On Information Measures for Bearings-only Estimation of a Random Walk Target

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Abstract
This report considers the bearings-only estimation problem of a random walk target. The estimation performance for a number of information measures in the Extended Kalman filter framework are investigated, both from a theoretical point of view and by simulation examples.

Keywords: bearings-only estimation, information measure, Extended Kalman filter
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Chapter 1

Introduction

Optimal trajectory for bearings-only tracking is a classical nonlinear estimation problem. The problem is to estimate the state of a target given a number of noisy measurements. The sensor platform is free to maneuver, and the problem is to find the optimal trajectory that maximizes the tracking and estimation performance.

The problem can be divided into subgroups depending on assumptions about the sensor observation model, the target motion model, the sensor motion model, etc. The most common assumptions are that target and observer are moving in the same plane and that the target travels on straight lines with constant velocity [5]. It can be shown that the observer motion is important to obtain a unique solution [7]. In [3] the observability of the three-dimensional problem is analyzed and conditions on the observability of an $n$-th order target dynamic model is given in [1].

Many researchers are defining an optimization problem with an information theoretic utility function, see e.g. [6], [10], [2], [8], [9]. A common choice is to use Extended Kalman Filter and a utility criterion from experimental design.

In estimation with a bearings-only sensor we intuitively realize that the closer the target, the less the measurement noise will affect the estimate. In addition, observability aspects of the estimation problem play an important role in the overall result. In the bearings-only sensor case, the triangulation of the target becomes better if the relative angle of the two observation positions w.r.t. the target position, is near $\pi/2$ rad, than if the relative angle is small. One may say that the optimal position of an observation is a combination of minimizing the distance and good triangulation. Furthermore, the initial covariance of the target and the objective function affect the result. In this paper we investigate the relationship between these different types of behavior, both from a theoretical point of view, and by simulation examples.

The estimation problem is first defined in Section 2. In Section 3 we derive objective functions based on some well-known information criteria. Finally we show some simulations examples that illustrate the theoretical results in Section 5.
Chapter 2

Problem Definition

In this work we consider the bearings-only estimation problem of a random walk target. The problem is illustrated in Figure 2.1, where a sensor platform is receiving measurements of a target.

2.1 Target Model

Let the state vector be the position of the target

\[ x = [x, \ y]^T \]  \hspace{1cm} (2.1)

and assume that its covariance matrix is

\[ P = \begin{bmatrix} p_{xx} & 0 \\ 0 & p_{yy} \end{bmatrix}. \]  \hspace{1cm} (2.2)
Assuming zero cross-correlation is no restriction since we can always find a reference system where the covariance matrix is diagonal.

2.2 Sensor Platform Model

Let the position of the bearings-only sensor be

\[ x^s = [x^s, y^s]^T \]  

(2.3)

and we assume that this position is known. We parameterize the next sensor position with a simple circle representation

\[ x^s_+ = x^s + v^s T \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \]  

(2.4)

where \( v^s \) is the speed of the sensor platform and \( T \) is the sample time. Thus, by using this geometrical model we ignore all dynamic constraints, i.e. the sensor is free to move in any direction \( \theta \).

2.3 Observation Model

The bearings-only observation model is expressed as

\[ z = h(\hat{x}, x^s, e) = \arctan \left( \frac{\hat{y} - y^s}{\hat{x} - x^s} \right) + e \]  

(2.5)

where the measurement noise is \( e \sim \mathcal{N}(0, R) \). The Jacobian of the observation model is

\[ C = \begin{bmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{bmatrix}_{x=\hat{x}, e=0} = \frac{1}{||r||^2} \begin{bmatrix} -(\hat{y} - y^s), \hat{x} - x^s \end{bmatrix} \]  

(2.6)

where

\[ r = \begin{bmatrix} \hat{x} - x^s \\ \hat{y} - y^s \end{bmatrix} \]  

(2.7)

For later use we also here define \( \bar{r} \) as the normalization of \( r \), i.e.,

\[ \bar{r} = r/||r||. \]  

(2.8)

2.4 EKF Covariance Update Equation

The well-known Kalman filter is the optimal filter, in the minimum square error sense, for linear and Gaussian systems [4]. For non-linear models the Smith Extended Kalman filter (EKF) is a very popular approach where linearizations of the models are used. (Extended) Kalman filter consists of a prediction step and a measurement update step. However, in this work we are mostly interested in the update step and therefore we assume that \( \hat{x} \) is the predicted state and \( P \) is its predicted covariance.

