Power Series Solution of the Hamilton-Jacobi-Bellman Equation for Time-Varying Differential-Algebraic Equations

Johan Sjöberg, Torkel Glad

Division of Automatic Control

E-mail: johans@isy.liu.se, torkel@isy.liu.se

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Address:

Department of Electrical Engineering

Linköpings universitet

SE-581 83 Linköping, Sweden

WWW: <http://www.control.isy.liu.se>

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Abstract

Optimal control problems for a class of nonlinear time-varying differential-algebraic equations are considered. It is shown that they possess a well-defined feedback solution in a neighborhood of the origin. Explicit formulas for the series expansions of the cost function and control law are given.

Keywords: Optimal Control, Differential-Algebraic Equations

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Johan Sjöberg and Torkel Glad
 Division of Automatic Control
 Department of Electrical Engineering
 Linköpings universitet, SE-581 83 Linköping, SWEDEN
 {johans,torkel}@isy.liu.se

Abstract—Optimal control problems for a class of nonlinear time-varying differential-algebraic equations are considered. It is shown that they possess a well-defined feedback solution in a neighborhood of the origin. Explicit formulas for the series expansions of the cost function and control law are given.

I. INTRODUCTION

This paper deals with the control problem of nonlinear time-varying differential-algebraic equations (DAEs). The area of nonlinear DAEs has also been studied in for example [1], [2], [3]. More specifically, we focus on finite horizon optimal feedback control and the goal is to calculate a series solution. A truncation of this series solution can then be used as an approximative optimal feedback law.

The infinite horizon optimal control problem for time-invariant state-space systems was first considered in [4]. There, it was shown that the solution could be obtained in form of power series, the terms of which could be sequentially obtained through solution of a quadratic optimal control problem for the linearized system and subsequent solution of a series of linear partial differential equations. Further, a formal proof of the convergence of the power series was presented in the case when the input signal is scalar and the system has the form $\dot{x} = f(x) + Bu$. In [5] these results were extended to general time-invariant state-space systems, $\dot{x} = f(x, u)$, and this work was extended even more in [6]. More recent works which extends and relaxes the earlier works for state-space systems in various directions are, for example, [7], [8], [9], [10]. The same method has been generalized by [11] to handle also nonlinear time-invariant differential-algebraic systems.

In [12], the power series expansion method was developed to also incorporate the finite horizon case and time-varying state-space systems. In our contribution, we generalize the result in [12] to handle also nonlinear DAEs using a similar method to what is used in [11].

For a good overview of more recent methods to find approximative solutions of the Hamilton-Jacobi-Bellman equation, see [13] and references therein.

Notation: The notation in this paper is fairly standard. The Jacobian matrix $\frac{\partial h_h}{\partial x}$ will be denoted $h_{h;x}$ and $(\cdot)^{[i]}$ will be used to denote the terms of order i in a power series expansion. $Q \succ (\succeq) 0$ means that Q is a real positive (semi)definite matrix. $[m]$ will denote the integer part of m .

II. PROBLEM DESCRIPTION

The optimal control problem is to minimize the criterion

$$G(x(T)) + \int_{\tau}^T L(t, x, u) dt \quad (1)$$

where $x = (x_1^T, x_2^T)^T$ satisfies the differential-algebraic system

$$\dot{x}_1 = F_1(t, x_1, x_2, u) \quad (2a)$$

$$0 = F_2(t, x_1, x_2, u) \quad (2b)$$

with some initial condition $x_1(\tau) = x_{1,0}$. As usual we define the optimal return function

$$V(\tau, x_{1,0}) = \inf_{u(\cdot)} \left(G(x(T)) + \int_{\tau}^T L(t, x, u) dt \right) \quad (3)$$

We will consider the optimal control problem for τ in an interval $[T_0, T]$ and $x_{1,0}$ in some neighborhood of the origin. We make the following assumption.

Assumption 1: The constraint equation (2b) can be solved to give

$$x_2 = \varphi(t, x_1, u) \quad (4)$$

for all $t \in [T_0, T]$ and all x_1 and u in some open set Ω containing the origin.

The computational challenge lies in the fact that the function φ , although guaranteed to exist, can usually not be given an explicit form.

Assumption 2: The functions $F(t, x, u)$, $L(t, x, u)$ and $G(x_1)$ are analytical around the origin $(x, u) = 0$ for all $t \in [T_0, T]$.

This assumption makes it possible to express these functions as power series

$$F(t, x, u) = A(t)x + B(t)u + F_h(t, x, u) \quad (5a)$$

$$L(t, x, u) = x^T Q(t)x + u^T R(t)u + 2x^T S(t)u + L_h(t, x, u) \quad (5b)$$

$$G(x_1) = x_1^T M x_1 + G_h(x_1) \quad (5c)$$

which are uniformly convergent in a neighborhood around the origin $(x, u) = 0$, for all $t \in [T_0, T]$.

