), (2), and (6) has been used. Note the similarity with the standard first order ILC updating equation [4],

\[ u_{k+1}(t) = Q(q)(u_k(t) + L(q)e_k(t)) \]  

(8)

where \( e_k(t) \) is the error. In the disturbance rejection approach \( e_k(t) \) is simply the output \( y_k(t) \). In (7) the \( Q \)-filter is chosen as \( Q \equiv 1 \) and the \( L \)-filter is an iteration dependent filter since the gain is reduced every iteration. This means that \( L_k(q) = \frac{1}{(k+1)G(q)} \)

which, normally, is a non-causal filter. Since \( e_k(t) \), \( 0 \leq t \leq n-1 \), is available when calculating \( u_{k+1}(t) \) this is not a problem. The fact that the ILC algorithm can utilize non-causal filters is the reason why it is possible to achieve such good results with a quite simple control structure. This is also explored in [6].

By just observing the output in the first iteration it is possible to find an estimate of \( d \). Since there is a measurement disturbance the estimate can however be improved and this is what the algorithm iteratively will do. Note that since the gain of the \( L_k \)-filter is reduced with \( k \) the algorithm will not work very well if \( d(t) \) is varying as a function of iteration. The gain of the \( L \)-filter will actually tend to zero when \( k \to \infty \). The analysis of the proposed algorithm is done in the next sections. An ILC method that can work also when the disturbance is \( k \)-dependent is presented in [10, 11]. This method relies on an estimation based on a Kalman filter with adaptive gain.

Before doing the analysis of the proposed algorithm, some assumptions on the disturbances and the system is presented.

### 3 Assumptions

The system description is given from (1) and the ILC updating equation from (7). It is assumed that \( u_0(t) \) is chosen as \( u_0(t) = 0, t \in [0, t_j] \). In the system description in (1) it is clear that the system disturbance \( d(t) \) is repetitive with respect to the iterations, i.e., does not depend on the iteration \( k \). Notice however that the system does not have to start and end at the same condition and it is therefore not a repetitive control (RC) problem but instead an ILC problem. The measurement disturbance \( n_k(t) \) is assumed to be equal to \( \nu(t) \) where \( t = k \cdot t \) and \( \nu(t) \) represents a white stationary stochastic process with zero mean and variance \( r_n \). The expected value, \( E\{n_k(t)\} \), is therefore with respect to the underlying process \( \nu \), and

\[ E\{n_k(t)\} = 0 \]

The variance becomes

\[ Var\{n_k(t)\} = r_n \]

and since \( \nu \) is white \( E\{n_i(t)n_j(t)\} \) equals \( r_n \) if and only if \( i = j \) and it equals 0 otherwise. This is true also for different \( t \) in the same iteration, i.e., \( E\{n_k(t_1)n_k(t_2)\} \).

Note that since \( d(t) \) is a deterministic signal, the expected value becomes \( E\{d(t)\} = d(t) \).

The goal for the ILC algorithm applied to the system in (1) is to find an input signal \( u_k(t) \) such that the disturbance \( d(t) \) is completely compensated for. Clearly the optimal solution is to find a \( u_k \) such that

\[ u_k(t) = (G^0(q))^{-1} d(t) \]

which has also been discussed in Section 1. In the next sections, different iterative solutions to this problem will be discussed.

### 4 Analysis

#### 4.1 Notation

Before doing the analysis of the proposed ILC scheme some comments on the notation. In general when using ILC the error, i.e., the difference between a reference and the actual output of the system is studied, see for example [12]. For the disturbance rejection approach discussed here there is no reference since the goal is to have \( z_k(t) \) equal to zero. This can also be expressed in a more classical ILC framework as the error

\[ e_k(t) = r(t) - z_k(t) = -z_k(t) \]  

(9)

should be as close to zero as possible. The ILC algorithm does not have \( e_k(t) \) available, instead

\[ e_k(t) = r(t) - y_k(t) = -y_k(t) \]  

(10)
has to be used. Notice that with this definition of the error (7) becomes the same as (8). To make it easier to compare the results presented here with the previous results from, e.g., Chapter 4 of [9], [12] or [11], the notation in (9) and (10) will be used here.

