Comparison of two structure-exploiting optimization algorithms for integral quadratic constraints

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Abstract

As the semidefinite programs that result from integral quadratic constraints are usually large it is important to implement efficient algorithms. The interior-point algorithms in this paper are primal-dual potential reduction methods and handle multiple constraints. Two approaches are made. For the first approach the computational cost is dominated by a least-squares problem that has to be solved in each iteration. The least-squares problem is solved using an iterative method, namely the conjugate gradient method. The computational effort for the second approach is dominated by forming a linear system of equations. This system of equations is used to compute the search direction in each iteration. If the number of variables are reduced by solving a smaller subproblem the resulting system has a very nice structure and can be solved efficiently. The first approach is more efficient for larger problems but is not as numerically stable.

1 Introduction

A unified approach to robustness analysis in control using Integral Quadratic Constraints (IQCs) was presented in (Megretski and Rantzer, 1997). Applying the Kalman-Yakubovich-Popov lemma, e. g. (Rantzer, 1996), it can be shown how IQCs can be formulated as Linear Matrix Inequalities (LMIs), e. g. (Boyd *et al.*, 1994a). The corresponding optimization problem is a Semidefinite Program (SDP).

Efficient interior-point programs for general SDPs have been presented in (Nesterov and Nemirovskii, 1994). These algorithms are still not efficient enough for the large SDPs that result from IQCs. It is, however, possible to increase the efficiency by utilizing the special problem structure inherited from the IQCs.

Previous work on implementing efficient solvers for LMIs originating from IQCs are based on cutting plane algorithms, (Kao *et al.*, 2000) and outer approximation methods, (Parrilo, 1999). Recently, two interior path-following algorithms with frequency dependent barrier functions were presented in (Kao, 2002). Initial results, where the problem structure is exploited, have been presented in (Hansson and Vandenberghe, 2000a), (Hansson and Vandenberghe, 2001), (Wallin *et al.*, 2001) and (Vandenberghe, 2001). In this paper we will expand on the results presented in (Wallin *et al.*, 2001).

2 Primal and dual problems

The solution to an IQC problem can be obtained by solving an SDP with the following structure:

$$\begin{aligned} & \text{minimize } c^T x \\ & \text{s. t. } X_i = \bar{A}_i P_i \bar{B}_i + \bar{B}_i^T P_i \bar{A}_i^T + M_i(x) \\ & \quad X = \oplus_{i=1}^N X_i \geq 0 \\ & \quad i = 1, 2, \dots, N \end{aligned} \tag{1}$$

where $\bar{A}_i^T = [I \ 0]$, $\bar{B}_i = [A_i \ B_i]$ and $A_i \in \mathbf{R}^{n_i \times n_i}$, $B_i \in \mathbf{R}^{n_i \times m_i}$. The matrix $M_i(x) = M_{i0} + \sum_{j=1}^K x_j M_{ij}$ is symmetric. The constraint (1) may be rewritten using linear operators as

$$X_i = M_0 + \mathcal{F}_i(P_i) + \mathcal{G}_i(x).$$

The most efficient algorithms for solving SDPs are primal-dual algorithms. Thus, we will formulate the dual of the SDP (1).

$$\begin{aligned} & \text{maximize } - \sum_{i=1}^N \langle Z_i, M_{i0} \rangle \\ & \text{s. t. } \bar{B}_i Z_i \bar{A}_i + \bar{A}_i^T Z_i \bar{B}_i^T = 0 \end{aligned} \tag{2}$$

$$\begin{aligned} & \sum_{i=1}^N \langle M_{ij}, Z_i \rangle = c_j \\ & Z = \oplus_{i=1}^N Z_i \geq 0 \\ & i = 1, 2, \dots, N; \quad j = 1, 2, \dots, K \end{aligned} \tag{3}$$

