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On-line Identification and Adaptive Trajectory Tracking for Nonlinear Stochastic Continuous Time Systems Using Differential Neural Networks

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Abstract

Identification of nonlinear stochastic processes via differential neural networks is discussed. A new "dead-zone" type learning law for the weight dynamics is suggested. By a stochastic Lyapunov-like analysis the stability conditions for the identification error as well as for the neural network weights are established. The adaptive trajectory tracking using the obtained neural network model is realized for the subclass of stochastic completely controllable processes linearly dependent on control. The upper bounds for the identification and adaptive tracking errors are established.

Keywords: Dynamic Neural Networks, Stochastic Processes, Identification, Adaptive Control.

1. Introduction

Due to the big enthusiasm generated by the successful applications, the use of *Neural Network technique* seems to be a very effective tool to control a wide class of complex nonlinear systems in the absence of a complete model information or, even, considering a controlled plant as "a black box" (Hunt *et al.*, 1992). The NN's can be qualified as static (feedforward) and as dynamic (recurrent or differential) nets. The most of recent publications (see, for example, Haykin, 1994; Agarval, 1997; Parizini and Zoppoli, 1994) deals with *Static* Neural Nets providing the appropriate approximation of a nonlinear operator functions in the right-hand side of dynamic model equations. Although they have been used successfully, the major disadvantages are a slow learning rate (the weights updates do not utilize the information on the local NN structure) and a high sensitivity of the function approximation to the training

data (they do not have memory, so their outputs are uniquely determined by the current inputs and weights). *Dynamic Neural Networks* can successfully overcome these drawbacks and demonstrate a workable behavior in the presence of essential unmodelled dynamics because their structure incorporate feedback. They have been introduced by Hopfield, 1984 and then studied in (Sandberg, 1991 (the approximation aspects), Rivithakis and Cristodoulou, 1994 (the direct adaptive regulation via DNN), Poznyak *et al.*, 1999 (the adaptive tracking using DNN)). Several advance results concerning DNN have been recently obtained in (Narendra and Li, 1998 (the identification and control), Lewis and Parisini, 1998 (NN feedback control: the stability analysis) and in (Rivithakis and Cristodoulou, 2000). All of these publications concern NN applications considered specifically only for the external environment of the deterministic nature, that is, all dynamic trajectories as well as external perturbations and internal uncertainties are assumed to be deterministic functions of time. On our knowledge, there are a few publications dealing with the identification or control of stochastic processes with unknown dynamics via NN. So, in (Kosmatopulos and Christodoulou, 1994) the Boltzmann type recurrent high-order NN (g-RHONN without a hidden layer) was applied for the estimation of the unknown density of a "transmitted" stochastic signal.

This paper is our first attempt to spread the ideas of the Neuro Control approach to the class of continuous time stochastic processes with incomplete dynamic model description. Dynamic Neural Networks given by differential equations (*Differential Neural Networks*) are applied for the identification and adaptive trajectory tracking purposes. The main goal of the paper is to show that practically the same DNN, designed for the deterministic systems with bounded perturbations, is robust with respect to perturbations of a *stochastic nature* which are unbounded (with probability one). The *stochastic calculus* (such as Itô formula and *the martingale technique* together with *the strong large number law* implementation are shown to be the effective instrument for the investigation of such systems.

2. Study Motivation

State feedback control is one of the topic acquiring big importance and attention in engineering publications (Isidori, 1995). To realized *the deterministic trajectory tracking* of a nonlinear controlled system given by

$$\dot{x} = f(x, t) + g(x, t)u \quad (1)$$

based on the available information on the reference dynamics $\dot{x}^* = \bar{\varphi}(x^*, t)$, the successful feedback control u (compensating arising nonlinear effects) can be designed as the nonlinear feedback satisfying

$$g(x, t)u = [\bar{\varphi}(x^*, t) - f(x, t) + A^*(x - x^*)], \quad A^* \text{ - stable} \quad (2)$$

The control (3) is realizable if and only if the following "controllability-rank" condition is fulfilled. A possible control actions (having the minimal norm) and verifying (2) is as follows¹

$$u = g^+(x, t) [\bar{\varphi}(x^*, t) - f(x, t) + A^*(x - x^*)] \quad (3)$$

It provides the global *exponential stability property* for the corresponding closed-loop system as

$$\begin{aligned} \frac{d}{dt} \|x - x^*\|_P^2 &= 2(x - x^*)^\top P (f(x, t) + g(x, t)u - \bar{\varphi}(x^*, t)) = \\ &= (x - x^*)^\top (PA^* + A^{*\top}P) (x - x^*) = - (x - x^*)^\top Q (x - x^*) \\ PA^* + A^{*\top}P &= -Q < 0 \end{aligned}$$

and, as the result, $\|x - x^*\|_P^2 \leq \|x_0 - x^*\|_P^2 \exp(-\lambda_{\min}(P^{-1/2}QP^{-1/2})t)$. To realize (3) and the most of other feedback controls, the only one condition must be fulfilled: *the complete information on the given dynamics should be available*.

The question is: what to do in the case when we deal with a partially or completely *unknown dynamics*, that is, the "black-box" or "grey-box" (there is an unmodelled dynamic part) situations are of interest or, may be, the dynamics of a given system is disturbed by an external noise of a stochastic nature. One of the possible solutions of this problems is related to *Adaptive Control* implementation (Krstić *et al.*, 1995; Duncan *et al.*, 1999) based on *Identification Technique* (Ljung, 1999; Ljung and Gunnarson, 1990) providing a designer some sort of model estimates $\hat{x}_t$ generated by the model equation

$$\frac{d}{dt} \hat{x} = \hat{f}(\hat{x}, t \mid \mathbf{c}_t) + \hat{g}(\hat{x}, t \mid \mathbf{c}_t)u_t \quad (4)$$

The right-hand side functions can be used in an applied control law, for example, in (3) instead of unknown nonlinear functions participating in the dynamic model description, that is,

$$u = \hat{g}^+(\hat{x}, t \mid \mathbf{c}_t) \left[\bar{\varphi}(x^*, t) - \hat{f}(\hat{x}, t \mid \mathbf{c}_t) + A^*(\hat{x} - x^*) \right] \quad (5)$$

or, the same expression, but only x is used instead of $\hat{x}$ (since x is measurable). The adjusting or *learning law* $\dot{\mathbf{c}}_t = \mathbf{c}_t(\hat{x}_t, x_t, u_t)$ to be designed for the parameters $\mathbf{c}_t$ realizes, so-called, *the identification process* providing the "closeness" of the model states $\hat{x}_t$ to the real system state vector x_t .

The specific features of the problem, tackled in this paper, are as follows: first, the class of unknown dynamic systems is an extension of (1) including ones given by *Stochastic*

¹ $[\cdot]^+$ is the pseudoinverse (in Moor-Penrose sense) operator acting as $AA^+A = A$, $A^+AA^+ = A^+$, $0^+ = 0$, $c^+ = c^\top / \|c\|^2$ (if c is a non-zero vector).

Nonlinear Differential Equations with "smooth enough" regular part; second, the class of the model structures (4), used for the identification purposes, includes, so-called, *Differential Neural Networks* (Dynamic Neural Networks functioning in continuous time) which weights are adjusted on-line by a special "differential learning procedure" (Poznyak *et al.*, 1999) and used, simultaneously, in the corresponding control law to provide a qualitative trajectory tracking process.

