Recursive Parameter Estimation
Using Closed-loop Observations

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June 12, 2003

Report no.: LiTH-ISY-R-2509

Technical reports from the Control & Communication group in Linköping are available at http://www.control.isy.liu.se/publications.
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Abstract

The aim of the given paper is development of a joint input-output approach for the identification of closed-loop systems in the case of an additive correlated noise acting on the output of the system. Here the ordinary prediction error method is applied to solve the closed-loop identification problem by processing observations. In the case of the known regulator, the two-stage method, which belongs to the ordinary joint input-output approach, reduces to the one-stage method. In such a case, the open-loop system could be easily determined after some extended rational transfer function has been identified. In the case of the unknown regulator, the estimate of the extended transfer function is used to generate an auxiliary input. The form of an additive noise filter, that ensures the minimal value of the mean square criterion, is determined. The results of numerical simulation and identification of the open-loop system by computer, using the two-stage method and closed-loop observations are given.

1 INTRODUCTION

The closed-loop systems identification approaches can be divided into three main groups: a direct approach, an indirect approach, and a joint input-output, which are worked out to identify the open-loop system. The direct approach is realized, using the input and noisy output observations when the feedback is ignored. In such a case, the open-loop system is identified if the respective identifiability conditions are satisfied [1]. The indirect approach is used, first, to identify some closed-loop system transfer function and, second, to determine the open-loop system parameters, assuming that the regulator is known beforehand [2]. The joint input-output approach regards the input and output both together as the output of some augmented system excited by some extra input or a set-point signal and noise. It determines the open-loop system parameters, applying the estimate of the transfer function of the augmented system [3]. In this connection, in the case of linear feedback, the two-stage method [4] is proposed, provided that the system is in the same set of models under consideration. Recently a projection method for closed-loop identification has been worked out by [5], which belongs to the framework of the two stage method, too. The joint input-output approach usually uses the well known ordinary prediction error method to solve the closed-loop identification problem [6]. Here, there arises a problem of well-grounded determination of the form of the input-output relationship of some system transfer function, because many of its models, such as the finite impulse response (FIR), a high order autoregressive model with external input, a finite number of alternative orthogonal functions, such as Laguerre functions or generalized versions, or a noncausal FIR model could be used in order to create an auxiliary input of the system [4], [5], [7]. It is also important here to analyze the effect of the correlated additive noise on the accuracy of the estimates of unknown parameters, obtained by processing observations.

In this paper, a two-stage method applying the prediction error model, will be analyzed in respect of the form of a sensitivity function of the augmented system applied to restore the true input, acting in a closed-loop. In the Section 2 the statement of the problem is given. Based on it in the Section 3 we determine the optimal sensitivity function used to generate the auxiliary input in the case of the known regulator as well as in the opposite case. Section 4 presents the recursive parameter estimation procedure based on the ordinary recursive least squares (RLS). In the Section 5 the efficiency of the estimation procedure is analyzed with respect of
different structures of the noise filter. A Monte Carlo simulation results, concerning the current estimation of parameters calculated by using the two-stage method are given in Section 6.

2 STATEMENT OF THE PROBLEM

Assume that a control system to be observed is causal, linear, and time-invariant (LTI) with one output \( y(k) \) and one input \( u(k) \) given by the equation

\[
y(k) = G_0(q, \theta)u(k) + H_0(q, \varphi)\xi(k),
\]

that consists of two parts (Fig. 1): a system model \( G_0(q, \theta) \) and a noise one \( H_0(q, \varphi) \). Here \( k \) is a current number of observations of respective signal, \( \theta, \varphi \) are unknown parameter vectors to be estimated, \( q \) is the backward time-shift operator such that \( q^{-1}u(k) = u(k-1) \). \( \{\xi(k)\} \) is used to generate immeasurable noise \( v(k) \) and it is assumed to be statistically independent and stationary with

\[
E[\xi(k)] = 0, E[\xi(k)\xi(k + r)] = \sigma^2_k \delta(r),
\]

where \( E[\xi(k)] \) is the mean value, \( \sigma^2_k \) is the variance, \( \delta(r) \) is the Kronecker delta function, and \( H_0(q, \varphi) \) is an inversely stable monic filter.