Let \( z \) and \( \hat{z} = h(\hat{x}, x^s, 0) \) be a measurement and the expected measurement, respectively. The covariance update step is

\[ P_+ = P - PC^T S^{-1} CP \]  

(2.9)
where the covariance of the innovation $z - \hat{z}$ is

$$S = CPC^T + R.$$  \hfill (2.10)

The update step can also be expressed in the so-called information form

$$P_+^{-1} = P^{-1} + C^T R^{-1} C.$$ \hfill (2.11)

### 2.5 Optimization Problem

The optimization problem is defined as

$$\min_{\theta} \quad L(P_+(\theta)).$$ \hfill (2.12)

The loss functions considered in this report are the information measures defined as

\begin{align*}
L_{\text{det}} P(P_+) & = \det P_+ \quad \hfill (2.13) \\
L_{\text{det}} P^{-1}(P_+) & = -\det P_+^{-1} \quad \hfill (2.14) \\
L_{\text{tr}} P(P_+) & = \text{tr} P_+ \quad \hfill (2.15) \\
L_{\text{tr}} P^{-1}(P_+) & = -\text{tr} P_+^{-1}. \quad \hfill (2.16)
\end{align*}
Chapter 3

Information Measures

This section presents some results for the information measures (2.13)-(2.16) and the corresponding optimization problem (2.12).

**Proposition 1 (det \( P_+ \) for Random Walk Target)** Let \( P > 0 \) be defined as (2.2) and assume that \( P_+ \) is given by the EKF update equations in (2.9), (2.10) and (2.6), then the determinant of \( P_+ \) can be expressed as

\[
\det P_+ = \frac{R \det P}{S}. \tag{3.1}
\]

**Proof:** First we note that \( S \) is a scalar and \( S > 0 \) since \( P > 0 \). By the use of the determinant rules \( \det(AB) = \det A \det B \) and \( \det(I + ab^T) = 1 + b^T a \) we can derive the determinant of \( P_+ \) in (2.9) as

\[
\begin{align*}
\det P_+ &= \det \left( P - \frac{1}{S} P C^T C P \right) \\
&= \det P \det \left( I - \frac{1}{S} C^T C P \right) \\
&= \det P \left( 1 - \frac{1}{S} C P C^T \right) = \det P \left( \frac{S - C P C^T}{S} \right) \\
&= \frac{R \det P}{S}.
\end{align*}
\]

\( \Box \)

**Remark 1** If \( S \) is scalar, the result 3.1 is general, i.e., independent on the observation model matrix \( C \), as long as \( C C^T > 0 \).

**Corollary 1 (min \( \det P_+ \) for Random Walk Target)** Assume that we have the same prerequisites as in Proposition 1. The the optimal \( \theta \) for the \( \det P_+ \)
criterium is
\[ \theta^* = \arg \min_{\theta} \det P_+ = \arg \max_{\theta} \frac{r^T(\theta) P^{-1} r(\theta)}{||r(\theta)||^2}. \tag{3.3} \]

Proof: Only \( S \) is dependent on \( \theta \) in (3.1). Rewrite \( S \) as
\[ S(\theta) = C(\theta) P C(\theta)^T + R \]
\[ = \frac{P_y (\hat{y} - y_s(\theta))^2 + P_y (\hat{x} - x_s(\theta))^2}{||r(\theta)||^4} + R \]
\[ = \det P r(\theta)^T P^{-1} r(\theta) \]
\[ = \frac{||r(\theta)||^4}{||r(\theta)||^4} + R. \]

Thus, minimizing \( \det P_+ \), w.r.t. \( \theta \), is equivalent problem of maximizing \( S \), which is an equivalent problem of maximizing
\[ \frac{r^T(\theta) P^{-1} r(\theta)}{||r(\theta)||^4}. \tag{3.5} \]

\[ \square \]

Corollary 2 (\( \det P_+^{-1} \) for Random Walk Target) Assume that we have the same prerequisites as in Proposition 1. Furthermore, assume that \( R > 0 \). The determinant of \( P_+^{-1} \) can then be expressed as
\[ \det P_+^{-1} = \frac{S}{R \det P}. \tag{3.6} \]

Proof: Obvious from Proposition 1 since \( \det P^{-1} = (\det P)^{-1} \).

\[ \square \]

Corollary 3 (\( \min - \det P_+^{-1} \) for Random Walk Target) Assume that we have the same prerequisites as in Proposition 1. The the optimal \( \theta \) for the \( \det P_+^{-1} \) criterium is
\[ \theta^* = \arg \min_{\theta} - \det P_+ = \arg \max_{\theta} \frac{r^T(\theta) P^{-1} r(\theta)}{||r(\theta)||^2}. \tag{3.7} \]

Proof: Obvious from Corollary 1 and Corollary 2.

\[ \square \]

Remark 2 Since \( \det P^{-1} = (\det P)^{-1} \) is no surprise that the results (3.3) and (3.7) are equal.

