Variance Expressions for Spectra Estimated Using Auto-Regressions

Liang Liang Xie and Lennart Ljung

Department of Electrical Engineering
Linköping University, SE-581 83 Linköping, Sweden
WWW: <http://www.control.isy.liu.se>
Email: ljung@isy.liu.se

October 3, 2001

Report no.: LiTH-ISY-R-2361
Submitted to Journal of Econometrics

Technical reports from the Automatic Control group in Linköping are available by anonymous ftp at the address [ftp.control.isy.liu.se](ftp://ftp.control.isy.liu.se). This report is contained in the file 2361.pdf.

Variance Expressions for Spectra Estimated Using Auto-Regressions*

Liang-Liang Xie

Lennart Ljung

Institute of Systems Science

Dept of Electrical Engineering

Chinese Academy of Sciences

Linköping University

100080, Beijing, China

581 83, Linköping, Sweden

June 28, 2001

Abstract

An expression for the variance of the estimated spectrum based on auto-regressions is developed. This expression is asymptotic in the number of data, but exact in the model order. As the order tends to infinity it converges to the well known result that the variance is proportional to the model order times the square of the spectrum itself. The exact expression gives insight into the character of this convergence, its speed and its dependence on the poles of the underlying AR-process.

*The work of the first author was completed while visiting Linköping University as Guest Researcher. Please address all correspondence to the second author Professor L. Ljung. E-mail: ljung@isy.liu.se

1 Introduction

In this paper we will study the asymptotic properties of spectral estimates based on Autoregressive (AR) models of time series. This is of course a much-studied subject with many well known results. Our contribution will be an exact expression for the asymptotic variance (or rather, the variance of the asymptotic distribution) of the spectral estimate for finite order AR models. In particular, this result illuminates the character of the convergence to the well known asymptotic expressions as the order of the AR model tends to infinity. The role of the poles of the underlying true AR-process will also be displayed.

The setup is as follows: Consider a time series $y(t), t = 1, 2, \dots$. Denote its spectral density by $\Phi_y(\omega)$. From N observations from the time series, an AR-model of order n is estimated by minimizing

$$\sum_{t=n+1}^N (y(t) + a_1 y(t-1) + \dots + a_n y(t-n))^2, \quad (1)$$

w.r.t. $a_k, 1 \leq k \leq n$, yielding the estimates $\hat{a}_k(N)$. The variance of the innovation process is estimated by

$$\hat{\lambda}_N = \frac{1}{N-n} \sum_{t=n+1}^N (y(t) + \hat{a}_1(N)y(t-1) + \dots + \hat{a}_n(N)y(t-n))^2. \quad (2)$$

Then the estimate of the spectral density of y is formed as

$$\hat{\Phi}_N(\omega) = \frac{\hat{\lambda}_N}{|\hat{A}_N(\omega)|^2}, \quad (3)$$

$$\hat{A}_N(\omega) = 1 + \hat{a}_1(N)e^{-j\omega} + \dots + \hat{a}_n(N)e^{-jn\omega}. \quad (4)$$

The asymptotic properties of this estimate are well known, e.g. [2] or [3]: Suppose that

the given process $y(t)$ indeed can be described by a r :th order AR-process:

$$y(t) + a_1 y(t-1) + \dots + a_r y(t-r) = e(t); \quad (5)$$

$$E e^2(t) = \lambda; \quad (6)$$

$$E(e^2(t) - \lambda)^2 = \mu, \quad (7)$$

where $e(t)$ is a sequence of i.i.d. random variables. Then, if $n \geq r$ and we denote

$$\hat{\theta}_N = \begin{pmatrix} \hat{a}_1(N) \\ \vdots \\ \hat{a}_n(N) \end{pmatrix} \quad (8)$$