The input \( \{u(k)\} \) is given by

\[
u(k) = [r(k) - y(k)]G_R(q, \alpha),
\]

where the reference signal \( \{r(k)\} \) is a quasi-stationary signal, independent of the stochastic disturbance \( \{v(k)\} \), and the controller \( G_R(q, \alpha) \), which is designed by minimizing the quadratic performance function, is exponentially stable. Here \( \alpha \) is the parameter vector of the controller.

The aim of the given paper is to investigate the two-stage approach in the case of additive correlated noise \( \{v(k)\} \), acting on the output of the system \( G_0(q, \theta) \) to be identified by closed-loop observations.

3 THE OPTIMAL MODEL

By the two-stage method one has to estimate the parameters \( \mu \) in the model [3] of the form

\[
u(k) = S(q, \mu)r(k) + H_1(q, \psi)\xi(k)
\]

in order to generate \( \hat{u}(k) = \hat{S}_N(q, \mu)r(k) \), and then to identify the parameter vector \( \theta \) of the open-loop system \( G_0(q, \theta) \), using auxiliary input \( \{\hat{u}(k)\} \) and the noisy output \( \{y(k)\} \). Here \( S(q, \mu) \) is the sensitivity function of some augmented system; \( \{\hat{u}(k)\} \) is the auxiliary input or the estimate of the true input \( \{u^*(k)\} \); \( \hat{S}_N(q, \mu) \) is the estimate of \( S(q, \mu) \), determined by using \( N \) pairs of observations of the reference signal \( \{r(k)\} \) and noisy input \( \{u(k)\} \) and \( H_1(q, \psi) \) is a noise model parameterized by \( \psi \).

First of all, there arises a problem to define the form of \( S(q, \mu) \) [8]. One can solve this, substituting (1) into (3) and rewriting it in such a form

\[
u(k) = \frac{G_R(q, \alpha)}{1 + G_R(q, \alpha)G_0(q, \theta)}[r(k) - H_0(q, \varphi)\xi(k)].
\]

Then rearranging it one could get (4) with \( S(q, \mu) = \{[G_R(q, \alpha)]^{-1} + Go(q, \theta)\}^{-1} \) and

\[
H_1(q, \psi) = -S(q, \mu)H_0(q, \varphi).
\]

Assuming that

\[
G_0(q, \theta) = \frac{B(q)}{A(q)} = \frac{b_0q^{-m} + \ldots + b_nq^{-n}}{1 + a_1q^{-1} + \ldots + a_nq^{-n}},
\]

\[
G_R(q, \alpha) = \frac{D(q)}{C(q)} = \frac{d_0 + d_1q^{-1} + \ldots + d_kq^{-k}}{1 + c_1q^{-1} + \ldots + c_kq^{-k}},
\]

\[
S(q, \mu) \text{ could be expressed by
}\]

\[
S(q, \mu) = \frac{D(q)A(q)}{C(q)A(q) + B(q)D(q)} = \frac{F(q)}{P(q)}.
\]

Here

\[
\frac{F(q)}{P(q)} = \frac{f_0f_1q^{-1} + \ldots + f_pq^{-p}}{1 + p_1q^{-1} + \ldots + p_kq^{-k}}
\]

is supposed to be inversely stable.

Corollary 1 The error \( \varepsilon(k) = P(q, P^*)u(k) - F(q, F^*)r(k) \) has the zero mean \( E[\varepsilon(k)] = 0 \), is non-correlated and its correlation function \( K_{\varepsilon\varepsilon}(\tau) = \sigma^2_\varepsilon \delta(\tau) \) if and only if

\[
H_1(q, \psi) = P^{-1}(q).
\]

Here \( F^*, P^* \) are true parameter vectors.

\[
\xi(k) \rightarrow H_0(q, \varphi) \rightarrow v(k)
\]

\[
r(k) \rightarrow e(k) \rightarrow u(k) \rightarrow G_R(q, \alpha) \rightarrow G_0(q, \theta) \rightarrow y(k)
\]

Figure 1: A closed-loop system to be observed
Proof: This is proven by substituting (7) and (9) into (4).