Proposition 2 (\( \tr P_+ \) for Random Walk Target) Assume that we have the same prerequisites as in Proposition 1. The trace of \( P_+ \) can then be expressed as
\[ \tr P_+ = \frac{R \tr P + \frac{1}{||r||^2} \det P}{S}. \tag{3.8} \]
Proof: As before we note that \( S \) is a scalar.

\[
\text{tr} \, P_+ = \text{tr} \left( P - \frac{1}{S} P C^T C \right) \tag{3.9}
\]

\[
= \frac{1}{S} \left( R \text{tr} \, P + C P^T \text{tr} \, P - \text{tr}(P C^T C) \right)
\]

\[
= \frac{1}{S} \left( R \text{tr} \, P + \frac{\det \, P}{||r||^2} \right)
\]

since

\[
C P C^T \text{tr} \, P = \frac{P_x (\hat{y} - y^s(\theta))^2 + P_y (\hat{x} - x^s(\theta))^2}{||r(\theta)||^4} \tag{3.10}
\]

and

\[
\text{tr}(P C^T C) = \frac{P_x^2 (\hat{y} - y^s(\theta))^2 + P_y^2 (\hat{x} - x^s(\theta))^2}{||r(\theta)||^4} \tag{3.11}
\]

Remark 3 In the case of a perfect measurement, i.e. \( R = 0 \), the determinant criterion (3.1) is zero, but the trace criterion (3.8) is not zero.

Corollary 4 (\( \min \text{tr} \, P_+ \) for Random Walk Target)

\[
\theta^* = \arg \min_\theta \text{tr} \, P_+
\]

\[
= \arg \max_\theta r^T (\theta) P^{-1} r(\theta) + \frac{1}{\det \, P} R ||r(\theta)||^2
\]

\[
1 + \frac{\text{tr} \, P}{\det \, P} R ||r(\theta)||^2.
\]

Proof: Use (3.4) in (3.8).

\[\square\]

Proposition 3 (\( \text{tr} \, P_+^{-1} \) for Random Walk Target) Assume that we have the same prerequisites as in Corollary 2. The trace of \( P_+^{-1} \) can then be expressed as

\[
\text{tr} \, P_+^{-1} = \frac{\text{tr} \, P}{\det \, P} + \frac{1}{R ||r||^2}. \tag{3.13}
\]

Proof: Note that

\[
\text{tr} \, A^{-1} = \frac{\text{tr} \, A}{\det \, A} \tag{3.14}
\]

for a general \((2 \times 2)\) matrix \( A \). Apply this rule to \( \text{tr} \, P_+^{-1} \) and use the results (3.1) and (3.8).

\[\square\]
Corollary 5 (\(\min - \text{tr} \ P_+^{-1}\) for Random Walk Target)

\[
\theta^* = \arg \min_{\theta} - \text{tr} \ P_+^{-1} = \arg \min_{\theta} ||r(\theta)|| \tag{3.15}
\]

Proof: Obvious from (3.13), since \(r\) is the only variable dependent on \(\theta\). □

Remark 4 Minimizing \(- \text{tr} \ P_+^{-1}\) is the same as minimizing the distance between the target and the sensor platform.
Chapter 4

Analysis of $\det P_{+}^{-1}$

Consider the formula in (3.7)

$$\arg\min_\theta - \det P_{+}^{-1} = \arg\max_\theta \frac{\bar{r}^T(\theta)P^{-1}\bar{r}(\theta)}{\|r(\theta)\|^2}$$ (4.1)

and note that

$$\frac{1}{\lambda^P_{\text{max}}} \leq \frac{\bar{r}^T(\theta)P^{-1}\bar{r}(\theta)}{\|r(\theta)\|^2} \leq \frac{1}{\lambda^P_{\text{min}}}$$ (4.2)

where $\lambda^P_{\text{min}}$ and $\lambda^P_{\text{max}}$ are the minimum and the maximum eigenvalues of $P$. (Thus, since $P$ is diagonal, the eigenvalues are the diagonal elements.) Hence

$$R_1 \triangleq \max_\theta \frac{\bar{r}^T(\theta)P^{-1}\bar{r}(\theta)}{\min_\theta \frac{\bar{r}^T(\theta)P^{-1}\bar{r}(\theta)}{\|r(\theta)\|^2}} = \frac{\lambda^P_{\text{max}}}{\lambda^P_{\text{min}}}.$$ (4.3)