The inner product $\langle \cdot, \cdot \rangle$ is the trace of the product of the matrices. The constraints (2) and (3) can be rewritten using adjoint linear operators

$$\mathcal{F}_i^*(Z_i) = 0; \quad \sum_{i=1}^N \mathcal{G}_i^*(Z_i) = c.$$

3 Primal-dual potential reduction

Potential reduction methods start at a strictly feasible point and reduce the potential function

$$\Phi(X, Z) = \nu\sqrt{n}\langle X, Z \rangle + \log \det X^{-1} + \log \det Z^{-1} + n \log \langle X, Z \rangle - n \log n$$

by at least a fixed amount in each iteration. The potential function combines the duality gap, $\langle X, Z \rangle$, with the deviation from the central path. The optimal value of the duality gap is known to be zero. The duality gap is an upper bound on the difference between the value of the primal objective and the optimal value of the primal problem. The constant $\nu \geq 1$ sets the relative weight of how important it is to reduce the duality gap and how important it is that the iterate is close to the central path. For more details on how to choose this parameter see e. g. (Vandenberghe and Boyd, 1996). The parameter n is the dimension of X . The iterates remain strictly feasible and converge to an optimum in polynomial time. The method works as follows

given strictly feasible X and Z .

repeat

1. Find directions δX and δZ . We will use the Nesterov-Todd direction.
2. Find $p, q \in \mathbf{R}$ that minimize $\Phi(X + \delta X, Z + \delta Z)$ doing a plane search.
3. Update $X = X + p\delta X$ and $Z = Z + q\delta Z$.

until $\langle X, Z \rangle \leq \epsilon$.

The primal-dual potential reduction method used is due to Nesterov and Todd, e. g. (Todd *et al.*, 1998), and it is the same method that is used in SP (Vandenberghe and Boyd, 1998).

3.1 The Nesterov-Todd search direction

The Nesterov-Todd direction makes the primal direction primal-dual symmetric. Due to the structure of the problem we can work on one constraint at a time. The following steps is a way to compute the Nesterov-Todd direction (Frenk *et al.*, 1999):

- Compute a Cholesky factorization:

$$X_i = R_i^T R_i.$$

The cost for this is $O(n_i^3)$.

- Compute an eigenvalue decomposition:

$$R_i Z_i R_i^T U_i = U_i \Lambda_i.$$

The cost for this is $O(n_i^3)$

- Compute the matrices:

$$\begin{aligned} W_i &= R_i^T U_i \Lambda_i^{-\frac{1}{4}} \\ S_i &= W_i W_i^T \\ D_i &= \frac{n + \nu \sqrt{n}}{\langle X, Z \rangle} X_i - Z_i^{-1} = W_i (\rho \Lambda_i^{\frac{1}{2}} - \Lambda_i^{-\frac{1}{2}}) W_i^T. \end{aligned}$$

The cost for this is $O(n_i^3)$

- Solve the linear system of equations:

$$-S_i \delta Z_i S_i + \underbrace{\mathcal{F}_i(\delta P_i) + \mathcal{G}_i(\delta x)}_{\delta X_i} = -D_i \quad (4)$$

$$\mathcal{F}_i^*(\delta Z_i) = 0 \quad (5)$$

$$\sum_{i=1}^N \mathcal{G}_i^*(\delta Z_i) = 0. \quad (6)$$

In this part of the algorithm there is much to be gained from an efficient implementation.

- Compute δX_i as

$$\delta X_i = S_i \delta Z_i S_i - D_i \quad (7)$$

4 Approach 1: Conjugate Gradient based method

To reformulate (4), (5) and (6) as a least-squares problem we first define

$$\begin{aligned} \delta Z &= \oplus_{i=1}^N \delta Z_i \\ \delta P &= \oplus_{i=1}^N \delta P_i \\ W &= \oplus_{i=1}^N W_i \\ D &= \oplus_{i=1}^N D_i \\ \mathcal{F}(\delta P) &= \oplus_{i=1}^N \mathcal{F}_i(\delta P_i) \\ \mathcal{G}(\delta x) &= \oplus_{i=1}^N \mathcal{G}_i(\delta x) \\ \mathcal{A}(\delta P, \delta x) &= \mathcal{F}(\delta P) + \mathcal{G}(\delta x) \end{aligned}$$

If (4) is premultiplied by W_i^{-1} and postmultiplied by W_i^{-T} and the above definitions are used we can reformulate the problem as

$$\begin{aligned} \underbrace{W^T \delta Z W}_{\delta \tilde{Z}} - \underbrace{W^{-1} \mathcal{A}(\delta P, \delta x) W^{-T}}_{\tilde{\mathcal{A}}(\delta P, \delta x)} &= W^{-1} D W^{-T} \\ \underbrace{\mathcal{A}^*(W^{-T} \delta \tilde{Z} W^{-1})}_{\tilde{\mathcal{A}}^*(\delta \tilde{Z})} &= 0. \end{aligned}$$