4.2 $G^0(q)$ is known
Consider the estimator from (4),
\[
\hat{d}_k(t) = \frac{1}{k} \sum_{j=0}^{k-1} (y_k(t) - G(q)u_k(t))
\]
When the system is known, i.e., $G(q) = G^0(q)$, and the disturbance $n_k(t)$ is defined as in Section 3, then this estimator will asymptotically give an unbiased estimate of the disturbance $d(t)$,
\[
\lim_{k \to \infty} \hat{d}_k(t) = \lim_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k-1} (d(t) + n_j(t)) = d(t) \tag{11}
\]
From the ILC perspective this implies that the algorithm will converge to zero error. Obviously, it is not only the fact that the estimate is unbiased that is of interest. Also the variance of the estimate is an important property. The variance is given by,
\[
\text{Var}(\hat{d}_k(t)) = E\{\hat{d}_k^2(t)\} - (E\{\hat{d}_k(t)\})^2 = \frac{1}{k^2} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} (d(t) + n_i(t))(d(t) + n_j(t))
\]
\[
- d^2(t) = \frac{1}{k^2} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} E\{d^2(t) + d(t)n_i(t) + n_j(t) + n_i(t)n_j(t)\} - d^2(t) = \frac{r_n}{k} \tag{12}
\]
where the last equality follows from the fact that $d(t)$ is deterministic, $E\{n_i(t)\} = 0$, and that $E\{n_i(t)n_j(t)\} = r_n$ if $i = j$ and 0 otherwise, see also Section 3.

Interesting is also to see how the resulting control input, $u_k(t)$, develops. Using the updating equation in (7) with the true system $G^0(q)$ instead of $G(q)$ and $u_0(t) = 0$ the error $\epsilon_1(t)$ becomes,
\[
\epsilon_1(t) = -G^0(q)u_1(t) - d(t)
\]
\[
= G^0(q) \frac{1}{G^0(q)}(d(t) + n_0(t)) - d(t) \tag{13}
\]
\[
= n_0(t)
\]
This means that $d(t)$ is completely compensated for and the mathematical expectation of $\epsilon_1(t) = 0$ when the system is known. What can be improved is however the variance of $\epsilon_k(t)$. The variance of $\epsilon_1$ is readily calculated as
\[
\text{Var}(\epsilon_1(t)) = r_n \tag{14}
\]

The best result that can be achieved is when the disturbance $d(t)$ is perfectly known. This gives
\[
\epsilon(t) = G^0(q) \frac{1}{G^0(q)} d(t) - d(t) = 0 \tag{15}
\]
i.e., zero variance.

The proposed algorithm from (7) is evaluated in a simulation. The measure utilized in the evaluation is
\[
V_k = \frac{1}{r_n} \cdot \frac{1}{n-1} \sum_{t \in [0, t_f]} \epsilon_k^2(t) \tag{16}
\]
i.e., the variance of $\epsilon_k$ normalized with the variance of the measurement disturbance. From (14) it is clear that $V_1 = 1$ which is also shown in Figure 2. For $k = 0$ the measure $V_0$ does not correspond to a variance since $\epsilon_0(t) = -d(t)$. $V_0$ therefore depends only on the size of the disturbance $d(t)$. The simulation is however done to show what happens with the variance of the output $\epsilon_k(t)$ for $k \geq 1$.

A rapid decrease of $V_k$ can be seen in the first iterations. After 10 iterations, for example, $V_k$ is reduced to 0.1. To reduce the last 0.1 units down to 0, however, takes infinitely many iterations. The conclusion from this simulation is that the use of the proposed ILC algorithm gives an increased performance in the case when the system is completely known but the disturbance is unknown. In the next section the properties of the method will be examined when the system is not completely known.