We can also define

$$R_2 \triangleq \max_\theta \frac{\|r(\theta)\|^2}{\min_\theta \frac{\|r(\theta)\|^2}{\|\bar{r}(\theta)\|^2}} = \frac{(d + v^sT)^2}{(d - v^sT)^2}$$ (4.4)

where $d = \sqrt{x^2 + y^2}$ is the distance of the sensor to the target. The $\hat{d}$ values where $R_1 = R_2$ have important implications with this reward function. Equating $R_1 = R_2$,

$$\frac{(d + v^sT)^2}{(d - v^sT)^2} = \frac{\lambda^P_{\text{max}}}{\lambda^P_{\text{min}}}$$ (4.5)

and taking square roots of the both sides

$$\frac{d + v^sT}{|d - v^sT|} = \sqrt{\frac{\lambda^P_{\text{max}}}{\lambda^P_{\text{min}}}}$$ (4.6)

Now we can obtain the lower and upper solutions $\hat{d}_l$ and $\hat{d}_u$ to this equation according to whether $d - v^sT \geq 0$ or otherwise.

$$\hat{d}_l = \frac{\lambda^P_{\text{max}}/\lambda^P_{\text{min}} - 1}{\sqrt{\lambda^P_{\text{max}}/\lambda^P_{\text{min}} + 1}} v^sT$$ (4.7)

$$\hat{d}_u = \frac{\lambda^P_{\text{max}}/\lambda^P_{\text{min}} + 1}{\sqrt{\lambda^P_{\text{max}}/\lambda^P_{\text{min}} - 1}} v^sT.$$ (4.8)
A typical case of $R_1$ and $R_2$ with respect to target-to-sensor distance $\hat{d}$ for $v^*T = 1$ and $\lambda_{\text{max}}^P/\lambda_{\text{min}}^P = 4$.

A typical case of $R_1$ and $R_2$ are illustrated in Figure 4.1 for $v^*T = 1$ and $\lambda_{\text{max}}^P/\lambda_{\text{min}}^P = 4$.

Note that since $R_2$ goes to infinity when $\hat{d}$ goes to $v^*T$, it is reasonable to expect that $R_2 \gg R_1$ in the range $\hat{d}_l \leq \hat{d} \leq \hat{d}_u$. If $R_2 \gg R_1$, the reward will be quite sensitive to the denominator. Therefore, when the target-to-sensor distance $\hat{d}$ is in the range $\hat{d}_l \leq \hat{d} \leq \hat{d}_u$, then the UAV will directly go towards the target. Note that the range $\hat{d}_l \leq \hat{d} \leq \hat{d}_u$ gets smaller and smaller when $\lambda_{\text{max}}^P/\lambda_{\text{min}}^P \rightarrow \infty$. Hence, this behavior is common only for a limited range of initial target uncertainties. The most important of these cases is the case where $\lambda_{\text{max}}^P/\lambda_{\text{min}}^P \approx 1$ in which case the motion directly towards target is global.

Outside the range $\hat{d}_l \leq \hat{d} \leq \hat{d}_u$ we can assume $R_1 \gg R_2$ and in this case, the reward will be sensitive to changes in numerator and it is going to be maximized when the normalized vector $\bar{v}(\theta)$ points to the least uncertain (most certain) direction of the target due to the numerator term. In other words, the UAV will go in $x$ direction if $P_x < P_y$ and in $y$ direction otherwise. This type of behavior is the common motion mode when $\lambda_{\text{max}}/\lambda_{\text{min}} \rightarrow \infty$. 

Figure 4.1: A typical case of $R_1$ and $R_2$ with respect to target-to-sensor distance $\hat{d}$ for $v^*T = 1$ and $\lambda_{\text{max}}^P/\lambda_{\text{min}}^P = 4$.
Chapter 5

Simulations

5.1 2D Simulation Examples

In this section we illustrate the equations above by showing some examples in Figures 5.1 and 5.2 where we use the determinant (3.6) and the trace criterion (3.13), respectively. Each figure consists of 6 subfigures. Two different initial covariance matrices are used in each case, $P = \text{diag}(2^2, 1^2)$ in the subfigures to the left and $P = \text{diag}(2^2, 0.1^2)$ in the subfigures to the right. The initial covariance is shown as a black ellipse.