**Remark 1** If (9) is valid, then substituting (7) and (9) into (5) we have

\[ H_0(q, \varphi) = -[D(q)A(q)]^{-1}. \]  

(10)

**Corollary 2** If (10) is satisfied, then the error \( \tilde{e}(k) = A(q,a^*)y(k) - B(q,b^*)u(k) \) has the zero mean \( E(\tilde{e}(k)) = 0 \), is non-correlated and its correlation function \( K_F = \sigma^2 \delta(t) \) when \( y(k) = D(q,d^*)y(k) \) and \( \tilde{u}(k) = D(q,d^*)u(k) \). Here \( a^*, b^*, d^* \) are true parameter vectors.

Proof: This is proven using (1), where instead of \( u(k) \) and \( H_0(q, \varphi) \) the \( \tilde{u}(k) \) and (10) are substituted.

**Conclusion 1** Under considered conditions the criterion

\[ I_N(f^*, p^*) = \frac{1}{N} \sum_{k=1}^{N} \tilde{e}^2(k) \]  

(11)

and

\[ \bar{I}_N(b^*, a^*) = \frac{1}{N} \sum_{k=1}^{N} \tilde{e}^2(k), \]  

(12)

to be minimized acquire the minimums if and only if the equality (9) and (10) are satisfied, respectively.

In the case of the known regulator \( G_R(q, \alpha) \) the two-stage method reduces to the one-stage method. In such a case, the open-loop system \( G_0(q, \theta) \) could be easily determined after the extended rational transfer function (7) has been identified.

**4 RECURSIVE PROCEDURE**

The first step is to estimate the unknown parameters of \( S(q, \mu) \). For this purpose the ordinary prediction error method, based on the recursive LS (RLS) of the form

\[ \hat{\mu}(k) = \hat{\mu}(k - 1) + \frac{\Gamma(k - 1)z(k - 1)}{1 + z^T(k)\Gamma(k - 1)z(k)} \tilde{e}(k) \]  

(13)

\[ \Gamma(k) = \Gamma(k - 1) - \frac{\Gamma(k - 1)z(k)z^T(k)\Gamma(k - 1)}{1 + z^T(k)\Gamma(k - 1)z(k)} \]

could be used with the vector of observations \( z^T(k) = [-r(k-1), \ldots, -r(k-\alpha), u(k-1), \ldots, u(k-\kappa)] \), and some initial values of the vector \( \hat{\theta}(0) \) and matrix \( \Gamma(0) \). Here

\[ \hat{\mu}^T(k) = [f^T(k), \hat{p}^T(k)] \]  

(14)

\[ \tilde{e}(k) = P(q, \hat{\mu}(k - 1))u(k) - F(q, \hat{f}(k - 1))\nu(k) \]

are the current estimate of the parameter vector \( \mu^T = (f^T, p^T) = (f_0, \ldots, f_{\alpha}, p_1, \ldots, p_\kappa) \) and the prediction error on the current \( k \)-th iteration, respectively.

The next step is to restore the current \( k \)-th value of the auxiliary input according to the formula

\[ \hat{u}(k) = S(q, \hat{\mu}(k))\nu(k). \]  

(15)

Then, the current estimate of the parameter vector

\[ \hat{\theta}(k) = \hat{\theta}(k - 1) + \frac{\Pi(k - 1)\tilde{e}(k) - \tilde{e}(k)}{1 + z^T(k)\Pi(k - 1)z(k)} \tilde{e}(k), \]  

(17)

with the vector of data of input-output signals \( z^T(k) = (\hat{u}(k-1), \ldots, \hat{u}(k-m), -\hat{y}(k-1), \ldots, -\hat{y}(k-n)) \) and some initial values of the vector \( \hat{\theta}(0) \) and matrix \( \Pi(0) \). Here

\[ \hat{\theta}^T(k) = (b^T(k), \hat{a}^T(k)) \] is the estimate of the parameter vector (16),

\[ \tilde{e}(k) = A(q, \hat{a}(k-1))y(k) - B(q, \hat{b}(k-1))\hat{u}(k) \]

is the prediction error on the current \( k \)th iteration.