4.3 Some notes on the asymptotic and transient behavior
In practice it is clear that a model of the true system has to be used in the ILC algorithm. In this section some results based on simulations will be discussed. The transient behavior of the proposed algorithm is highlighted and compared with another algorithm. In many applications it is often required that the algorithm should give a small error after, perhaps, the first
Consider the ILC updating scheme
\[ u_{k+1}(t) = u_k(t) + L_k(q)e_k(t) \] (17)
from (7) with \( e_k(t) \) defined as in (10) applied to the system in (1). The filters \( L_k(q) \) are chosen as
\[
L_k(q) = (G^0(q))^{-1} \quad (18a) \\
L_k(q) = \mu \cdot (G^0(q))^{-1} \quad (18b) \\
L_k(q) = (G(q))^{-1} \quad (18c) \\
L_k(q) = \frac{1}{k+1} (G^0(q))^{-1} \quad (18d) \\
L_k(q) = \frac{1}{k+1} (G(q))^{-1} \quad (18e)
\]
Assume that the ILC updating scheme in (17) gives a stable ILC system for all the different choices of \( L_k \)-filters in (18). The system \( G^0 \) is given by
\[ G^0(q) = \frac{0.07q^{-1}}{1 - 0.93q^{-1}} \] (19)
and the model \( G \) by
\[ G(q) = \frac{0.15q^{-1}}{1 - 0.9q^{-1}} \] (20)
To compare the transient behavior of the four ILC schemes created by using the updating scheme from (17) and the filters from (18) a simulation is performed. The system used in the simulation is given by (1) and the actual system description by (19). The model of the system, available for the ILC control scheme, is given by (20). The variance of the additive noise, \( n_k(t) \), is set to \( 10^{-3} \).
To evaluate the result from the simulations the following measure is used
\[ V(\epsilon_k) = \frac{1}{n-1} \sum_{t=1}^{n} \epsilon_k^2(t) \] (21)
which is an estimate of the variance if \( \epsilon_k \) is a random variable with zero mean. In Figure 3 the results from the simulations are shown. In the first iteration \( V(\epsilon_0) \) contains only the value of \( V(d) \) and for the \( d \) used in the simulations \( V(d) = 0.179 \). Obviously the ILC schemes, (18a) and (18d), give similar results in iteration 1 since both use the inverse of the true system to find the next control input. The pair, (18c) and (18e), give for the same reason similar results after one iteration. With the \( L_k \)-filter from (18b) having \( \mu = 0.25 \) the error reduces less rapidly than all the other choices of filters because of the low gain of the \( L \)-filter.

Figure 3 shows the general behavior that can be expected from the different ILC approaches covered by (17) and (18). It is clear that among the methods described here the approach given by (18d) is the best choice, although this method requires that the system description is completely known. If the system is not known as in (18e) the result may be not so good, cf. Figure 3.

For the asymptotic analysis the case when \( L_k \) is chosen according to (18a) is first considered. Since \( u_0(t) = 0 \), this means that \( \epsilon_0(t) = -d(t) \). From (1) it now follows that
\[ y_0(t) = d(t) + n_k(t) \]
and therefore
\[ u_1(t) = -(G^0)^{-1}(d(t) + n_0(t)) \]
The corresponding \( \epsilon_1(t) \) becomes \( \epsilon_1(t) = -n_0(t) \). This means that
\[ u_2(t) = -(G^0)^{-1}(d(t) + n_1(t)) \]
and \( \epsilon_2(t) = n_1(t) \). The asymptotic value of \( V(\epsilon_k) \) for \( k > 0 \) therefore becomes equal to \( r_n \), since with this approach \( \epsilon_k(t) = -n_k(t) \), \( k > 0 \).

A more general case is when a model of the system \( G^0 \) is used in the \( L \)-filter. This corresponds to the choice of filter for the ILC algorithm according to (18c). Obviously it is true that \( \epsilon_0(t) = -d(t) \) again. Now assume that the relation between the true system and the model can be described according to

\[
G^0(q) = (1 + \Delta G(q))G(q)
\]

where \( \Delta G(q) \) is a relative model uncertainty. Using the ILC updating scheme in (17) and the filter in (18c), it is straightforward to arrive at

\[
\epsilon_1(t) = \Delta G(q)d(t) + (1 + \Delta G(q))n_0(t)
\]

\[
= -\Delta G(q)\epsilon_0(t) + (1 + \Delta G(q))n_0(t)
\]

and in the general case

\[
\epsilon_{k+1}(t) = -\Delta G(q)\epsilon_k(t) + (1 + \Delta G(q))n_k(t)
\]

which can be expanded into the following finite sum

\[
\epsilon_{k+1}(t) = (1 + \Delta G(q))\sum_{j=0}^{k} (-\Delta G(q))^{j-1}n_{k-j}(t)
\]

\[
- (-\Delta G(q))^k d(t)
\]