and let θ_0 be the corresponding vector of the true AR-parameters, we have that

$$\sqrt{N}(\hat{\theta}_N - \theta_0) \in \text{AsN}(0, P); \quad (9a)$$

$$P = \lambda[E\psi_t\psi_t^T]^{-1}; \quad (9b)$$

$$\psi_t = \begin{pmatrix} y(t-1) \\ \vdots \\ y(t-n) \end{pmatrix}; \quad (9c)$$

$$\sqrt{N}(\hat{\lambda}_N - \lambda) \in \text{AsN}(0, \mu). \quad (9d)$$

Here $x_N \in \text{AsN}(a, B)$ means that the random variable x_N converges in distribution to the normal distribution with mean a and covariance matrix B as N tends to infinity. Moreover, the joint distribution of the two estimates is normal with $\hat{\theta}$ independent of $\hat{\lambda}$.

From (9) all results about the asymptotic distribution of $\hat{\Phi}_N(\omega)$ readily can be derived. We will be particularly interested in the variance of the asymptotic distribution of $\sqrt{N}(\hat{\Phi}_N(\omega) - \Phi_y(\omega))$ and denote it by $\Lambda_n(\omega)$:

$$\sqrt{N}(\hat{\Phi}_N(\omega) - \Phi_y(\omega)) \in \text{AsN}(0, \Lambda_n(\omega)) \quad (10)$$

It can be computed from (9), but this gives a rather implicit and complex expression. As the order of the AR-model, n , tends to infinity, the expression for $\Lambda_n(\omega)$ simplifies considerably:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \Lambda_n(\omega) = \begin{cases} 2\Phi_y^2(\omega) & \text{if } \omega \neq 0, \omega \neq \pi; \\ 4\Phi_y^2(\omega) & \text{if } \omega = 0 \text{ or } \omega = \pi. \end{cases} \quad (11)$$

This has been proved by [5], for the case of a fixed regression order, and by [1] for the case that n tends to infinity at the same time as N . The case of an input (exogenous process) present is treated in e.g., [6] and [4].

The purpose of the current paper is to give an explicit expression for $\Lambda_n(\omega)$ as a function of n . This will display the nature of the convergence taking place in (11), the convergence rate, and also the role that is played by the poles of the true underlying AR-process.

The main technical result is stated and proved in the next section. Some discussion of the result then follows in Section 3. Throughout the remainder of the paper, we will for simplicity abuse notation and write

$$\lim_{N \rightarrow \infty} N \cdot \text{Var } \hat{x}_N = P \quad (12)$$

to mean

$$\sqrt{N}(\hat{x}_N - E\hat{x}_N) \in \text{AsN}(0, P) \quad (13)$$

2 The Main Result

We will use polynomials in the shift operator q for efficient notation and write

$$A_0(q)y(t) = e(t) \quad (14)$$

for the true AR-representation (5). This means that

$$A_0(q) = 1 + a_1 q^{-1} + \dots + a_r q^{-r}.$$

Similarly the n th order model used in (1) – (4) will be written as

$$A(q, \theta)y(t) = e(t), \quad (15)$$

where θ collects the parameters a_k , $1 \leq k \leq n$ in a column vector. The corresponding estimate is denoted $\hat{\theta}_N$ as in (8).

Thus, from (4)

$$\hat{A}_N(\omega) = A(e^{j\omega}, \hat{\theta}_N) = 1 + \sum_{k=1}^n \hat{a}_k(N) e^{-j\omega k} = 1 + W^T(e^{j\omega}) \hat{\theta}_N, \quad (16)$$

where $\hat{\theta}_N$ contains the LS-estimated AR-parameters and with an obvious definition of $W(e^{j\omega})$.

We have the following result.