**5 DETERMINATION OF EFFICIENCY**

In some cases the recursive estimation procedure (13)–(17) will become inefficient, and estimates of parameters (16) will be biased. It is obvious, that the bias of the estimates also depends on the choice of a recursive algorithm used for the unknown parameter estimation. On the other hand, it is related with our assumption on the structure of the filter \( H_0(q, \varphi) \), generating a noise process \( \{e(k)\} \). It is known, that choice of an algorithm depends on the form of \( H_0(q, \varphi) \). For instance, if equality (10) is valid, then to estimate the parameter vector (16), the recursive generalized least squares (RGLS) could be used. If, for simplicity, we assume that in (10) the polynomial \( D(q, d^*) \equiv 1 \), then the RLS will be efficient. Therefore, if the initial model of the noise process is accurate (e.g., we assume the second case while actually the first one is valid), then the recursive parameter estimation algorithm was chosen by us incorrectly and the above-mentioned estimation procedure fails. No doubt it is necessary to determine the efficiency of the parameter estimation procedure during recursive calculations. In this paper the efficiency is checked by comparing current estimates \( \hat{a}_1(k), \ldots, \hat{a}_n(k) \) of parameters of the open-loop system \( G_0(q, \theta) \) with the respective estimates of the parameter...
of the denominator of the filter $H_0(q, \varphi)$. First of all, assume that the filter $H_0(q, \varphi)$ is of the general form
\[
H_0(q, \varphi) = \frac{N(q, \varrho)}{M(q, \nu)} = \frac{1 + \varrho_1 q^{-1} + \ldots + \varrho_a q^{-a}}{1 + \nu_1 q^{-1} + \ldots + \nu_o q^{-o}}.
\] (18)

In this case, the necessary calculations are as follows:
a) reconstruction of the current value of the correlated noise $\{v(k)\}$ according to
\[
\hat{v}(k) = y(k) - \hat{y}(k),
\] (19)
\[
\hat{y}(k) = \frac{\hat{b}_1(q)q^{-1} + \ldots + \hat{b}_m(q)q^{-m}}{1 + \hat{a}_1(q)q^{-1} + \ldots + \hat{a}_n(q)q^{-n}}\hat{u}(k);
\] (20)
b) calculation of estimates $\hat{\phi}(k) = (\hat{\varphi}^T(k), \hat{v}^T(k))$ of the parameters $\varphi = (\varphi^T, v^T)$ of the noise filter $H_0(q, \varphi)$ of form (18) by the recursive equations
\[
\dot{\hat{\phi}}(k) = \dot{\hat{\varphi}}(k - 1) + \frac{\Lambda(k - 1)\rho(k - 1)}{1 + \rho^T(k)\Lambda(k - 1)\rho(k)}\hat{v}(k),
\] (21)
\[
\Lambda(k) = \Lambda(k - 1) - \frac{\Lambda(k - 1)\rho(k)\rho^T(k)\Lambda(k - 1)}{1 + \rho^T(k)\Lambda(k - 1)\rho(k)}.
\] (22)