3. Uncertain Stochastic System

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbf{P})$ be a given filtered probability space ($(\Omega, \mathcal{F}, \mathbf{P})$ is complete, the $\mathcal{F}_0$ contains all the $\mathbf{P}$ -null sets in $\mathcal{F}$, the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is right continuous: $\mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t$). On this probability space define an m -dimensional standard Brownian motion, that is, $(\bar{W}_t, t \geq 0)$ (with $\bar{W}_0 = 0$) is an $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted $\mathbb{R}^m$ -valued process such that

$$\begin{aligned} E\{\bar{W}_t - \bar{W}_s \mid \mathcal{F}_s\} &= 0 \quad \mathbf{P} - a.s., \quad \mathbf{P}\{\omega \in \Omega : \bar{W}_0 = 0\} = 1 \\ E\{[\bar{W}_t - \bar{W}_s][\bar{W}_t - \bar{W}_s]^\top \mid \mathcal{F}_s\} &= (t - s)I \quad \mathbf{P} - a.s. \end{aligned}$$

Consider the stochastic nonlinear controlled continuous-time system with the dynamics x_t :

$$x_t = x_0 + \int_{s=0}^t b(s, x_s, u_s) dt + \int_{s=0}^t c(s, x_s, u_s) d\bar{W}_s \quad (6)$$

or, in the abstract (symbolic) form,

$$dx_t = b(t, x_t, u_t) dt + c(t, x_t, u_t) d\bar{W}_t, \quad x_0 = x_{given}, \quad t \in [0, \infty) \quad (7)$$

The first integral in (6) is an stochastic ordinary integral and the second one is an Itô integral. In the above $u_t \in U \subseteq \mathbb{R}^p$ is a control at time t , $b : [0, \infty) \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$, and $c : [0, \infty) \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^{n \times m}$. It is assumed that

A1: $\{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration generated by $(\bar{W}_t, t \geq 0)$ and augmented by the $\mathbf{P}$ -null sets from $\mathcal{F}$.

A2: (U, d) is a separable metric space with a metric d .

Definition 1. The function $f : [0, \infty) \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^{n \times m}$ is said to be an $L_\phi(C^2)$ -**mapping** if it is Borel measurable and is C^2 in x (almost everywhere) for any $t \in [0, \infty)$ and any $u \in U$ and also there exist a constant L and a modulus of continuity $\phi : [0, \infty) \rightarrow [0, \infty)$ such that for any $t \in [0, \infty)$ and for any $x, u, \hat{x}, \hat{u} \in \mathbb{R}^n \times U \times \mathbb{R}^n \times U$

$$\begin{aligned} \|f(t, x, u) - f(t, \hat{x}, \hat{u})\| &\leq L\|x - \hat{x}\| + \phi(d(u, \hat{u})), \quad \|f(t, 0, u)\| \leq L \\ \|f_x(t, x, u) - f_x(t, \hat{x}, \hat{u})\| &\leq L\|x - \hat{x}\| + \phi(d(u, \hat{u})) \\ \|f_{xx}(t, x, u) - f_{xx}(t, \hat{x}, \hat{u})\| &\leq \phi(\|x - \hat{x}\| + d(u, \hat{u})) \end{aligned}$$

(here $f_x(\cdot, x, \cdot)$ and $f_{xx}(\cdot, x, \cdot)$ are the partial derivatives of the first and the second order).

A3: both $b(t, x, u)$ and $c(t, x, u)$ are $L_\phi(C^2)$ -mappings.

Notice that from the definition given above it follows that $\|f(t, x, u)\| \leq L_0 + L_1 \|x\|$. So,

$$\|b(t, x, u)\|^2 \leq L_{b0} + L_{b1} \|x\|^2, \quad \|c(t, x, u)\|^2 \leq L_{c0} + L_{c1} \|x\|^2 \quad (8)$$

Definition 2. A stochastic control u_t is called a **feasible** in the stochastic sense (or, s -feasible) for the system (7) if

1.

$$\begin{aligned} u_t \in \mathcal{U}[0, \infty) &:= \left\{ u : [0, \infty) \times \Omega \rightarrow U \mid u_t \text{ is } \{\mathcal{B}_t\}_{t \geq 0} \text{-adapted} \right\} \\ \{\mathcal{B}_t\}_{t \geq 0} &\text{ is the filtration generated by } ((x_\tau, u_\tau : \tau \in [0, t]), t \geq 0) \\ &\text{and augmented by the } \mathbf{P} \text{ - null sets from } \mathcal{F} \end{aligned}$$

2. x_t is the unique solution of (7) in the sense that for any x_t and $\hat{x}_t$, satisfying (7), $\mathbf{P}\{\omega \in \Omega : x_t = \hat{x}_t\} = 1$.

The set of all s -feasible controls is denoted by $\mathcal{U}_{feas}^s[0, \infty)$. The pair $(x_t; u_t)$, where x_t is the solution of (7) corresponding to this u_t , is called an s -**feasible** pair. The assumptions **A1-A3** guarantee that any u_t from $\mathcal{U}[0, \infty)$ is s -feasible. It is assumed also that *the past information about* $(x_\tau, u_\tau : \tau \in [0, t])$ *is available* for the controller: $u_t := u(t, x_{\tau \in [0, t]}, u_{\tau \in [0, t]})$ implying that u_t becomes $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted too.

The only sources of uncertainty in this system description are the system random noise $\bar{W}_t$ and the priori unknown $L_\phi(C^2)$ -functions $b(t, x, u)$ and $c(t, x, u)$. The *class* Σ_{un} of *uncertain stochastic systems* (7) satisfying **A1-A3** is considered below as the main object of the investigation.

4. Differential Neural Networks

Consider the following DNN (Poznyak *et al.*, 1999) with a single hidden layer²:

$$d\hat{x}_t = [A\hat{x}_t + W_{1,t}\sigma(V_{1,t}\hat{x}_t) + W_{2,t}\varphi(V_{2,t}\hat{x}_t)\gamma(u_t)]dt \quad (9)$$

where $\hat{x}_t \in \mathbb{R}^n$ is its state vector at time t , $\sigma : \mathbb{R}^r \rightarrow \mathbb{R}^k$ is the given $L_\phi(C^2)$ -mapping, $\varphi : \mathbb{R}^s \rightarrow \mathbb{R}^l$ is the given $L_\phi(C^2)$ -mapping, $V_{1,t} \in \mathbb{R}^{r \times n}$, $V_{2,t} \in \mathbb{R}^{s \times n}$ are a $\{\mathcal{B}_t\}_{t \geq 0}$ -adapted adjustable internal hidden layer weight matrices, $W_{1,t} \in \mathbb{R}^{n \times k}$, $W_{2,t} \in \mathbb{R}^{n \times l}$ are a $\{\mathcal{B}_t\}_{t \geq 0}$ -adapted adjustable the output layer weight matrices, $\gamma : \mathbb{R}^p \rightarrow \mathbb{R}^q$ the given $L_\phi(C^2)$ -mapping satisfying $\|\gamma(u)\| \leq \gamma_0 + \gamma_1 \|u\|$ for any $u \in U$, $A \in \mathbb{R}^{n \times n}$ is a constant matrix. The initial state vector $\hat{x}_0$ is supposed to be quadratically integrable $(E\{\|\hat{x}_0\|^2\} < \infty)$.

²A multi layer DNN can be represented in a single layer form by the corresponding matrix dimension extension.

Remark 1. The mappings σ and φ are usually assumed to be of the main diagonal structure with the bounded sigmoidal elements σ_{ii} and φ_{ii} , that is,

$$\sigma_{ii} := \frac{a_i}{1 + e^{-b_i^\top \hat{x}}} - c_i \quad (i = 1, \min\{r; k\}), \quad \bar{\sigma} := \sup_{x \in \mathbb{R}^n} \|\sigma(x)\|, \quad \bar{\varphi} := \sup_{x \in \mathbb{R}^n} \|\varphi(x)\|$$

which obviously belong to the class of $L_\phi(C^2)$ -functions.

5. The Learning Law for Identification Process

The matrices collection $(A, V_{1,t}, V_{2,t}, W_{1,t}, W_{2,t})$ should be selected to provide "a good enough" identification process for any possible uncertain stochastic system from Σ_{un} .

Accept several additional assumptions concerning the stability property of the process to be identified and the structure of the applied control.

A4: the applied s -feasible control $u_t \in U$ is *quadratically integrable* (uniformly on t) as well the corresponding stochastic system dynamics x_t , that is, $\limsup_{t \rightarrow \infty} \left(E \left\{ \|u_t\|^2 \right\} + E \left\{ \|x_t\|^2 \right\} \right) < \infty$. In fact, this assumption restricts the consideration by the class of quadratically stable systems.