Theorem 2.1 *Consider the AR-process (14) of order r . Let the A -polynomial be estimated as an n :th order AR model (15) and assume that $n \geq r$. Let the roots of the true AR-polynomial $z^r A_0(z)$ be α_k , $k = 1, \dots, r$. Suppose that the $r - 2\rho$ first ones are real and the remaining 2ρ ones are complex conjugated: $\alpha_{r-2\rho+2k-1} = \beta_k$, $\alpha_{r-2\rho+2k} = \bar{\beta}_k$, $k = 1, \dots, \rho$. From the roots define*

$$\Pi_k \triangleq e^{-2(n-r)j\omega} \prod_{m=1}^k \frac{(1 - \bar{\alpha}_m e^{j\omega})^2}{(e^{j\omega} - \alpha_m)^2}, \quad 0 \leq k \leq r-1, \quad (17)$$

and from the complex conjugated roots define for $1 \leq i \leq \rho$,

$$C(\beta_i) \triangleq (1 - |\beta_i|^2) \cdot \frac{(1 - \beta_i e^{j\omega})(1 - \bar{\beta}_i e^{j\omega}) + (e^{j\omega} - \beta_i)(e^{j\omega} - \bar{\beta}_i)}{(e^{j\omega} - \beta_i)^2 (e^{j\omega} - \bar{\beta}_i)^2}; \quad (18)$$

Then the variance of the $A(e^{j\omega}, \hat{\theta}_N)$ -function is given by

$$\lim_{N \rightarrow \infty} N \cdot \text{Var}\{A(e^{j\omega}, \hat{\theta}_N)\} = |A_0(e^{j\omega})|^2 \left[(n-r) + \sum_{m=1}^r \frac{1 - |\alpha_m|^2}{|e^{j\omega} - \alpha_m|^2} \right], \quad (19)$$

and the variance of its squared amplitude is given by

$$\lim_{N \rightarrow \infty} N \cdot \text{Var}\{|A(e^{j\omega}, \hat{\theta}_N)|^2\} = 2|A_0(e^{j\omega})|^4 S_n(\omega, A_0), \quad (20)$$

where,

$$\begin{aligned} S_n(\omega, A_0) = & \left\{ (n-r) + \sum_{k=1}^r \frac{1 - |\alpha_k|^2}{|e^{j\omega} - \alpha_k|^2} + \sum_{p=1}^{n-r} \cos(2p\omega) \right. \\ & \left. + \text{Re} \left\{ \sum_{k=1}^{r-2\rho} \frac{1 - |\alpha_k|^2}{(e^{j\omega} - \alpha_k)^2} \Pi_{k-1} + \sum_{i=1}^{\rho} C(\beta_i) \Pi_{r-2(\rho-i+1)} \right\} \right\}, \end{aligned} \quad (21)$$

with $C(\beta_i)$ Π_k defined in (18) and (17).

Proof. From (9) the covariance matrix of the LS-estimated AR-parameters $\hat{\theta}_N$ is given by

$$\lim_{N \rightarrow \infty} N \cdot \text{Cov} \hat{\theta}_N = \lambda R^{-1}, \quad (22)$$

where

$$R = E\{\psi_t \psi_t^T\}, \quad \psi_t \triangleq \frac{d}{d\theta} [A(q, \theta)y(t)] = [y(t-1), y(t-2), \dots, y(t-n)]^T. \quad (23)$$

By Parseval's relationship and equation (14),

$$R = T(\Phi_y(\omega)), \quad \Phi_y(\omega) = \frac{\lambda}{|A_0(e^{j\omega})|^2}, \quad (24)$$

where

$$T(f(\omega)) \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} W(e^{j\omega}) W^*(e^{j\omega}) f(\omega) d\omega, \quad (25)$$

with $W(e^{j\omega})$ defined in (16) and $(\cdot)^*$ denoting the conjugate transpose.

Then by (16) the variance of the A -function estimate can be calculated by

$$\lim_{N \rightarrow \infty} N \cdot \text{Var}\{A(e^{j\omega}, \hat{\theta}_N)\} = \lambda W^*(e^{j\omega}) T^{-1}(\Phi_y(\omega)) W(e^{j\omega}). \quad (26)$$

But it is hard to convert (26) into a simple analytic form, which obviously should only depend on A_0 .