The function $F_h(t, x, u)$ contains terms of order two and higher in x and u , while $L_h(t, x, u)$ and $G_h(x_1)$ contain

terms of at least degree three. The matrices $A(t)$, $B(t)$, $Q(t)$ and $R(t)$ are continuous real matrix functions partitioned as

$$A(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix}, \quad B(t) = \begin{pmatrix} B_1(t) \\ B_2(t) \end{pmatrix}$$

$$Q(t) = \begin{pmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{pmatrix}$$

Assumption 3: The matrix $A_{22}(t)$ is invertible for all $t \in [T_0, T]$. The weight matrices satisfy

$$\begin{pmatrix} Q(t) & S(t) \\ S^T(t) & R(t) \end{pmatrix} \succeq 0, \quad R(t) \succ 0, \quad M \succeq 0 \quad (6)$$

for $t \in [T_0, T]$.

The corresponding expression for the optimal return function is

$$V(t, x) = x^T P(t)x + V_h(t, x) \quad (7)$$

where $V_h(t, x)$ contains the terms of order three and higher in x . We will consider the class of feedback laws which can be expressed as

$$u(t, x) = D(t)x + u_h(t, x) \quad (8)$$

where $D(t)$ is a continuous matrix function. The function $u_h(t, x)$ consists of terms of degree two or higher in x and is uniformly convergent about the origin for $t \in [T_0, T]$.

III. A SERIES SOLUTION WHEN THERE ARE NO ALGEBRAIC EQUATIONS.

If the equation $F_2 = 0$ is absent, we write (2) in the form

$$\dot{x} = F(t, x, u) \quad (9)$$

The problem is now a standard optimal control problem with the following Hamilton-Jacobi equation [14], [15].

$$-V_t(t, x) = \min_u L(t, x, u) + V_x(t, x)F(t, x, u) \quad (10)$$

It follows that the optimal feedback law $u_*(t, x)$ has to satisfy the following equations.

$$0 = L(t, x, u_*(t, x)) + V_t(t, x) + V_x(t, x)F(t, x, u_*(t, x))$$

$$0 = L_u(t, x, u_*(t, x)) + V_x(t, x)F_u(t, x, u_*(t, x)) \quad (11)$$

If (5), (7) and (8) are inserted into (11), we obtain two polynomial equations in x , where the coefficients in the polynomials are differential equations in t . Since these polynomial equations must hold for all x in a neighborhood of the origin, the coefficients corresponding to different orders in x will yield separate differential equations in t . Solving these differential equations will then yield the optimal solution. Willenstein showed the following basic result, see [12].

Theorem 1: Consider a system described by (9). Then the optimal feedback control $u_*(t, x)$ exists and is the unique solution of (11) for small $|x|$ and $t \in [T_0, T]$. Furthermore, the optimal return function and optimal feedback are of the form (7) and (8), respectively. In these expressions, $P(t)$ and $D_*(t)$ are given by

$$0 = \dot{P}(t) + P(t)\tilde{A}(t) + \tilde{A}(t)^T P(t) - P(t)B(t)R(t)^{-1}B(t)^T P(t) + \tilde{Q}(t) \quad (12a)$$

$$0 = P(T) - M \quad (12b)$$

and

$$D_*(t) + R^{-1}(t)(S^T(t) + B(t)^T P(t)) = 0 \quad (12c)$$

where

$$\tilde{A}(t) = A(t) - B(t)R^{-1}(t)S^T(t)$$

$$\tilde{Q}(t) = Q(t) - S(t)R^{-1}(t)S^T(t)$$

The higher order terms in (7), (8) can be calculated from the following expressions.

$$V_x^{[m]}(t, x)A_c(t)x + V_t^{[m]}(t, x) =$$

$$- \sum_{k=3}^{m-1} V_x^{[k]}(t, x)B(t)u_*^{[m-k+1]}(t, x)$$

$$- \sum_{k=2}^{m-1} V_x^{[k]}(t, x)F_h^{[m-k+1]}(t, x, u_*) - L_h^{[m]}(t, x, u_*)$$

$$- 2 \sum_{k=2}^{\lfloor \frac{m-1}{2} \rfloor} u_*^{[k]}(t, x)^T R(t)u_*^{[m-k]}(t, x)$$

$$- u_*^{[m/2]}(t, x)^T R(t)u_*^{[m/2]}(t, x) \quad (13a)$$

where $m = 3, 4, \dots$ and $A_c(t) = A(t) + B(t)D_*(t)$,

$$V^{[m]}(T, x) = G^{[m]}(x) \quad (13b)$$

and

$$u_*^{[k]}(t, x) = -\frac{1}{2}R(t)^{-1} \left\{ V_x^{[k+1]}(t, x)B(t) \right.$$

$$\left. + \sum_{i=1}^{k-1} V_x^{[k-i+1]}(t, x)F_{h;u}^{[i]}(t, x, u_*) + L_{h;u}^{[k]}(x, u_*) \right\} \quad (13c)$$

for $k = 2, 3, \dots$. In (13) the convention that $\sum_k^l = 0$ for $l < k$ is used and the terms $u^{[m/2]}$ are to be omitted if m is odd.