Here the vector of observations
\[
\rho^T(k) = (\hat{v}(k), \ldots, \hat{v}(k - \omega), -\hat{\xi}(k - 1), \ldots, -\hat{\xi}(k - o)),
\]
assuming that
\[
\hat{\xi}(k) = \frac{1 + \hat{v}_1(k)q^{-1} + \ldots + \hat{v}_\omega(k)q^{-\omega}}{1 + \hat{\nu}_1(k)q^{-1} + \ldots + \hat{\nu}_o(k)q^{-o}}\hat{v}(k),
\] (23)
with
\[
\hat{\varphi}(k) = (\hat{\varphi}_1(k), \ldots, \hat{\varphi}_\omega(k)), \dot{\hat{v}}^T(k) = (\hat{v}_1(k), \ldots, \hat{v}_\omega(k));
\]
c) check-up of the conditions
\[
\frac{1}{\omega} \left( \sum_{j=1}^\omega \hat{\varphi}_j^2(k) \right)^{\frac{1}{2}} < \delta_1, \quad \frac{1}{n} \left( \sum_{j=1}^n (\hat{\varphi}_j(k) - \hat{\varphi}_j(k))^2 \right)^{\frac{1}{2}} < \delta_2,
\] (24)
if in (10) $D(q, d^*) \equiv 1$, and the ordinary RLS is used for the parameter estimation (in this case $\omega \equiv n$). Here $\delta_1$ and $\delta_2$ are respective thresholds. If inequalities (24) are satisfied, then the RLS (17) turns out to be efficient and therefore the computational process can be continued. In the opposite case, it is necessary to choose another parameter estimation algorithm.

6 NUMERICAL SIMULATION

The closed-loop system is described by [9]
\[
y(k) = 0.75u(k - 1) + 0.985y(k - 1) + \xi(k),
\] (25)
where $\{\xi(k)\}$ is a sequence of independent identically distributed variables. In such a case,
\[
v(k) = (1 - 0.985q^{-1})^{-1}\xi(k).
\] (26)

The controller design equation is
\[
u(k) = \epsilon(k) + 0.1005u(k-1) - 0.1016u(k-2).
\] (27)

In Fig. 2 the simulated input $\{u(k)\}$, noisy output $\{y(k)\}$ and the reference signal $\{r(k)\}$ of the closed-loop system are presented. The sequences, including that of the noise $\{\xi(k)\}$ with different signal-to-noise ratios (SNR—the square root of the ratio of true output and noise variances) were generated, using Matlab.

The 40 pairs of signals $\{r(k)\}$ and $\{u(k)\}$, presented in Fig. 2, were processed by the non-recursive LS in order to obtain the initial off-line estimates of the transfer function
\[
S(q, \mu) = \frac{f_0 + f_1 q^{-1}}{1 + p_1 q^{-1} + p_2 q^{-2} + p_3 q^{-3}}.
\] (28)

Afterwards, the recursive estimation was performed. The recursive estimates and the true values of the parameters
\[
f_0 = 1; \quad f_1 = -0.98; \quad p_1 = -0.34; \quad p_2 = 0.2; \quad p_3 = -0.1
\] (29)
are shown in Fig. 3b—f, respectively. The initial and

Figure 2: Signals of the system (25)–(27) in the presence of additive noise (SNR=5) on the output: (a) noisy input, (b) the reference signal (in gray) and the noisy output (in black). Axes: $x$—numbers of observations, $y$—amplitudes.
Estimates of parameters

Simplified because of the only one estimated parameter.

Recursive estimates were substituted into (28) instead of the unknown true values of parameters. Afterwards, the current value of the auxiliary input \( \{ \tilde{u}(k) \} \) was generated by filtering the reference signal \( \{ r(k) \} \) by means of the filter \( S(q, \hat{p}) \). The restored input for the initial values of estimates (29), the input (27), and the true one, obtained by substituting the true parameter values (29) in (28) are shown in Fig. 4. The next step is estimation of the parameters \( \mathbf{b}^T = (b_0, a_1) \) when \( a_1 = -0.985 \) and \( b_0 = 0.75 \) according to the recursive estimation procedure, using the RLS of the form (17). The current pairs of signals \( \{ \tilde{u}(k) \}, \{ y(k) \} \) for \( k = 42, \ldots, 200 \) were used to calculate the recursive estimates \( \hat{a}_1(k), \hat{b}_0(k), \hat{v}_1(k) \), when the additive noise is of the form (26)(Fig. 5a) and of the form

\[
v(k) = \xi(k)
\]