This is equivalent to the least-squares problem

$$\min_{\delta P, \delta x} \|\tilde{\mathcal{A}}(\delta P, \delta x) + W^{-1} D W^{-T}\|$$

and

$$\delta \tilde{Z} = \tilde{\mathcal{A}}(\delta P, \delta x) + W^{-1}DW^{-T}$$

The normal equations are solved using the conjugate gradient method with a block-Jacobi preconditioner. Utilizing the structure of the problem is in this case the same as choosing an appropriate preconditioner. To make the algorithm efficient it is essential to terminate prior to convergence. Notice that it is sufficient to have δX , δZ to be a descent direction for the potential function (Vandenberghe and Boyd, 1995), (Boyd *et al.*, 1994b). The computational complexity is experimentally found to be $O(\max[n_i]^3)$. For more details see (Hansson and Vandenberghe, 2000b).

5 Approach 2: Reducing the number of variables

In this approach, first introduced in (Vandenberghe, 2001), we try to reduce the number of variables. This is done by calculating basis matrices for δZ_i that fulfill equation (5) of the Nesterov-Todd search direction. Having these basis matrices it is also possible to eliminate δP_i from (4). This allows us to solve a much smaller system of equations when we are to calculate the search direction. If we partition δZ_i as

$$\delta Z_i = \begin{bmatrix} \delta Z_{i(11)} & \delta Z_{i(12)} \\ \delta Z_{i(12)}^T & \delta Z_{i(22)} \end{bmatrix}$$

then $\mathcal{F}_i^*(\delta Z_i)$ can explicitly be written

$$\mathcal{F}_i^*(\delta Z_i) = A_i \delta Z_{i(11)} + \delta Z_{i(11)} A_i^T + B_i \delta Z_{i(12)}^T + \delta Z_{i(12)} B_i^T.$$

A basis for δZ_i that satisfies $\mathcal{F}_i^*(\delta Z_i) = 0$ is

$$F_{ik} = \begin{cases} \begin{bmatrix} E_{ik(11)} & E_{ik(12)} \\ E_{ik(12)}^T & 0 \end{bmatrix} & k = 1, 2, \dots, m_i n_i \\ \begin{bmatrix} 0 & 0 \\ 0 & E_{ik(22)} \end{bmatrix} & k = m_i n_i + 1, \dots, k_{i_{max}} \end{cases}.$$

where $k_{i_{max}} = m_i n_i + \frac{m_i(m_i+1)}{2}$, $E_{ik(12)}$ is the standard basis for unstructured $n_i \times m_i$ matrices and where $E_{ik(22)}$ is the standard basis for symmetric $m_i \times m_i$ matrices. The matrix $E_{ik(11)}$ is related to $E_{k(12)}$ through (5), that is

$$A_i E_{ik(11)} + E_{ik(11)} A_i^T + B_i E_{ik(12)}^T + E_{ik(12)} B_i^T = 0.$$

We can express δZ_i as

$$\delta Z = \sum_{k=1}^{k_{i_{max}}} \delta z_{ik} F_{ik}.$$

The cost for finding the basis is $O(\max[n_i]^4)$. In case the A_i s are diagonal or can be made diagonal the cost is, however, only $O(\max[n_i]^3)$. Observe that the basis matrices only have to be computed once and that this can be done in advance. Every basis matrix satisfies (5) and thus

$$\langle F_{ik}, \mathcal{F}_i(\delta P_i) \rangle = \underbrace{\langle \mathcal{F}_i^*(F_{ik}), \delta P_i \rangle}_{=0} = 0, \quad \forall \delta P_i.$$

This fact can be used to eliminate δP_i from (4). If we take the inner product between F_{ij} and (4) we get

$$\begin{aligned} & -\langle F_{ij}, S_i \delta Z_i S_i \rangle + \underbrace{\langle F_{ij}, \mathcal{F}_i(\delta P_i) \rangle}_{=0} + \langle F_{ij}, G_i(\delta x) \rangle \\ &= - \sum_{k=1}^{m_i n_i + \frac{m_i(m_i+1)}{2}} \delta z_{ik} \langle F_{ij}, S_i F_{ik} S_i \rangle + \sum_{k=1}^K \delta x_k \langle F_{ij}, M_{ik} \rangle = -\langle F_{ij}, D_i \rangle \end{aligned}$$