A5: it is assumed that $\|u_t\| \leq \delta_0 + \delta_1 \|x_t\|$. This class of controllers includes practically all known ones widely used in different applications (Linear (PID-type), Sliding Mode, HJ, Locally Optimal and others).

Accept also some technical assumption which will be use in the main result formulation.

A6: There exist a stable matrix A and positive definite matrices $\Lambda_\sigma, \Lambda_{\eta_\sigma}, \Lambda_f, \Lambda_\gamma, Q_0$ such that the matrix algebraic Riccati equation

$$\begin{aligned} G := & PA + A^\top P + \left(L_\sigma^2 V_{1,0}^\top W_{1,0} \Lambda_\sigma W_{1,0}^\top V_{1,0} + Q_0 \right) \\ & + P \left(\Lambda_\sigma^{-1} + \Lambda_{\eta_\sigma}^{-1} + \Lambda_f^{-1} + 8\bar{\sigma}^2 \|\Lambda_\gamma^{-1}\| W_{2,0} W_{2,0}^\top \right) P = 0 \end{aligned} \quad (10)$$

has a positive solution $P = P^\top > 0$.

A7. The large number law (Poznyak, 2000) is valid for the considered processes

$$\begin{aligned} T^{-1} \int_{t=0}^T \left[s_\mu^\varepsilon(P^{1/2} \Delta_t) \|x_t\|^2 - E \left\{ s_\mu^\varepsilon(P^{1/2} \Delta_t) \|x_t\|^2 \right\} \right] dt &\xrightarrow{a.s.} 0 \\ T^{-1} \int_{t=0}^T \left[s_\mu^\varepsilon(P^{1/2} \Delta_t) - E \left\{ s_\mu^\varepsilon(P^{1/2} \Delta_t) \right\} \right] dt &\xrightarrow{a.s.} 0 \end{aligned} \quad (11)$$

where $\varepsilon > 0$, the "dead-zone" function $s_\mu(z) : \mathbb{R}^n \rightarrow \mathbb{R}^1$ is defined as follows:

$$s_\mu(z) := [1 - \mu / \|z\|]_+ = \begin{cases} 1 - \mu / \|z\|, & \mu / \|z\| \leq 1 \\ 0, & \mu / \|z\| > 1 \end{cases} \quad (12)$$

and $\Delta_t := \hat{x}_t - x$. The last assumptions is not so restrictive since the only decreasing of the autocorrelation functions of the considered processes is required to fulfill (11) that

automatically holds for the accepted constructions (see Poznyak, 2000 and Poznyak *et al*, 2000).

The weights in (9) are adjusted by the following *Differential Learning Law*:

$$\left\{ \begin{array}{l} L_{1,t}^W := k_{w,1}^{-1} \dot{W}_{1,t} - \\ S_{\mu,\varepsilon} (P^{1/2} \Delta_t) P \Delta_t [\sigma^\top (V_{1,t} \hat{x}_t) - \hat{x}_t^\top (V_{1,t} - V_{1,0})^\top D_\sigma^\top (V_{1,t} \hat{x}_t)] = 0 \\ L_{2,t}^W := k_{w,2}^{-1} \dot{W}_{2,t} - k_{w,2} S_{\mu,\varepsilon} (P^{1/2} \Delta_t) P \Delta_t [\gamma^\top (u_t) \varphi^\top (V_{2,t} \hat{x}_t) \\ - \hat{x}_t^\top (V_{2,t} - V_{2,0})^\top \left(\sum_{i=1}^q \gamma_i (u_t) D_{i\varphi}^\top (V_{2,t} \hat{x}_t) \right)] = 0 \\ L_{1,t}^V := k_{v,1}^{-1} \dot{V}_{1,t} - \\ S_{\mu,\varepsilon} (P^{1/2} \Delta_t) \left[D_\sigma^\top (V_{1,t} \hat{x}_t) W_{1,t}^\top P \Delta_t + 2L_\sigma^2 \Lambda_{\eta_\sigma} (V_{1,t} - V_{1,0}) \hat{x}_t \right] \hat{x}_t^\top = 0 \\ L_{2,t}^V := k_{v,2}^{-1} \dot{V}_{2,t} - S_{\mu,\varepsilon} (P^{1/2} \Delta_t) \left[\left(\sum_{i=1}^q \gamma_i (u_t) D_{i\varphi}^\top (V_{2,t} \hat{x}_t) \right) W_{2,t}^\top P \Delta_t \right. \\ \left. + 4 \|\Lambda_\gamma^{-1}\| \|W_{2,0}^\top P \Delta_t\|^2 L_\varphi^2 (V_{2,t} - V_{2,0}) \hat{x}_t \right] \hat{x}_t^\top = 0 \end{array} \right. \quad (13)$$

with any nonzero initial weights $V_{1,0}$ and $V_{2,0}$ and

$$S_{\mu,\varepsilon} (z) := \left(s_\mu^{1+\varepsilon} (z) \left[2s_\mu (z) + (2 + \varepsilon) \frac{\mu}{\|z\|} \right] \right) \quad (14)$$

$$\begin{aligned} \mu^2 &:= \frac{(C_0 + C_1 \rho)}{(2 + \varepsilon) \lambda_{\min}(P^{-1/2} Q_0 P^{-1/2})}, \quad f_1 := L_1 + \|A\| + \|W_{2,0}\| \bar{\varphi} \gamma_1 \delta_1 \\ f_0 &:= L_0 + \|W_{1,0}\| \bar{\sigma} + \|W_{2,0}\| \bar{\varphi} (\gamma_0 + \gamma_1 \delta_0) \\ C_i &:= 2(2 + \varepsilon) f_i^2 (\|\Lambda_f\| + \|\Lambda_\gamma\|) + (2 + \varepsilon)^2 L_{ci} \|P\| \end{aligned} \quad (15)$$

$$\rho \geq \frac{1}{2} \limsup_{T \rightarrow \infty} \left[T^{-1} \int_{t=0}^T E \left\{ s_\mu^\varepsilon (P^{1/2} \Delta_t) \|x_t\|^2 \right\} \right] \left[T^{-1} \int_{t=0}^T E \left\{ s_\mu^{1+\varepsilon} (P^{1/2} \Delta_t) \right\} dt \right]^{-1} \quad (16)$$

Theorem 1 (Main result on Learning Law). *If, under the assumptions A1-A7, the learning law (13) is applied, then the following properties of the corresponding identification process are guaranteed:*

1) *the asymptotic convergence with the probability one to the dead-zone takes place:*

$$s_\mu (P^{1/2} \Delta_t) := \left[1 - \mu / \left\| P^{1/2} \Delta_t \right\| \right]_+ \xrightarrow{a.s.} 0 \quad (17)$$

2) *the weight matrices as well as the identification error remain bounded (in mean square sense) during all learning process, that is, for any $t \geq 0$*

$$\begin{aligned} \sum_{i=1}^2 \left(\text{tr} E \left\{ V_{i,t}^\top V_{i,t} \right\} + \text{tr} E \left\{ W_{i,t}^\top W_{i,t} \right\} \right) + \text{tr} E \left\{ V_0 (P^{1/2} \Delta_t) \right\} &\leq C < \infty \\ V_0 (z) &:= \|z\|^2 s_\mu^{2+\varepsilon} (z) \end{aligned} \quad (18)$$

The proof of the theorem is given in Appendix.