(30)

(Fig. 5b), respectively. In both cases, the current estimates \( \hat{v}_1(k) \) were calculated, using the efficiency determination procedure (19)–(24), which essentially simplified because of the only one estimated parameter. In order to determine how different measurement noise (26) realizations affect the input signal (27) restoration and the system (25) parameter estimation we have used a Monte Carlo simulation, with 10 data sets, each containing 200 auxiliary input-output observations pairs. In each ith experiment the estimates of parameters \( a_1 = -0.985 \) and \( b_0 = 0.75 \), using the RLS of the form (17), were determined. Table 1 illustrates the values \( \bar{a}_1, \bar{b}_0 \) of estimates \( \hat{a}_1(k), \hat{b}_0(k) \), (averaged by 10 experiments), and their confidence intervals

\[
\Delta_1 = \pm t_\alpha \frac{\hat{\sigma}_{b_0}}{\sqrt{L}}, \quad \Delta_2 = \pm t_\alpha \frac{\hat{\sigma}_{a_1}}{\sqrt{L}}, \quad \forall k = 1, 200.
\]

Here \( \hat{\sigma}_{b_0}, \hat{\sigma}_{a_1} \) are estimates of the standard deviations \( \sigma_{b_0}, \sigma_{a_1} \), respectively; \( \alpha = 0.05 \) is the significance level; \( t_\alpha = 2.26 \) is the 100(1 - \( \alpha \))% point of Student’s distribution with \( L - 1 \) degrees of freedom; \( L = 10 \) is the number of experiments. From the analysis of the estimates, presented in Fig. 3b–f, it follows that estimates of the parameter of filter (28) converge to the true ones (29), respectively. The auxiliary input approximates the true input more accurately, comparing with the input (27), generated in the presence of \( \{ v(k) \} \) (see Fig. 4). It should be noted, that with an increase of a number of processed observations \( k \) the accuracy of es-

![Figure 3: processed signals \( \{ u(k) \} \) (in black) and \( \{ r(k) \} \) (in gray), (b–f) current estimates (in black) and true values (in gray) of parameters \( f_0, f_1, p_1, p_2 \) and \( p_3 \) (29), respectively. Axes: \( x \)--numbers of observations, \( y \)--amplitudes.](image1)

![Figure 4: Input sequences of the closed-loop system (25)--(27) in comparison with the true input (in gray): (a) the auxiliary input (in black), (b) the input (27), generated in the presence of additive noise on the output (in black). Axes: \( x \)--numbers of observations, \( y \)--amplitudes.](image2)

Table 1: The averaged estimates of parameters and their confidence intervals for different \( k \)

<table>
<thead>
<tr>
<th>Observations</th>
<th>Estimates of parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>( \bar{a}_1 )</td>
</tr>
<tr>
<td>50</td>
<td>( -0.95 \pm 0.03 )</td>
</tr>
<tr>
<td>100</td>
<td>( -0.95 \pm 0.02 )</td>
</tr>
<tr>
<td>150</td>
<td>( -0.95 \pm 0.02 )</td>
</tr>
<tr>
<td>200</td>
<td>( -0.95 \pm 0.02 )</td>
</tr>
</tbody>
</table>
The current estimates of $\hat{a}_1(k)$ (in solid line), $b_0(k)$ (in dash-dotted line), $\hat{v}_1(k)$ (in dashed line) and true values $b_a$ (in gray) depending on the number of processed observations and on the form of the additive noise $\{v(k)\}$: (a) (26), (b) (30).

Axes: $x$—numbers of observations, $y$—amplitudes.

Authors would like to thank for the financial support The Royal Swedish Academy of Sciences and The Swedish Institute—New Visby project Ref No. 2473/2002 (381/T81).

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