The subfigures in the top row show the information value, given by the current criterion, as a contour plot. The information value is computed for each position given one observation in that position.

The subfigures in the second row show the information gradient as a vector field plot. For each position, the arrow shows the direction where the best information will be obtained. In practice, the information value on a small circle around the current position is evaluated given one observation on the circle.

Finally, the subfigures in the third row are showing an information gradient as the former one, but the covariance is first updated by an observation in the initial position, i.e., the center position of the circle. This is a more realistic case than the one-observation case, since we in real world experiments often get a sequence of several observations.

5.2 3D Simulation Examples

The work can of course be extended to bearings-only in 3D with some modifications of the models. The state vectors of the target and the sensor are augmented by a third dimension state $z$ and $z^s$, respectively. The target motion model is a 3D random walk model, analogous to the former model. However, we assume that the sensor is moving in a plane with constant height $z$, thus, the dynamic model of the sensor can still be expressed as (2.4). The observation model is
Figure 5.1: The result of the 2D determinant information criterion $\text{det} P^{-1}$. Initial covariances are shown as black ellipses. First row: information value given one observation. Second row: information gradient. Third row: information gradient given an initial observation.
Figure 5.2: The result of the 2D trace information criterion $\text{tr} \, P$. Initial covariances are shown as black ellipses. First row; information value given one observation. Second row; information gradient. Third row; information gradient given an initial observation.
augmented by a second angle measurement

\[ z = h(\hat{x}, \mathbf{x}^s, e) = \begin{pmatrix} \arctan\left(\frac{\hat{y} - y^s}{\hat{x} - x^s}\right) \\ \arctan\left(\frac{\hat{z} - z^s}{\sqrt{(\hat{x} - x^s)^2 + (\hat{y} - y^s)^2}}\right) \end{pmatrix} + e \quad (5.1) \]

where the measurement noise is \( e \sim \mathcal{N}(0, \text{diag}(R, R)). \)

Unfortunately, intuitively nice expressions can not, at least not by us, be obtained as in the 2D case. However, we show some simulations examples in Figures 5.3 and 5.4 that can be compared to the 2D case. The height of the sensor is in these examples constant 2 units and the initial covariance matrices are \( P = \text{diag}(2^2, 1^2, 0.1^2) \) and \( P = \text{diag}(2^2, 0.1^2, 0.1^2) \). We see that the behavior is similar to the 2D case "far" from the target. There is a singularity in the target location that can be reached in the 2D case but not in the 3D case when the sensor platform travels on a different altitude. This causes that the information surface close to the target is finite and more "flat" in the 3D case.
Figure 5.3: The result of the 3D determinant information criterion $\det P^{-1}$. Initial covariances are shown as black ellipses. First row; information value given one observation. Second row; information gradient. Third row; information gradient given an initial observation.
Figure 5.4: The result of the 3D trace information criterion $\text{tr}\, P$. Initial covariances are shown as black ellipses. First row; information value given one observation. Second row; information gradient. Third row; information gradient given an initial observation.
Chapter 6

Conclusions

The conclusions of this work are

- The determinant expression (3.1) and (3.6) are not dependent on the observation model.

- The trace of the inverse of the covariance (3.13) is not that useful, the sensor platform will go towards the target regardless of the initial prediction covariance.

- In Chapter 4 the determinant criterion (3.1) is analyzed. We showed that for a certain distance range the sensor platform should go towards the target. Outside this range the sensor platform should go in the $x$ direction if $P_x > P_y$, otherwise in the $y$ direction.

- The simulation examples shows that the determinant (inverse) criterion (3.6) and trace criterion (3.8) are significantly different. The determinant criterion wants the sensor to go slightly more towards the target than the trace criterion. However, in practice the optimization result will be similar since they are similar in the area in the major uncertainty direction of the covariance.

- The simulation examples shows that the information curves for the 3D case are similar to the 2D case when the target and the sensor are "far" apart. When the sensor is "closer" to the target the information surface is more flat since the sensor and target are on different altitudes and the singularity at the target position can not be reached.

- The trace of the covariance is maybe the hardest to analyze of the loss functions considered in this work, but we still think that this loss function is the most useful in the bearings-only estimation case. For instance, one issue with the determinant criterion (see Remark 3) is that it can be zero if just only one eigenvalue is zero.
Bibliography


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