6. Adaptive Trajectory Tracking

In this section we consider the adaptive trajectory problem formulated for the subclass of completely controllable stochastic nonlinear processes with a linear forcing control to avoid the discussions concerning the controllability properties and simplify the main idea of the presentation given below. Let the random process x_t be the subclass of (7) and is given by

$$dx_t = [f(t, x_t) + u_t] dt + c(t, x_t, u_t) d\bar{W}_t, \quad x_0 = x_{\text{given}}, \quad t \in [0, \infty) \quad (19)$$

where $u_t \in \mathbb{R}^n$, $f : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $c : [0, \infty) \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^{n \times m}$ are $L_\phi(C^2)$ -mappings such that

A3':

$$\|f(t, x)\| \leq L_{0,f} + L_{1,f} \|x\|, \quad \|c(t, x, u)\| \leq \bar{c} \quad (20)$$

The *nominal trajectory* x_t^* to be tracked is assumed to be bounded that, for the functions $\bar{\varphi}(x^*, t)$ uniformly bounded on t and having summed bounded variation on x^* , implies

$$\sup_t \|\bar{\varphi}(x_t^*, t)\| = \bar{\varphi}_0 < \infty \quad (21)$$

To simplify the further presentation, select the DNN structure as follows

$$\frac{d}{dt} \hat{x}_t = \hat{f}_t(\hat{x}_t) + u_t, \quad \hat{f}_t(x) := Ax + W_{1,t} \sigma(V_{1,t} x) \quad (22)$$

Realize the adaptive trajectory tracking control u_t as

$$u_t = \hat{u}_t := \bar{\varphi}(x^*, t) - \hat{f}_t(\hat{x}_t) + A^*(\hat{x}_t - x_t^*), \quad A^* - \text{a stable matrix} \quad (23)$$

Remark 2. Since the trajectory x_t is assumed to be completely measurable, their use in (23) instead of $\hat{x}_t$ is preferable, that is, the control u_t can be formed as

$$u_t = \hat{u}_t := \bar{\varphi}(x^*, t) - \hat{f}_t(x_t) + A^*(x_t - x_t^*) \quad (24)$$

In this case the proof of the stability for the close-loop system remains practically the same. The only one thing is changed: in the corresponding Lyapunov function (see the proof of Theorem 2) instead of $\Delta_t^* := \hat{x}_t - x_t^*$ should be used $\Delta_t^* := x_t - x_t^*$.

Remark 3. We realize that in this simplified case ($g(x, t) = I$) the high-gain controller also can be applied to provide the robustness property for the closed-loop system that corresponds to a negative linear feed-back given by

$$u_t = \hat{u}_t := -K(x_t - x_t^*), \quad 0 < K - \text{large enough} \quad (25)$$

Such class of robust controllers "ignores" the nonlinearities which are partially or completely unknown and, as a result (the simulations below) shows the less quality of tracking with respect to (24) where the nonlinear effects are partially compensated by the term $\hat{f}_t(x_t)$ obtained by DNN implementation.

Select the stable matrix A and the positive definite matrices $\Lambda_\sigma^{-1}, \Lambda_{\eta_\sigma}^{-1}, \Lambda_{\Delta f}^{-1}$ and Q_1 such that the following algebraic matrix Riccati equation

$$\begin{aligned} Ric := P_1 A + A^\top P_1 + P_1 \left[\Lambda_\sigma^{-1} + \Lambda_{\eta_\sigma}^{-1} + \Lambda_{\Delta f}^{-1} \right] P_1 + 3 \|\Lambda_{\Delta f}\| C_1^2 I + Q_1 \\ + \left[L_\sigma^2 V_{1,0}^\top W_{1,0} \Lambda_\sigma W_{1,0}^\top V_{1,0} \right] = 0, \quad C_1 := L_{1,f} + \|A\| \end{aligned}$$

has the positive solution $P_1 = P_1^\top > 0$. Let P_2 satisfies the following Lyapunov matrix equation

$$P_2 A^* + A^{*\top} P_2 + 3 \|\Lambda_{\Delta f}\| C_1^2 I = -Q_2$$

where A^* the stable matrix participationg in (23).

Let the weight matrices of the DNN (22) be adjusted according to the following *Differential Learning Law*:

$$\left\{ \begin{aligned} L_{w1,t} &:= 2S_{\mu,\varepsilon}(z_t) [\sigma(V_{1,t}\hat{x}_t) \Delta_t^\top P - D_\sigma(V_{1,t}\hat{x}_t) (V_{1,t} - V_{1,0}) \hat{x}_t \Delta_t^\top P] dt \\ &\quad + 2(4 + \varepsilon) k_{w,1}^{-1} dW_{1,t}^\top = 0 \\ L_{v1,t} &:= 2S_{\mu,\varepsilon}(z_t) [\hat{x}_t \Delta_t^\top P W_{1,t} D_\sigma(V_{1,t}\hat{x}_t) + 2L_\sigma^2 \hat{x}_t \hat{x}_t^\top (V_{1,t} - V_{1,0})^\top \Lambda_{\eta_\sigma}] dt \\ &\quad + 2(4 + \varepsilon) k_{v,1}^{-1} dV_{1,t}^\top = 0 \end{aligned} \right. \quad (26)$$

where the "dead-zone" function $S_{\mu,\varepsilon}(z_t)$ is defined by (14), $z_t := \begin{bmatrix} P_1^{1/2} & 0 \\ 0 & P_2^{1/2} \end{bmatrix} \begin{pmatrix} \Delta_t \\ \Delta_t^* \end{pmatrix}$

and the number μ satisfies $\mu^2 \geq \left[\frac{\beta}{2\rho} \limsup_{T \rightarrow \infty} \frac{\int_{t=0}^T E\{s_\mu^\varepsilon(z_t)\} dt}{\int_{t=0}^T E\{s_\mu^{1+\varepsilon}(z_t)\} dt} \right]$ with β and ρ defined as

$$\begin{aligned} \beta &:= \left[(2 + \varepsilon)^2 \text{tr} \{P_1 + P_2\} + (2 + \varepsilon) 3 \|\Lambda_{\Delta f}\| C_0^2 \right] \bar{c}^2 \\ C_0 &:= L_{0,f} + \|W_{1,0}\| \bar{\sigma} + (L_{1,f} + \|A\|) \sup_t \|x_t^*\| \\ \rho &:= \min \left\{ \lambda_{\min} \left(P_1^{-1/2} Q_1 P_1^{-1/2} \right); \lambda_{\min} \left(P_2^{-1/2} Q_2 P_2^{-1/2} \right) \right\} \end{aligned}$$

6.4. The main result on adaptive tracking with on-line identification

Theorem 2. *The application of the adaptive control (23) together with the learning law (26) under the assumptions A1-A2 and A3' (instead of A3) implies with probability one and in mean square sense the convergence of the vector process $(\Delta_t^\top, \Delta_t^{*\top})^\top : \Delta_t = \hat{x}_t - x_t, \Delta_t^* = \hat{x}_t - x_t^*$ to the "dead-zone" defined by $\Delta_t^\top P_1 \Delta_t + \Delta_t^{*\top} P_2 \Delta_t^* \leq \mu$ keeping the weights $W_{i,t}, V_{i,t}$ bounded.*

The proof of the theorem is given in the Appendix.

Corollary 1. *The stability of unknown stochastic system follows from the boundness of the nominal trajectory x_t^* and the inequality $\|x_t - x_t^*\| = \|(x_t - \hat{x}_t) + (\hat{x}_t - x_t^*)\| \leq \|\Delta_t\| + \|\Delta_t^*\|$. The tracking upper bound can be estimated (for large enough t) as*

$$\begin{aligned} \|x_t - x_t^*\|^2 &\leq \|\Delta_t\|_{P_1}^2 + \left\| P_2^{-1/2} P_1^{-1} P_2^{-1/2} \right\| \cdot \|\Delta_t^*\|_{P_2}^2 \\ &\leq \max \left\{ 1; \left\| P_2^{-1/2} P_1^{-1} P_2^{-1/2} \right\| \right\} \mu^2 \end{aligned}$$

7. Simulation Results

7.1. Identification of Unforced Stochastic Systems

Consider the nonlinear stochastic process generated by the *Van der Pole* oscillator with a noisy component, i.e., $x_{1,0} = 2$, $x_{2,0} = 0$ and

$$\begin{cases} dx_{1,t} = x_{2,t} dt \\ dx_{2,t} = 2.5 \left[(1 - x_{1,t}^2) x_{2,t} - x_{1,t} \right] dt + 0.075 d\bar{W}_t \end{cases}$$