In order to get a simple analytic expression, we introduce a virtual time series $\{v(t)\}$ defined by

$$A_1(q)y(t) = v(t). \quad (27)$$

Then by (15), we have a new model

$$\frac{A(q, \theta)}{A_1(q)}v(t) = e(t). \quad (28)$$

The model (28) may not be actually used, but using it will result in the same variance as using (15), and the expression may be easier to simplify. That is indeed the case if $A_1 = A_0$. Then $v(t) = e(t)$ is i.i.d. and the new model (28) becomes

$$\frac{A(q, \theta)}{A_0(q)}v(t) = e(t). \quad (29)$$

Now, supposing α_k , $1 \leq k \leq r$ are the roots of $A_0(q)$ ($\max_{1 \leq k \leq r} |\alpha_k| < 1$ for the AR-process (14) to be stationary), if we reparameterize model (29) using the orthonormal basis (see [7])

$$\Gamma_{n,r}(q) = [\gamma_1(q), \gamma_2(q), \dots, \gamma_n(q)]^T, \quad (30)$$

where, $\gamma_i(q)$, $1 \leq i \leq n$ is defined by

$$\gamma_i(q) \triangleq \begin{cases} q^{-i}, & 1 \leq i \leq n-r; \\ b_{i-(n-r)}(q) q^{-(n-r)}, & n-r < i \leq n, \end{cases} \quad (31)$$

with

$$b_k(q) \triangleq \frac{\sqrt{1 - |\alpha_k|^2}}{q - \alpha_k} \prod_{m=1}^{k-1} \frac{1 - \bar{\alpha}_m q}{q - \alpha_m}, \quad 1 \leq k \leq r, \quad (32)$$

then we have a new equivalent model

$$A'(q, \theta')v(t) = e(t), \quad (33)$$

$$\begin{aligned} \text{with } A'(q, \theta') &\triangleq \frac{1}{A_0(q)} + [\gamma_1(q), \gamma_2(q), \dots, \gamma_n(q)] \cdot \theta' \\ &= \frac{1}{A_0(q)} + \theta'_1 q^{-1} + \dots + \theta'_{n-r} q^{-(n-r)} \\ &\quad + \theta'_{n-r+1} b_1(q) q^{-(n-r)} + \dots + \theta'_n b_r(q) q^{-(n-r)}, \end{aligned} \quad (34)$$

and there exists some linear relationship between θ and θ' . Note that θ' would be complex parameters if A_0 has complex roots.

Let $\hat{\theta}'_N$ denote the LS-estimate of the parameters θ' in model (33)-(34). Denote $\tilde{\theta}'_N = \hat{\theta}'_N - E\hat{\theta}'_N$. It is apparent that

$$\lim_{N \rightarrow \infty} N \cdot \text{Var}\{A(e^{j\omega}, \hat{\theta}_N)\} = |A_0(e^{j\omega})|^2 \lim_{N \rightarrow \infty} N \cdot \text{Var}\{A'(e^{j\omega}, \hat{\theta}'_N)\}; \quad (35)$$

$$\lim_{N \rightarrow \infty} N \cdot E\{[A(e^{j\omega}, \hat{\theta}_N) - A_0(e^{j\omega})]^2\} = [A_0(e^{j\omega})]^2 \lim_{N \rightarrow \infty} N \cdot E\{[A'(e^{j\omega}, \hat{\theta}'_N) - 1]^2\} \quad (36)$$

By the same arguments as arriving at (22), the covariance matrix of $\hat{\theta}'_N$ is given by

$$\lim_{N \rightarrow \infty} N \cdot \text{Cov}\hat{\theta}'_N \triangleq \lim_{N \rightarrow \infty} N \cdot E(\tilde{\theta}'_N)(\tilde{\theta}'_N)^* = \lambda(R')^{-1}, \quad (37)$$

where,

$$R' = E\{\psi'_t \psi'^{*}_t\}, \quad \psi'_t \triangleq \frac{d}{d\theta'} [A'(q, \theta')v(t)] = [\gamma_1(q), \gamma_2(q), \dots, \gamma_n(q)]^T v(t). \quad (38)$$