This allows us to rewrite (4) and (6) on the form

$$\begin{bmatrix} -H_1 & 0 & \dots & 0 & G_1 \\ 0 & -H_2 & \dots & 0 & G_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -H_N & G_N \\ G_1^T & G_2^T & \dots & G_N^T & 0 \end{bmatrix} \begin{bmatrix} \delta z_1 \\ \delta z_2 \\ \vdots \\ \delta z_N \\ \delta x \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_N \\ 0 \end{bmatrix}, \quad (8)$$

where δz_i is a vector with all the unknown δz_{ik} stacked on top of each other and δx is a vector with all the unknown δx_j . The first block rows correspond to (4), the last block row corresponds to (6). Each element of a block H_i is

$$H_{i(jk)} = \langle F_{ij}, S_i F_{ik} S_i \rangle$$

and the elements of the blocks G_i and g_i are

$$\begin{aligned} G_{i(jk)} &= \langle F_{ij}, M_{ik} \rangle \\ g_{ij} &= -\langle F_{ij}, D_i \rangle. \end{aligned}$$

Note, as G_i does not depend on the current iterate it can be computed beforehand. The H_i s are positive definite as

$$\begin{aligned} x^T H_i x &= \left\langle \sum_j \sum_k x_j x_k F_{ij} S_i F_{ik} S_i, I \right\rangle \\ &= \left\langle \sum_j x_j S_i^{\frac{1}{2}} F_{ij} S_i^{\frac{1}{2}}, \sum_k x_k S_i^{\frac{1}{2}} F_{ik} S_i^{\frac{1}{2}} \right\rangle \\ &= \left\| \sum_k x_k S_i^{\frac{1}{2}} F_{ik} S_i^{\frac{1}{2}} \right\|_F^2. \end{aligned}$$

The block arrow system (8) with positive definite H_i s can be solved efficiently. The dominating cost for finding the search direction is the computation of the H_i s. This cost is $O(\max[n_i]^4)$ and is actually larger than the cost for solving

the system (8). The search direction δX_i is evaluated as in (7). Once the algorithm has converged the P_i s can be computed by solving the Lyapunov equation corresponding to the (1,1)-block of (1).

It is tempting to further reduce the number of variables by finding a basis that fulfills both (5) and (6). This can be done for example by computing a QR-factorization of G (Nocedal and Wright, 1999) and would result in a system

$$-\tilde{H}\delta\tilde{z} = \tilde{g}$$

where $\tilde{H}$ is slightly smaller in size than the blockdiagonal H of (8) but has no structure at all. Hence, we cannot solve this system of equations as efficiently as (8).

6 Practical aspects

As mentioned before the algorithms in this paper are feasible algorithms, i. e. every iterate is strictly feasible. It is usually not trivial to find a strictly feasible initial point. We have chosen to use a two phased method to get around this difficulty. In the first phase we have the primal problem

$$\begin{aligned} & \text{minimize } \bar{M}t \\ & \text{s. t. } X_i = \bar{A}_i P_i \bar{B}_i + \bar{B}_i^T P_i \bar{A}_i^T + M_i(x) + tI \\ & \quad X = \oplus_{i=1}^N X_i \geq 0 \\ & \quad x_j + \bar{M} \geq 0 \\ & \quad -x_j + \bar{M} \geq 0 \\ & \quad t + \alpha \geq 0 \\ & \quad i = 1, 2, \dots, N \end{aligned}$$

The artificial constraints on the x_j s makes it trivial to find a feasible point for the dual problem. The restrictions on the x_j s means that we are only looking in a hypercube. If we can find a feasible solution to this problem with a negative primal objective function we have found a feasible primal point to the original primal problem (1) and can proceed to the second phase. On the other hand, if we find a feasible solution with a positive dual objective function there is no strictly feasible solution inside the hypercube. Of course, we may always try a larger value of $\bar{M}$. In the second phase we use the P_i s and the x_j s that we got from the first phase as initial points and have the primal problem

$$\begin{aligned} & \text{minimize } c^T x \\ & \text{s. t. } X_i = \bar{A}_i P_i \bar{B}_i + \bar{B}_i^T P_i \bar{A}_i^T + M_i(x) \\ & \quad X = \oplus_{i=1}^N X_i \geq 0 \\ & \quad x_j + \bar{M} \geq 0 \\ & \quad -x_j + \bar{M} \geq 0 \\ & \quad i = 1, 2, \dots, N \end{aligned}$$

Again, the artificial bounds on the x_j s makes it trivial to find a feasible point to the dual problem. The optimal solution to this SDP is also an optimal solution to (1), provided the constraints involving $\bar{M}$ are not active. In case the constraints are active we may always try a larger value of $\bar{M}$.