The DNN applied for the identifications is as follows

$$\begin{aligned} d\hat{x}_t &= [A\hat{x}_t + W_{1,t}\sigma(V_{1,t}\hat{x}_t)] dt, \quad \hat{x}_{1,0} = 0.1, \quad \hat{x}_{2,0} = 0.5 \\ A &= -5I, \quad \sigma_i(z) = 1/(1 + \exp(-0.2z_i)) - 0.5 \end{aligned}$$

with the weights $W_{1,t}, V_{1,t} \in \mathbb{R}^{2 \times 2}$ adjusted by

$$\begin{cases} \dot{W}_{1,t} = -k_{w,1} s_\mu^\varepsilon (P^{1/2} \Delta_t) P \Delta_t [\sigma^\top(V_{1,t}\hat{x}_t) - \hat{x}_t^\top (V_{1,t} - V_{1,0})^\top D_\sigma^\top(V_{1,t}\hat{x}_t)] \\ \dot{V}_{1,t} = -k_{v,1} s_\mu^\varepsilon (P^{1/2} \Delta_t) \left[D_\sigma^\top(V_{1,t}\hat{x}_t) W_{1,t}^\top P \Delta_t + 2L_\sigma^2 \Lambda_{\eta_\sigma} (V_{1,t} - V_{1,0}) \hat{x}_t \right] \hat{x}_t^\top \end{cases}$$

The selected parameter values are $P = 60I$, $W_{1,0} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.2 \end{pmatrix}$, $V_{1,0} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $k_{1,w} = 2, k_{1,v} = 0.3, \varepsilon = \mu = 0.01$, $\Lambda_{\eta_\sigma} = \lambda_{\eta_\sigma} I$, $2L_\sigma^2 \lambda_{\eta_\sigma} = 0.05$. The correspondent simulation results, realized only with the use of two neurons ($i = 1, 2$), are depicted at the figures Fig.1-Fig.4.

Fig.1: x_1 and $\hat{x}_1$ (dashed).

Fig.2: x_2 and $\hat{x}_2$ (dashed).

Fig.3: The elements of W_1 .

Fig.4: The elements of V_1 .

The value of the performance index $I_t = (t + 0.001)^{-1} \int_{\tau=0}^t \|\Delta_\tau\|^2 d\tau$ obtained after $t = 20 \text{ sec.}$ is equal to $I_{t=20} = 0.4609$.

7.2. Identification of Controlled Stochastic Systems

The nonlinear stochastic process generated by

$$\begin{cases} dx_{1,t} = [-5x_{1,t} + 3|x_{2,t}| + u_{1,t}]dt + 0.088d\bar{W}_{1,t}, & x_{1,0} = 0.1 \\ dx_{2,t} = [-10x_{2,t} + 2|x_{1,t}| + u_{2,t}]dt + 0.088d\bar{W}_{2,t}, & x_{2,0} = -0.2 \end{cases} \quad (27)$$

is considered. The programmed controls are the sine-wave and the saw-tooth periodic functions, that is, $u_{1,t} = u_{1,0} \sin(\omega t)$, $u_{1,0} = \omega = 1$ and $u_{2,t} = u_{2,0} \cdot (t - k\tau)$, $t \in [k\tau, (k+1)\tau)$, $u_{2,0} = \tau = 0.2$, $k = 0, 1, 2, \dots$. The DNN applied for the identifications is

$$d\hat{x}_t = [A\hat{x}_t + W_{1,t}\sigma(V_{1,t}\hat{x}_t) + W_{2,t}\varphi(V_{2,t}\hat{x}_t)u_t]dt, \quad \hat{x}_{1,0} = 0.1, \quad \hat{x}_{2,0} = 0.05 \quad (28)$$

$$\begin{aligned} \sigma_i(z) &= 2/(1 + \exp(-r_s z_i)) - 0.5, \quad r_s = 0.025, \quad r_f = 0.3 \\ \varphi_{1,2} &= \varphi_{2,1} = 0, \quad \varphi_{i,i}(z) = 0.5/(1 + \exp(-r_f z_i)) - 0.05 \end{aligned} \quad (29)$$

and the weights $W_{1,t}, V_{1,t}, W_{2,t}, V_{2,t} \in \mathbb{R}^{2 \times 2}$ are adjusted by the learning law (13) with $q = 2$, $\gamma_i(u) = u_i$ ($i = 1, 2$). The selected parameter values are $P = \begin{pmatrix} 15 & -4 \\ -4 & 15 \end{pmatrix}$,

$A = \begin{pmatrix} -3 & -0.01 \\ -0.01 & -2 \end{pmatrix}$, $k_{1,w_1} = k_{1,w_2} = 0.3$, $k_{1,v_1} = 25$, $k_{1,v_2} = 2.5$, $\varepsilon = 0.01$, $\mu = 0.001$, $2L_\sigma^2 \lambda_{\eta_\sigma} = 0.1$, $\Lambda_\gamma^{-1} = \lambda_\gamma I$, $4L_\varphi^2 \lambda_\gamma = 0.1$ and the initial weight values equal to $W_{1,0} = \begin{pmatrix} 0.5 & 0.1 \\ 0 & 2 \end{pmatrix}$, $V_{1,0} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $W_{2,0} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.2 \end{pmatrix}$, $V_{2,0} = \begin{pmatrix} 1 & 0.1 \\ 0.1 & 2 \end{pmatrix}$. The correspondent simulation results, realized only with the use of 4 neurons (2 in the output and 2 in the hidden layer), are depicted at the figures Fig.7-Fig.8. The value of the performance index I_t , obtained after $t = 20 \text{ sec.}$, is equal to $I_{t=20} = 0.008087$.

Fig.7: x_1 and $\hat{x}_1$ (dashed).

Fig.8: x_2 and $\hat{x}_2$ (dashed).

7.3. Trajectory tracking based on DNN identifier

The same plant (27) (with very high noise intensity: $\sigma_1 = \sigma_2 = 0.5$) has been selected as the stochastic nonlinear system to be forced to track the nominal dynamics (the saw-tooth and the sine processes) given by

$$x_1^* = 1 + (10k - t)/5, \quad k = 0, 1, \dots; \quad x_2^* = \sin(2\pi t/10)$$

The DNN (9) ($W_{2,t}\varphi(x_t) \equiv I$) has been applied with the following parameters: $P = \begin{pmatrix} 1 & 0.25 \\ 0.25 & 1 \end{pmatrix}$,

$$A = \begin{pmatrix} -5 & -0.2 \\ -0.2 & -4 \end{pmatrix}, \quad r_s = 25, \quad r_f = 3, \quad k_{1,w_1} = 0.5, \quad k_{1,v_1} = 1, \quad \varepsilon = 0.01, \quad \mu = 0.001,$$

$$2L_\sigma^2\lambda_{\eta_\sigma} = 0.1, \quad \Lambda_\gamma^{-1} = \lambda_\gamma I. \quad \text{The initial weights were } W_{1,0} = \begin{pmatrix} 0.5 & 0 \\ 0 & 2 \end{pmatrix}, \quad V_{1,0} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

The adaptive control law $u(t)$ is given by (24) with $A^* = 75 \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. The corresponding adaptive tracking results are given in the Figures 9 and 10.

Fig.9: Tracking for x_1 .

Fig.10: Tracking for x_2 .

The tracking performance index I_t , obtained after $t = 20$ sec., is equal to $I_{t=20}^{ad-track} = 0.006665$. The index corresponding to the high-gain controller (25) with $K = 75I$ (but "ignoring" the nonlinear effects) is equal to $I_{t=20}^{high-gain} = 0.01155$, that is, twice worse with respect to the adaptive controller (24) using on-line DNN estimates.

8. Conclusion

In this paper a new adaptive neuro controller is proposed to force a given (but unknown) continuous time stochastic process to track a nominal trajectory. It is based on the DNN - identifier adjusting the corresponding weight matrices by a special differential learning law. Even the sigmoidal nonlinearities are bounded and, in the first glance, the hidden layer weights variation may not effect the identification quality, the presented theoretical study (as well as the corresponding simulations) show that the hidden layer weights updating significantly improves the identification process. Stochastic Lyapunov analysis (with the special form of Lyapunov function with "dead-zone" multipliers) is applied to state the existence of the identification and adaptive tracking error bounds which incorporate all characteristics of uncertainties (including noise variance).