Still by Parseval's relationship and that $v(t) = e(t)$,

$$R' = M_{n,r}(\Phi_v(\omega)), \quad \Phi_v(\omega) = \lambda, \quad (39)$$

where

$$M_{n,r}(f(\omega)) \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} \Gamma_{n,r}(e^{j\omega}) \Gamma_{n,r}^*(e^{j\omega}) f(\omega) d\omega, \quad (40)$$

with $\Gamma_{n,r}(q)$ defined in (30).

One advantage with expression (39)-(40) is that $\Phi_v(\omega)$ is constant. Since $\Gamma_{n,r}(q)$ is an orthonormal basis, it is easy to see that $M_{n,r}(1) = I_n$. Hence $R' = \lambda I_n$. Then by (37),

$$\lim_{N \rightarrow \infty} N \cdot \text{Cov} \hat{\theta}'_N = I_n. \quad (41)$$

As we mentioned before, θ' would be complex numbers if A_0 has complex roots. Hence, generally $E(\tilde{\theta}'_N)(\tilde{\theta}'_N)^T \neq \text{Cov} \hat{\theta}'_N$. Instead, we have

$$\lim_{N \rightarrow \infty} N \cdot E(\tilde{\theta}'_N)(\tilde{\theta}'_N)^T = \lambda R_1^{-1}, \quad (42)$$

where $R_1 = E\{\psi'_t \psi'^T_t\}$ with ψ'_t defined in (38). Moreover,

$$R_1 = \frac{\lambda}{2\pi} \int_{-\pi}^{\pi} \Gamma_{n,r}(e^{j\omega}) \Gamma_{n,r}^T(e^{-j\omega}) d\omega.$$

Note that A_0 has ρ pairs of complex conjugate roots: $\alpha_{r-2\rho+1} = \beta_1$, $\alpha_{r-2\rho+2} = \bar{\beta}_1$, $\dots$, $\alpha_{r-1} = \beta_\rho$, $\alpha_r = \bar{\beta}_\rho$. Then it is not difficult to calculate that

$$R_1 = \begin{bmatrix} I_{n-2\rho} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathcal{B}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{B}_\rho \end{bmatrix} \quad (43)$$

where for $i = 1, \dots, \rho$,

$$\mathcal{B}_i = \frac{1}{1 - \beta_i^2} \begin{bmatrix} 1 - |\beta_i|^2 & \beta_i - \bar{\beta}_i \\ \beta_i - \bar{\beta}_i & 1 - |\beta_i|^2 \end{bmatrix},$$

and

$$\mathcal{B}_i^{-1} = \frac{1}{1 - \bar{\beta}_i^2} \begin{bmatrix} 1 - |\beta_i|^2 & \bar{\beta}_i - \beta_i \\ \bar{\beta}_i - \beta_i & 1 - |\beta_i|^2 \end{bmatrix}.$$

From the model structure (34) and (41)-(43), it follows that

$$\begin{aligned}\lim_{N \rightarrow \infty} N \cdot \text{Var}\{A'(e^{j\omega}, \hat{\theta}'_N)\} &= \sum_{i=1}^n |\gamma_i(e^{j\omega})|^2; \\ \lim_{N \rightarrow \infty} N \cdot \text{E}\{[A'(e^{j\omega}, \hat{\theta}'_N) - 1]^2\} &= \sum_{i=1}^{n-2\rho} [\gamma_i(e^{j\omega})]^2 + \sum_{i=1}^{\rho} C(\beta_i) \Pi_{r-2(\rho-i+1)},\end{aligned}$$

where $C(\beta_i)$, Π_k are defined in (18) and (17).