7 Numerical examples

Example 1

This is the same example as Example 1 in (Parrilo, 1999). Consider the block diagram in Figure 1. We want to compute the worst case $\mathcal{L}_2$ induced norm between u and y .

Figure 1: Block diagram for Example 1.

The plant G is given by

$$G(s) = \begin{bmatrix} \frac{s+1}{s^2+2s+2} & 1 \\ 1 & 0 \end{bmatrix}.$$

The Δ -block is an uncertain contractive linear time-varying operator. It can be shown that finding the the worst case $\mathcal{L}_2$ induced norm is equivalent to solving the SDP

$$\begin{aligned} & \text{minimize } x_1 \\ \text{s. t. } & X_1 = \bar{A}_1 P_1 \bar{B}_1 + \bar{B}_1^T P_1 \bar{A}_1^T + M_1(x) \\ & X_2 = M_2(x) \\ & X = X_1 \oplus X_2 \geq 0 \\ & P_1 \in \mathbf{R}^{2 \times 2} \quad \text{and} \quad x \in \mathbf{R}^2 \end{aligned}$$

The matrices A_1 and B_1 in $\bar{B}_1$ come from a realization of G , i. e.

$$\begin{bmatrix} \dot{\xi}(t) \\ z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} \xi(t) \\ w(t) \\ u(t) \end{bmatrix}.$$

The matrices M_{ij} are defined such that

$$\begin{aligned} \begin{bmatrix} \xi(t) \\ w(t) \\ u(t) \end{bmatrix}^T M_{10} \begin{bmatrix} \xi(t) \\ w(t) \\ u(t) \end{bmatrix} &= -y^2(t) \\ \begin{bmatrix} \xi(t) \\ w(t) \\ u(t) \end{bmatrix}^T M_{11} \begin{bmatrix} \xi(t) \\ w(t) \\ u(t) \end{bmatrix} &= u^2(t) \\ \begin{bmatrix} \xi(t) \\ w(t) \\ u(t) \end{bmatrix}^T M_{12} \begin{bmatrix} \xi(t) \\ w(t) \\ u(t) \end{bmatrix} &= w^2(t) - z^2(t) \\ M_{20} &= 0 \\ M_{21} &= 0 \\ M_{22} &= 1 \end{aligned}$$

The solution is given by

$$P = - \begin{bmatrix} 3.495 & 0.667 \\ 0.667 & 0.667 \end{bmatrix} \quad x = \begin{bmatrix} 7.548 \\ 2.747 \end{bmatrix}$$

We ran the example for different choices of ν and $\bar{M}$. The results for the second phase are given in Table 1. The conjugate gradient approach is very sensitive to the choice of ν and $\bar{M}$ in the sense that the number of iterations for the conjugate gradient iterates and the number of outer iterates vary very much depending on those parameters. The elimination of variables approach, on the other hand, is rather robust to the choice of ν and $\bar{M}$. It is very unfortunate to have a method that is sensitive to $\bar{M}$ as the numeric value of this parameter is dependent on the problem. $\bar{M}$ has to be greater than the maximum absolute value of all x_j . Otherwise the optimal x is not in the hypercube.

ν	$\bar{M}$	# iter CG	CPU time CG [s]	# iter ELIM	CPU time ELIM [s]
3	10	16	11.68	8	0.45
3	100	32	41.56	11	0.53
3	1000	29	41.23	13	0.76
10	10	10	2.42	5	0.29
10	100	26	17.16	8	0.49
10	1000	35	46.47	9	0.51
30	10	15	4.00	6	0.33
30	100	16	20.89	7	0.42
30	1000	30	20.69	8	0.46

Table 1: Computational results for Example 1. The columns labeled CG are for the conjugate gradient based approach and the columns labeled ELIM are for the elimination of variables approach.