Appendix

Proof of Theorem 1. Represent (7) in the following form

$$\begin{aligned} dx_t &= [Ax_t + W_{1,0}\sigma(V_{1,0}x_t) + W_{2,0}\varphi(V_{2,0}x_t)\gamma(u_t) + \Delta f_t] dt + c(t, x_t, u_t) d\bar{W}_t \\ \Delta f_t &:= b(t, x_t, u_t) - [Ax_t + W_{1,0}\sigma(V_{1,0}x_t) + W_{2,0}\varphi(V_{2,0}x_t)\gamma(u_t)] \end{aligned} \quad (30)$$

By (9) and **A1-A4**, Δf_t verifies the inequality $\|\Delta f_t\| \leq f_0 + f_1 \|x_t\|$. Then from (7) and (9) it follows that

$$\begin{aligned} d\Delta_t &= [A\Delta_t + F_1 + F_2\gamma(u_t) - \Delta f_t] dt - c(t, x_t, u_t) d\bar{W}_t \\ F_1 &:= W_{1,t}\sigma(V_{1,t}\hat{x}_t) - W_{1,0}\sigma(V_{1,0}x_t) = \tilde{W}_{1,t}\sigma(V_{1,t}\hat{x}_t) + W_{1,0}[\hat{\sigma}_t + \check{\sigma}_t] \\ F_2 &:= (W_{2,t}\varphi(V_{2,t}\hat{x}_t) - W_{2,0}\varphi(V_{2,0}x_t))\gamma(u_t) \\ &= \left(\tilde{W}_{2,t}\varphi(V_{2,t}\hat{x}_t) + W_{2,0}[\hat{\varphi}_t + \check{\varphi}_t] \right) \gamma(u_t), \quad \tilde{W}_{i,t} := W_{i,t} - W_{i,0} \\ \hat{\sigma}_t &:= \sigma(V_{1,t}\hat{x}_t) - \sigma(V_{1,0}\hat{x}_t), \quad \check{\sigma}_t := \sigma(V_{1,0}\hat{x}_t) - \sigma(V_{1,0}x_t) \\ \hat{\varphi}_t &:= \varphi(V_{2,t}\hat{x}_t) - \varphi(V_{2,0}\hat{x}_t), \quad \check{\varphi}_t := \varphi(V_{2,0}\hat{x}_t) - \varphi(V_{2,0}x_t) \end{aligned} \quad (31)$$

In view of the differentiability and Lipschitz properties **A3**

$$\begin{aligned} \hat{\sigma}_t &= D_\sigma(V_{1,t}\hat{x}_t)\tilde{V}_{1,t}\hat{x}_t + \eta_{\sigma,t}, \quad \tilde{V}_{i,t} := V_{i,t} - V_{i,0} \\ \hat{\varphi}_t\gamma(u_t) &= \sum_{i=1}^q \gamma_i(u_t) D_{i\varphi}(V_{2,t}\hat{x}_t)\tilde{V}_{2,t}\hat{x}_t + \eta_{\varphi,t}\gamma(u_t) \\ D_\sigma(V_{1,t}\hat{x}_t) &:= \frac{\partial}{\partial z}\sigma(z) \big|_{z=V_{1,t}\hat{x}_t}, \quad D_{i\varphi}(V_{2,t}\hat{x}_t) := \frac{\partial}{\partial z}\varphi_i(z) \big|_{z=V_{2,t}\hat{x}_t} \end{aligned} \quad (32)$$

$$\begin{aligned}\|\eta_{\sigma,t}\| &\leq 2L_\sigma \left\| \tilde{V}_{1,t} \hat{x}_t \right\|, \|\check{\sigma}_t\| \leq \min\{2\bar{\sigma}; L_\sigma \|V_{1,0} \Delta_t\|\} \\ \|\eta_{\varphi,t}\| &\leq 2L_\varphi \left\| \tilde{V}_{2,t} \hat{x}_t \right\|, \|\check{\varphi}_t\| \leq \min\{2\bar{\varphi}; L_\varphi \|V_{2,0} \Delta_t\|\}\end{aligned}$$

Introduce the following Lyapunov function candidate

$$V_t := V_0 \left(P^{1/2} \Delta_t \right) + (2 + \varepsilon) \sum_{i=1}^2 \left(k_{v,i}^{-1} \text{tr} \left\{ \tilde{V}_{i,t}^\top \tilde{V}_{i,t} \right\} + k_{w,i}^{-1} \text{tr} \left\{ \tilde{W}_{i,t}^\top \tilde{W}_{i,t} \right\} \right) \quad (33)$$

where $0 < P = P \in \mathbb{R}^{n \times n}$ is a matrix to be selected, $\varepsilon, k_{w,i}, k_{v,i}$ are positive scalar parameters and the function $V_0(z)$ is defined by (18). Notice that

$$\begin{aligned}\frac{\partial}{\partial z} V_0(z) &= S_{\mu,\varepsilon}(z) := s_\mu^{1+\varepsilon}(z) \left[2s_\mu(z) + (2 + \varepsilon) \frac{\mu}{\|z\|} \right] z \\ \frac{\partial}{\partial \Delta} V_0(P^{1/2} \Delta) &= \left(s_\mu^{1+\varepsilon}(z) \left[2s_\mu(z) + (2 + \varepsilon) \frac{\mu}{\|z\|} \right] \right) \Big|_{z=P^{1/2} \Delta} P \Delta\end{aligned} \quad (34)$$

$$2s_\mu^{1+\varepsilon}(z) \leq S_{\mu,\varepsilon}(z) \leq (2 + \varepsilon) s_\mu^{1+\varepsilon}(z)$$

$$\begin{aligned}\frac{\partial^2}{\partial z^2} V_0(z) &= \left(2s_\mu^{2+\varepsilon}(z) + (2 + \varepsilon) \frac{\mu}{\|z\|} s_\mu^{1+\varepsilon}(z) \right) I \\ &+ (2 + \varepsilon) \left(\frac{\mu}{\|z\|} s_\mu^{1+\varepsilon}(z) + \frac{\mu^2}{\|z\|^2} (1 + \varepsilon) s_\mu^\varepsilon(z) \right) \frac{zz^\top}{\|z\|^2} \leq (2 + \varepsilon)^2 s_\mu^\varepsilon(z) I \\ \frac{\partial^2}{\partial \Delta^2} V_0(P^{1/2} \Delta) &\leq (2 + \varepsilon)^2 s_\mu^\varepsilon(z) \Big|_{z=P^{1/2} \Delta} P\end{aligned} \quad (35)$$

As $s_\mu(z) \in [0, 1]$, by (34), (35) and the Itô formula (Gard, 1988), it follows

$$\begin{aligned}dV_t &\leq 2(2 + \varepsilon) \sum_{i=1}^2 \left(k_{v,i}^{-1} \text{tr} \left\{ d\tilde{V}_{i,t}^\top \tilde{V}_{i,t} \right\} + k_{w,i}^{-1} \text{tr} \left\{ d\tilde{W}_{i,t}^\top \tilde{W}_{i,t} \right\} \right) + \\ &S_{\mu,\varepsilon}(P^{1/2} \Delta_t) 2\Delta_t^\top P d\Delta_t + (2 + \varepsilon)^2 s_\mu^\varepsilon(P^{1/2} \Delta) \text{tr} \{ c(t, x_t, u_t) P c^\top(t, x_t, u_t) \} dt\end{aligned} \quad (36)$$

The use of (36), (??), (32), (31), the inequalities

$$X^\top Y + Y^\top X \leq X^\top \Lambda^{-1} X + Y^\top \Lambda Y \quad (37)$$

valid for any $X, Y \in \mathbb{R}^{m \times n}$ and any $0 < \Lambda = \Lambda^\top \in \mathbb{R}^{m \times m}$, and $\|A^\top A\| \leq \text{tr} \{ A^\top A \}$, implies:

1)

$$2\Delta_t^\top P A \Delta_t = \Delta_t^\top (PA + AP) \Delta_t$$

2)

$$\begin{aligned}2\Delta_t^\top P F_1 &\leq \text{tr} \left\{ \left[2\sigma(V_{1,t} \hat{x}_t) \Delta_t^\top P - 2D_\sigma(V_{1,t} \hat{x}_t) \tilde{V}_{1,t} \hat{x}_t \Delta_t^\top P \right] \tilde{W}_{1,t} \right\} \\ &+ \text{tr} \left\{ \left[2\hat{x}_t \Delta_t^\top P W_{1,t} D_\sigma(V_{1,t} \hat{x}_t) + 4L_\sigma^2 \hat{x}_t \hat{x}_t^\top \tilde{V}_{1,t}^\top \Lambda_{\eta_\sigma} \right] \tilde{V}_{1,t} \right\} \\ &+ \Delta_t^\top \left(P \left[\Lambda_\sigma^{-1} + \Lambda_{\eta_\sigma}^{-1} \right] P + \left[L_\sigma^2 V_{1,0}^\top W_{1,0} \Lambda_\sigma W_{1,0}^\top V_{1,0} \right] \right) \Delta_t\end{aligned}$$

3)

$$\begin{aligned}
2\Delta_t^\top P F_2 &\leq \text{tr} \left\{ [2\varphi(V_{2,t}\hat{x}_t) \gamma(u_t) \Delta_t^\top P \right. \\
&\quad \left. \left\{ -2 \left(\sum_{i=1}^q \gamma_i(u_t) D_{i\varphi}(V_{2,t}\hat{x}_t) \right) \tilde{V}_{2,t} \hat{x}_t \Delta_t^\top P \right\} \tilde{W}_{2,t} \right\} \\
&\quad + \text{tr} \left\{ \left[2\hat{x}_t \Delta_t^\top P W_{2,t} \left(\sum_{i=1}^q \gamma_i(u_t) D_{i\varphi}(V_{2,t}\hat{x}_t) \right) \right] \tilde{V}_{2,t} \right\} \\
&\quad + \text{tr} \left\{ \left[8 \|\Lambda_\gamma^{-1}\| \|W_{2,0}^\top P \Delta_t\|^2 L_\varphi^2 \hat{x}_t \hat{x}_t^\top \tilde{V}_{2,t}^\top \right] \tilde{V}_{2,t} \right\} \\
&\quad + \Delta_t^\top P \left(8\bar{\sigma}^2 \|\Lambda_\gamma^{-1}\| W_{2,0} W_{2,0}^\top \right) P \Delta_t + 2 \|\Lambda_\gamma\| \left(f_0^2 + f_1^2 \|x_t\|^2 \right)
\end{aligned}$$

4)

$$-2\Delta_t^\top P \Delta f_t \leq \Delta_t^\top P \Lambda_f^{-1} P \Delta_t + 2 \|\Lambda_f\| \left(f_0^2 + f_1^2 \|x_t\|^2 \right)$$

Substituting these inequalities in to the right-hand side of (36), finally, it follows:

$$\begin{aligned}
dV_t &\leq -(2+\varepsilon) S_{\mu,\varepsilon} (P^{1/2} \Delta_t) \Delta_t^\top Q_0 \Delta_t dt - S_{\mu,\varepsilon} (P^{1/2} \Delta_t) \Delta_t^\top P c(t, x_t, u_t) d\bar{W}_t \\
&\quad + 2(2+\varepsilon) S_{\mu,\varepsilon} (P^{1/2} \Delta_t) (\|\Lambda_f\| + \|\Lambda_\gamma\|) \left(f_0^2 + f_1^2 \|x_t\|^2 \right) dt \\
&\quad + (2+\varepsilon)^2 s_\mu^\varepsilon (P^{1/2} \Delta_t) \text{tr} \{ c(t, x_t, u_t) P c^\top(t, x_t, u_t) \} dt
\end{aligned} \tag{38}$$

since, in view of the accepted learning laws (13), we have $G = 0$, $L_{i,t}^W = 0$ and $L_{i,t}^V = 0$. In view of $L_\phi(C^2)$ -property (8), from (38) it follows

$$\begin{aligned}
dV_t &\leq -(2+\varepsilon) \lambda_{\min}(Q_0) S_{\mu,\varepsilon} (P^{1/2} \Delta_t) \left(1 - \frac{\mu^2}{\|\Delta_t\|^2} \right) \|\Delta_t\|^2 dt + \\
C_1 &\left(s_\mu^\varepsilon (P^{1/2} \Delta_t) \|x_t\|^2 - 2s_\mu^{1+\varepsilon} (P^{1/2} \Delta_t) \rho \right) dt - S_{\mu,\varepsilon} (P^{1/2} \Delta_t) \Delta_t^\top P c(t, x_t, u_t) d\bar{W}_t
\end{aligned} \tag{39}$$

The integration of (39) implies

$$\begin{aligned}
0 &\leq \frac{2(2+\varepsilon)\lambda_{\min}(P^{-1/2}Q_0P^{-1/2})}{T} \int_{t=0}^T s_\mu^{1+\varepsilon} (P^{1/2} \Delta_t) \left(1 - \frac{\mu^2}{\|P^{1/2} \Delta_t\|^2} \right) \|\Delta_t\|^2 dt \\
&\stackrel{a.s.}{\leq} T^{-1} V_0 - T^{-1} V_t + C_1 T^{-1} \int_{t=0}^T \left(s_\mu^\varepsilon (P^{1/2} \Delta_t) \|x_t\|^2 - 2s_\mu^{1+\varepsilon} (P^{1/2} \Delta_t) \rho \right) dt \\
&\quad - T^{-1} \int_{t=0}^T S_{\mu,\varepsilon} (P^{1/2} \Delta_t) \Delta_t^\top P c(t, x_t, u_t) d\bar{W}_t
\end{aligned}$$

and, as a result,

$$\begin{aligned}
0 &\stackrel{a.s.}{\leq} T^{-1} V_0 + D_n - T^{-1} \int_{t=0}^T S_{\mu,\varepsilon} (P^{1/2} \Delta_t) \Delta_t^\top P c(t, x_t, u_t) d\bar{W}_t + \\
C_1 &\left(2T^{-1} \int_{t=0}^T E \{ s_\mu^{1+\varepsilon} (P^{1/2} \Delta_t) \} dt \right)^{-1} \left(\frac{T^{-1} \int_{t=0}^T E \{ s_\mu^\varepsilon (P^{1/2} \Delta_t) \|x_t\|^2 \}}{2T^{-1} \int_{t=0}^T E \{ s_\mu^{1+\varepsilon} (P^{1/2} \Delta_t) \} dt} - \rho \right) dt \\
&\stackrel{a.s.}{\leq} T^{-1} V_0 + D_n - T^{-1} \int_{t=0}^T S_{\mu,\varepsilon} (P^{1/2} \Delta_t) \Delta_t^\top P c(t, x_t, u_t) d\bar{W}_t
\end{aligned} \tag{40}$$

where

$$D_n := C_1 T^{-1} \int_{t=0}^T \left[s_\mu^\varepsilon (P^{1/2} \Delta_t) \|x_t\|^2 - E \left\{ s_\mu^\varepsilon (P^{1/2} \Delta_t) \|x_t\|^2 \right\} \right] dt \\ - 2C_1 \rho T^{-1} \int_{t=0}^T \left[s_\mu^{1+\varepsilon} (P^{1/2} \Delta_t) - E \left\{ s_\mu^{1+\varepsilon} (P^{1/2} \Delta_t) \right\} \right] dt$$