Clearly (23) and (24) are valid also for the case when there is no model error, i.e., \( \Delta G(q) = 0 \). Note that a sufficient stability condition for this choice of algorithm is that \( \|\Delta G\|_\infty < 1 \) which can be interpreted as that the model can have an error of 100\% but still give a stable ILC algorithm (see also [6]). If there is no measurement disturbance the error will also converge to zero if the model error is less than 100\%. This is probably why ILC has shown so successful in practical applications.

To understand why, in this case, using a model of the system gives better asymptotic performance compared to using the true system, consider (23) and (24). If \( \|\Delta G\| < 1 \) then, for a large enough \( k \), the influence of \( d(t) \) can be neglected, since \( \|\Delta G\|^k \) becomes small. Now assume that the model uncertainty is mainly a scaling error, i.e., the dynamics are captured by the model. This means that \( \Delta G(q) = \delta \) for some \( \delta \), with \( |\delta| < 1 \). Since the effect of \( d(t) \) in \( \epsilon_k(t) \) is neglected the expected value of \( \epsilon_k(t) \) becomes equal to 0. The variance expression is found using, e.g., (23) and

\[
r_{e,k+1} = E\{\epsilon^2_{k+1}(t)\} \approx \delta^2 r_{e,k} + (1 + \delta)^2 r_n
\]

Asymptotically this means that

\[
r_{e,\infty} \approx r_n \cdot \frac{1 + \delta}{1 - \delta}
\]

In the example when \( L_k(q) \) from (18c) is used, \( \delta \approx -\frac{1}{2} \) and using the result in (26) it follows that \( r_{e,\infty} \approx \frac{r_n}{4} \), i.e., \( r_{e,\infty} \approx 0.3 \cdot 10^{-3} \). In fact this is also what is shown in Figure 3. The conclusion from this is that it is possible to get a lower value of \( V(\epsilon_k) \) asymptotically by choosing a model such that \( G^0(q) = \kappa G(q) \) for some \( 0 < \kappa < 1 \) and let \( L(q) = (G(q))^{-1} \). Clearly this is the answer why it is, in this case, better to use a model of the system. When applying (18b) this is also shown explicitly since there the learning filter is chosen as a constant, \( \mu = 0.25 \) times the true system. Asymptotically this approach gives the second best level of \( V_k \). By comparing the result using (18b) and (18c) it also becomes clear that the asymptotic level is achieved by reducing the transient performance. For an iteration invariant learning filter it is therefore a balance between achieving a low asymptotic level of \( V_k \) and having a rapid reduction of \( V_k \) in the first iterations.

When the true model is known and used in the filter according to (18d), then the value of \( V(\epsilon_k) \) becomes equal to \( \frac{r_n}{\kappa} \) for \( k > 0 \). For \( k = 0 \), \( V(\epsilon_0) \) is equal to \( V(d) \) since \( \epsilon_0 = -d \). If the true system is not known and the model based approach in (18e) is used, then the equation corresponding to (23) becomes

\[
\epsilon_{k+1}(t) = \frac{k - \Delta G(q)}{k + 1} \epsilon_k(t) + \frac{1 + \Delta G(q)}{k + 1} n_k(t)
\]

with \( \epsilon_0(t) = -d(t) \). To prove stability and find the asymptotic value of \( V(\epsilon) \) for (27) is left for future work. From Figure 3 is however clear that this method does not always give a good transient behavior. This depends on the fact that the disturbance \( d(t) \) is not completely compensated for in the first iteration. Since the gain is decreased at every iteration the amount completely compensated for in the first iteration. Since the disturbance \( d(t) \) is not completely compensated for in the first iteration.

\[\text{5 Conclusions}\]

The major contribution of this paper is to show that introducing a measurement disturbance together with iterative learning control and taking this disturbance into account the filters in the ILC algorithm becomes iteration variant. When using iteration invariant filters the gain of the \( L \)-filter becomes important and by adjusting this gain it is possible to reach a lower norm of the asymptotic error than just applying the inverse system model as a learning filter.

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References