Noticing the relations (35) and (36), we have for the original model (15)

$$\lim_{N \rightarrow \infty} N \cdot \text{Var}\{A(e^{j\omega}, \hat{\theta}_N)\} = |A_0(e^{j\omega})|^2 \sum_{i=1}^n |\gamma_i(e^{j\omega})|^2; \quad (44)$$

$$\begin{aligned}\lim_{N \rightarrow \infty} N \cdot \text{E}\{[A(e^{j\omega}, \hat{\theta}_N) - A_0(e^{j\omega})]^2\} &= [A_0(e^{j\omega})]^2 \\ &\times \left\{ \sum_{i=1}^{n-2\rho} [\gamma_i(e^{j\omega})]^2 + \sum_{i=1}^{\rho} C(\beta_i) \Pi_{r-2(\rho-i+1)} \right\}.\end{aligned} \quad (45)$$

Based on the expressions (44) and (45), we now calculate the variance of the amplitude estimate $|\hat{A}_N|^2$. First it is easy to prove that

$$\begin{aligned}\text{Var}\{|\hat{A}_N|^2\} &= \text{E}(|\hat{A}_N|^2 - E|\hat{A}_N|^2)^2 = \text{E}(|\hat{A}_N|^2 - |A_0|^2 - E|\tilde{A}_N|^2)^2 \\ &= \text{E}|\tilde{A}_N|^4 + 2|A_0|^2 \text{E}|\tilde{A}_N|^2 + 2\text{Re}\{\bar{A}_0^2 \text{E}\tilde{A}_N^2\} + 4\text{Re}\{\bar{A}_0 \text{E}\tilde{A}_N |\tilde{A}_N|^2\} - (E|\tilde{A}_N|^2)^2\end{aligned} \quad (46)$$

where, $\tilde{A}_N \triangleq \hat{A}_N - A_0$, which tends to zero at the rate $\frac{1}{\sqrt{N}}$. Omitting higher order terms in (46), we have

$$\lim_{N \rightarrow \infty} N \cdot \text{Var}\{|\hat{A}_N|^2\} = 2|A_0|^2 \times \lim_{N \rightarrow \infty} N \cdot \text{E}|\tilde{A}_N|^2 + 2\text{Re}\{\bar{A}_0^2 \times \lim_{N \rightarrow \infty} N \cdot \text{E}\tilde{A}_N^2\}. \quad (47)$$

Now using (44) and (45), we have

$$\begin{aligned}
\lim_{N \rightarrow \infty} N \cdot \text{Var}\{|\hat{A}_N|^2\} &= 2|A_0(e^{j\omega})|^4 \left\{ \sum_{i=1}^n |\gamma_i(e^{j\omega})|^2 \right. \\
&\quad \left. + \text{Re} \left\{ \sum_{i=1}^{n-2\rho} [\gamma_i(e^{j\omega})]^2 + \sum_{i=1}^{\rho} C(\beta_i) \Pi_{r-2(\rho-i+1)} \right\} \right\} \\
&= 2|A_0(e^{j\omega})|^4 \left\{ (n-r) + \sum_{k=1}^r |b_k(e^{j\omega})|^2 \right. \\
&\quad \left. + \sum_{p=1}^{n-r} \cos(2p\omega) + \text{Re} \left\{ e^{-2(n-r)\omega j} \sum_{k=1}^{r-2\rho} [b_k(e^{j\omega})]^2 + \sum_{i=1}^{\rho} C(\beta_i) \Pi_{r-2(\rho-i+1)} \right\} \right\} \\
&= 2|A_0(e^{j\omega})|^4 \left\{ (n-r) + \sum_{k=1}^r \frac{1 - |\alpha_k|^2}{|e^{j\omega} - \alpha_k|^2} + \sum_{p=1}^{n-r} \cos(2p\omega) \right. \\
&\quad \left. + \text{Re} \left\{ \sum_{k=1}^{r-2\rho} \frac{1 - |\alpha_k|^2}{(e^{j\omega} - \alpha_k)^2} \Pi_{k-1} + \sum_{i=1}^{\rho} C(\beta_i) \Pi_{r-2(\rho-i+1)} \right\} \right\}.
\end{aligned}$$