Example 2

The setup in this example is the same as in Example 1 above except that u , w , y , and z are vector-valued, and that the Δ -block is a diagonal contractive linear

time-varying operator. The derivation is similar, but there are now instead as many IQCs associated with the Δ -block as the dimension $\bar{p}$ of w or equivalently z . This is the same example as Example 2 in (Parrilo, 1999) except for the numerical values.

For simplicity we will assume that u and y have the same dimensions as w and z , i.e. $\bar{p}$. We construct random systems in the following way. Let λ_{ij} , $j = 1, 2, \dots, 4$, $i = 1, 2, \dots, a$, be drawn from a uniform distribution on $[-2, -1]$, and define

$$\tilde{A}_i = \text{diag}_j(\lambda_{ij}) + \tilde{A}_0 \in \mathbf{R}^{4 \times 4}$$

where $\tilde{A}_0$ is a matrix of zeros except for the first super-diagonal which have all elements equal to one. Then let $\tilde{A} = \text{blockdiag}_i(\tilde{A}_i)$. Similarly define

$$\tilde{B}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbf{R}^{4 \times 1}$$

and let $B_w = B_u = \text{blockdiag}_i(\tilde{B}_i)$, and $\tilde{B} = [B_w \ B_u]$. For $\tilde{C}$ introduce

$$\tilde{C}_i = [1 \ 0 \ \dots \ 0] \in \mathbf{R}^{1 \times 4}$$

and let $C_y = C_z = \text{blockdiag}_i(\tilde{C}_i)$, and $\tilde{C}^T = [C_z^T \ C_y^T]$. The dimensions of $(\tilde{A}, \tilde{B}, \tilde{C})$ are $n_1 \times n_1$, $n_1 \times m_1$ and $m_1 \times n_1$, respectively, where $m_1 = 2\bar{p}$ and $n_1 = \sum_{i=1}^{\bar{p}} 4$. In order to couple all inputs with all outputs we will introduce random unitary state, input, and output transformations. To this end let $\tilde{S}_{n_1} \in \mathbf{R}^{n_1 \times n_1}$ and $\tilde{S}_{m_1} \in \mathbf{R}^{m_1 \times m_1}$ be matrices with entries drawn from a uniform distribution on $[0, 1]$. Let the singular value decompositions of these matrices be given by

$$\tilde{S}_{n_1} = U_{n_1} \Sigma_{n_1} V_{n_1}^T; \quad \tilde{S}_{m_1} = U_{m_1} \Sigma_{m_1} V_{m_1}^T$$

and define

$$A_1 = U_{n_1}^T \tilde{A} U_{n_1}; \quad B_1 = U_{n_1}^T \tilde{B} V_{m_1}; \quad C_1 = U_{m_1} \tilde{C} U_{n_1}$$

Now we let G be such that it has (A_1, B_1, C_1) as a realization. The matrices M_{ij} , $i = 1, 2, \dots, p$, where $p = 1 + \bar{p}$, are defined similarly as for Example 1. The computational results for the second phase are shown in Table 2. Here, $\nu = 3$ and the $\bar{M} = 10$. This choice made the conjugate gradient approach perform well. It is obvious that the conjugate gradient approach is more efficient than the elimination of variables approach for large problems. However, the parameter $\bar{M}$ and the parameter ν have to be well chosen if the conjugate gradient based approach is going to perform well.

8 Conclusions and future work

In this paper was presented two approaches to efficiently solve SDPs originating from IQCs. In the conjugate gradient based method the number of iterations is highly dependent on the parameter ν , which weights how important it is to

a	1	2	3	4	5	6
n	29	33	37	41	45	49
# variables	105	189	305	453	633	845
CPU time CG [s]	11.24	11.39	12.10	16.35	29.99	39.85
CPU time ELIM [s]	12.09	25.48	50.53	97.27	173.10	293.07

Table 2: Computational results for Example 2. The columns labeled CG are for the conjugate gradient based approach and the columns labeled ELIM are for the elimination of variables approach.

decrease the duality gap to how important it is to stay close to the central path, and the choice of M . For large problems this approach is more efficient.

The number of iterations to decrease the duality gap to a certain value is always smaller for the elimination of variables approach. This approach is also more reliable. Furthermore, there is potential to increase the efficiency of this method, using special structure of the A_i s and computing the matrix H in (8) using the fast Fourier transform (Vandenberghe, 2001).

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