In view of large number law for depended processes (Poznyak, 2000) and lemma 2 from (Poznyak *et al.*), it follows that $D_n \xrightarrow{a.s.} 0$ and $T^{-1} \int_{t=0}^T S_{\mu,\varepsilon} (P^{1/2} \Delta_t) \Delta_t^\top P c(t, x_t, u_t) d\bar{W}_t \xrightarrow{a.s.} 0$. So, (40) leads to the following conclusion that $s_\mu (P^{1/2} \Delta_t) \xrightarrow{a.s.} 0$ since

$$\limsup_{T \rightarrow \infty} T^{-1} \int_{t=0}^T s_\mu^{1+\varepsilon} (P^{1/2} \Delta_t) \left(1 - \frac{\mu^2}{\|P^{1/2} \Delta_t\|^2} \right) \|\Delta_t\|^2 dt \stackrel{a.s.}{\leq} 0$$

The property 1 is proven. To obtain the property 2 it is sufficient to apply the operator of mathematical expectation to both sides of the inequality

$$V_t \leq V_0 + T D_n - \int_{t=0}^T S_{\mu,\varepsilon} (P^{1/2} \Delta_t) \Delta_t^\top P c(t, x_t, u_t) d\bar{W}_t$$

resulted from (40) that implies $E \{V_t\} \leq E \{V_0\} < \infty$. ■

Proof of Theorem 2. 1) Define the "joint" Lyapunov function $\bar{V}_t$ as

$$\bar{V}_t := V_0 \left(\begin{bmatrix} P_1^{1/2} & 0 \\ 0 & P_2^{1/2} \end{bmatrix} \begin{pmatrix} \Delta_t \\ \Delta_t^* \end{pmatrix} \right) + \\ (2 + \varepsilon) \left(k_{w,1}^{-1} tr \left\{ \tilde{V}_{1,t}^\top \tilde{V}_{1,t} \right\} + k_{v,1}^{-1} tr \left\{ \tilde{W}_{1,t}^\top \tilde{W}_{1,t} \right\} \right) \quad (41)$$

The identification and tracking errors $\Delta_t = \hat{x}_t - x_t$ and $\Delta_t^* = \hat{x}_t - x_t^*$ satisfy

$$d\Delta_t = [A\Delta_t + F_1 - \Delta f_t] dt - c_t d\bar{W}_t \\ d\Delta_t^* = A^* \Delta_t^* dt + c_t d\bar{W}_t, \quad c_t := c(x, u, t) \quad (42)$$

where Δf_t verifies

$$\|\Delta f_t\| = \left\| f(t, x_t) - \hat{f}_0(x_t) \right\| \leq L_{0,f} + \|W_{1,0}\| \bar{\sigma} + (L_{1,f} + \|A\|) \|x_t\| \\ \leq C_0 + C_1 \|\Delta_t\| + C_1 \|\Delta_t^*\| - 2\Delta_t^\top P_1 \Delta f_t \\ \leq \Delta_t^\top P_1 \Lambda_{\Delta f}^{-1} P_1 \Delta_t + 3 \|\Lambda_{\Delta f}\| \left(C_0^2 + C_1^2 \|\Delta_t\|^2 + C_1^2 \|\Delta_t^*\|^2 \right) \quad (43)$$

By (34), (35) it follows

$$dV_t = S_{\mu,\varepsilon}(z_t) \left(2\Delta_t^\top P_1 d\Delta_t + 2\Delta_t^{*\top} P_2 d\Delta_t^* \right) + 2(2 + \varepsilon) k_{v,1}^{-1} tr \left\{ d\tilde{V}_{1,t} \tilde{V}_{1,t} \right\} + \\ 2(2 + \varepsilon) k_{w,1}^{-1} tr \left\{ d\tilde{W}_{1,t}^\top \tilde{W}_{1,t} \right\} + \frac{1}{2} tr \left\{ \begin{bmatrix} c_t & 0 \\ 0 & c_t \end{bmatrix} \frac{\partial^2}{\partial z^2} V_0(z_t) \begin{bmatrix} c_t & 0 \\ 0 & c_t \end{bmatrix}^\top \right\} dt \quad (44)$$

2) The next step consists of the substitution of (42) in (44). The upper estimates for the terms $2\Delta_t^\top P_1 A$ and $2\Delta_t^\top P_1 F_1$ remain absolutely the same as before in Theorem 1. Substituting them in (44) implies

$$\begin{aligned} d\bar{V}_t &\leq S_{\mu,\varepsilon}(z_t) \left(\Delta_t^\top (Ric - Q_1) \Delta_t + 3 \|\Lambda_{\Delta f}\| C_0^2 + tr \left\{ L_{w1,t} \tilde{W}_{1,t} \right\} \right. \\ &\quad \left. + tr \left\{ L_{v1,t} \tilde{V}_{1,t} \right\} + \Delta_t^{*\top} [P_2 A^* + A^{*\top} P_2 + 3 \|\Lambda_{\Delta f}\| C_1^2 I] \Delta_t^* \right) dt \\ &\quad + 2(2 + \varepsilon)^2 \bar{c}^2 tr \{P_1 + P_2\} s_\mu^\varepsilon(z_t) dt - S_{\mu,\varepsilon}(z_t) (\Delta_t^\top P_1 c_t - \Delta_t^{*\top} P_2 c_t) d\bar{W}_t \end{aligned}$$

and by the accepted learning law ($L_{w1,t} = 0$, $L_{v1,t} = 0$) we finally have

$$\begin{aligned} d\bar{V}_t &\leq -2\rho s_\mu^{1+\varepsilon}(z_t) \|z_t\|^2 \left(1 - \mu^2 / \|z_t\|^2 \right) dt - S_{\mu,\varepsilon}(z_t) (\Delta_t^\top P_1 c_t - \Delta_t^{*\top} P_2 c_t) d\bar{W}_t \\ &\quad \left(\left[(2 + \varepsilon)^2 tr \{P_1 + P_2\} + (4 + \varepsilon) 3 \|\Lambda_{\Delta f}\| C_0^2 \right] s_\mu^\varepsilon(z_t) \bar{c}^2 - 2\rho s_\mu^{1+\varepsilon}(z_t) \mu^2 \right) dt \end{aligned}$$

The integration of the last inequality implies

$$2\rho T^{-1} \int_{t=0}^T s_\mu^{1+\varepsilon}(z_t) \|z_t\|^2 \left(1 - \mu^2 / \|z_t\|^2 \right) dt \leq T^{-1} \bar{V}_0 + D_T + D_T' - D_T'' \quad (45)$$

where

$$\begin{aligned} D_T &:= 2\rho T^{-1} \int_{t=0}^T E \{ s_\mu^{1+\varepsilon}(z_t) \} dt \left[\frac{\beta \int_{t=0}^T E \{ s_\mu^\varepsilon(z_t) \} dt}{2\rho \int_{t=0}^T E \{ s_\mu^{1+\varepsilon}(z_t) \} dt} - \mu^2 \right] \\ D_T' &:= (2 + \varepsilon) 3\bar{c}^2 \|\Lambda_{\Delta f}\| C_0^2 T^{-1} \int_{t=0}^T [s_\mu^\varepsilon(z_t) - E \{ s_\mu^\varepsilon(z_t) \}] dt \\ &\quad - 2\rho \mu^2 T^{-1} \int_{t=0}^T [s_\mu^{1+\varepsilon}(z_t) - E \{ s_\mu^{1+\varepsilon}(z_t) \}] dt \\ D_T'' &:= T^{-1} \int_{t=0}^T S_{\mu,\varepsilon}(z_t) (\Delta_t^\top P_1 c_t - \Delta_t^{*\top} P_2 c_t) d\bar{W}_t \end{aligned}$$

By the definition of μ , the law of large numbers and the property of Itô integral

$$\begin{aligned} D_T &\leq 0, \quad D_T' \rightarrow 0, \quad D_T'' \rightarrow 0 \quad (T \rightarrow \infty) \quad a.s. \\ 0 &\leq 2\rho \limsup_{T \rightarrow \infty} T^{-1} \int_{t=0}^T s_\mu^{1+\varepsilon}(z_t) \|z_t\|^2 \left(1 - \mu^2 / \|z_t\|^2 \right) dt \stackrel{a.s.}{\leq} 0 \end{aligned}$$

that proves the results of the theorem dealing. The mean square stability follows directly from (45). ■

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