3 Discussion

First, from the theorem, we can compute the asymptotic variance of the spectral estimate by applying Gauss' Approximation formula to (3):

$$\text{Var}\hat{\Phi}_N(\omega) = \frac{1}{|A_0(\omega)|^4} \text{Var}\hat{\lambda}_N + \frac{\lambda^2}{|A_0(\omega)|^8} \text{Var}\{|\hat{A}_N(\omega)|^2\}. \quad (48)$$

Inserting (9d) and (20) gives

$$\lim_{N \rightarrow \infty} N \cdot \text{Var}\hat{\Phi}_N(\omega) = \Lambda_n(\omega) = \frac{\mu}{\lambda^2} \Phi_y^2(\omega) + 2\Phi_y^2(\omega) S_n(\omega, A_0), \quad (49)$$

where S_n is defined by (21).

Moreover, it follows from (21) that S_n is of the form

$$S_n(\omega, A_0) = n + g_n(\omega) + f_{r,n}(\omega), \quad (50)$$

where,

$$g_n(\omega) = \sum_{p=1}^n \cos(2p\omega) = \begin{cases} \operatorname{Re} \left\{ \frac{e^{2jn\omega} - 1}{e^{-2j\omega} - 1} \right\} & \text{if } \omega \neq 0, \pi; \\ n & \text{else.} \end{cases} \quad (51)$$

The term $f_{r,n}$ is $-r - \sum_{p=n-r+1}^n \cos(2p\omega)$ plus the second, fourth and fifth terms of (21). These sums contain no more than $3r$ terms together, and $f_{r,n}$ depends on n only via $\sum_{p=n-r+1}^n \cos(2p\omega)$ and Π_k . Obviously, $|\sum_{p=n-r+1}^n \cos(2p\omega)| < r$. On the other hand, $|\Pi_k|$ is n -independent, so it is clear that $f_{r,n}$ is bounded by an n -independent constant.

Now, for fixed r it is clear from (21) and (51) that we have the limits

$$\lim_{n \rightarrow \infty} \frac{1}{n} S_n(\omega) = \begin{cases} 1 & \text{if } \omega \neq 0, \pi; \\ 2 & \text{else.} \end{cases} \quad (52)$$

So the classical result (11) is re-established. It also shows that the convergence rate in (52) is always like $1/n$.

It is also clear from the expressions that the convergence will depend on the pole location of the true AR-process A_0 . For example, if all poles are in the origin, i.e., $\alpha_k \equiv 0$, it follows that the second term of (21) will be r and it can be verified that the sum of the fourth and fifth terms will be $\sum_{p=n-r+1}^n \cos(2p\omega)$. Generally speaking, a pole close to the origin ($\alpha_k \approx 0$) will make $\frac{1}{n} S_n$ deviate less from its limit as $n \rightarrow \infty$.

References

- [1] K.N. Berk. Consistent autoregressive spectral estimates. *Ann. Stat.*, 2:489–502, 1974.
- [2] E. J. Hannan. *Multiple Time Series*. Wiley, New York, 1970.
- [3] E. J. Hannan and M. Deistler. *The Statistical Theory of Linear Systems*. John Wiley, New York, 1988.

- [4] E.J. Hannan and L. Kavalieris. Multivariate liner time series models. *Adv. Appl. Prob.*, 16:492–561, 1984.
- [5] R.E. Kromer. *Asymptotic properties of the autoregressive spectral estimator*. PhD thesis, Stanford University, 1970.
- [6] L. Ljung and B. Wahlberg. Asymptotic properties of the least-squares method for estimating transfer functions and disturbance spectra. *Adv. Appl. Prob.*, 24:412–440, 1992.
- [7] B.Ninness, H. Hjalmarsson, and F. Gustafsson. Generalized Fourier and Toeplitz results for rational orthonormal bases. *SIAM J. Contr. Optim.*, 37(2):429–460, 1999.