Proof: See [12]. ■

We see that the first terms of $u_*(t, x)$ and $V(t, x)$,

$$u_*^{[1]}(t, x) = D_*(t)x, \quad V^{[2]}(x) = x^T P(t)x \quad (14)$$

are given by the solution to the Riccati Differential Equation (RDE) in (12a) together with the boundary condition (12b) and the equation (12c).

In order to obtain higher order approximations of the optimal feedback law, $u_*(t, x)$, and the corresponding cost function, $V(t, x)$, we need to solve (13). First note that

$$F_h^{[k]}(t, x, u_*) = F_h^{[k]}(t, x, u_*^{[1]} + u_*^{[2]} + \dots + u_*^{[k-1]})$$

$$L_h^{[k]}(t, x, u_*) = L_h^{[k]}(t, x, u_*^{[1]} + u_*^{[2]} + \dots + u_*^{[k-2]})$$

since $F_h(t, x, u)$ and $L_h(t, x, u)$ are power series beginning with terms of order two and three respectively. Based on this it can be seen that the right-hand side of (13a) only depends on the terms

$$u_*^{[1]}, \dots, u_*^{[m-2]}, V^{[2]}, \dots, V^{[m-1]} \quad (15)$$

and the right-hand side of (13c) only depends on

$$u_*^{[1]}, \dots, u_*^{[k-1]}, V^{[2]}, \dots, V^{[k+1]} \quad (16)$$

It follows that the equations for the higher order terms can be solved recursively. At each step, (13a) can be regarded as a linear differential equation for the coefficients of $V^{[m]}$, with the right hand side known and with the boundary condition $V^{[m]}(T, x) = G^{[m]}(x)$. Therefore, by starting with $u_*^{[1]}(t, x) = D_*(t)x$ and $V^{[2]}(t, x) = x^T P(t)x$ it is possible to consecutively calculate the terms

$$V^{[3]}(t, x), u_*^{[2]}(t, x), V^{[4]}(t, x), u_*^{[3]}(t, x), \dots$$

and thereby generating power series for $u_*(t, x)$ and $V(t, x)$.

IV. THE DIFFERENTIAL-ALGEBRAIC SYSTEM CASE

In principle the results of the previous section can be used to solve the optimal control problem for (2). Using Assumption 1 we can write the system as

$$\dot{x}_1 = \hat{F}_1(t, x_1, u) = F_1(t, x_1, \varphi(t, x_1, u), u) \quad (17a)$$

and the cost function as

$$\hat{L}(t, x_1, u) = L(t, x_1, \varphi(t, x_1, u), u) \quad (17b)$$

The optimal control problem is thus transformed to an ordinary problem in the state variables x_1 and Theorem 1 is directly applicable. Since φ is usually not explicitly given we can however not use the series expansion expressions of the previous section directly. To circumvent this difficulty we note that in order to calculate the approximative solutions of $V(t, x)$ and $u_*(t, x)$, only the series expansions in x and u of the functions involved, *i.e.*, $F(t, x, u)$, $L(t, x, u)$ and $u(t, x)$, are needed. Thus, in order to determine an approximative solution to the DAE problem the series expansions around $(x_1, u) = (0, 0)$ of $F(t, x_1, u)$ and $\hat{L}(t, x_1, u)$ in (17) are needed. The idea, used in this paper, is to utilize that the series expansions of the composite functions $\hat{F}_1(t, x_1, u)$ and $\hat{L}(t, x_1, u)$ can be computed based on the series expansions of $F_1(t, x_1, x_2, u)$, $L(t, x_1, x_2, u)$, $u(t, x_1)$ and $\varphi(t, x_1, u)$. Therefore, the power series of $\varphi(t, x_1, u)$ is needed. We use the notation

$$x_2 = \varphi(t, x_1, u) = \varphi^{[1]}(t, x_1, u) + \varphi_h(t, x_1, u) \quad (18)$$

where $\varphi_h(t, x_1, u)$ contains terms beginning with degree two in x_1 and u . From (5a) we have that the series expansion of (2b) is given by

$$\begin{aligned} 0 &= F_2(t, x_1, x_2, u) \\ &= A_{21}(t)x_1 + A_{22}(t)x_2 + B_2(t)u + F_{2h}(t, x_1, x_2, u) \end{aligned} \quad (19)$$

If (18) is combined with (19) the expression obtained is

$$\begin{aligned} 0 &= A_{21}(t)x_1 + A_{22}(t)\{\varphi^{[1]}(t, x_1, u) + \varphi_h(t, x_1, u)\} \\ &\quad + B_2(t)u + F_{2h}(t, x_1, \varphi^{[1]}(t, x_1, u) + \varphi_h(t, x_1, u), u) \end{aligned}$$

Since the equation above must hold for all (x_1, u) close to the origin the first order term of $\varphi(t, x_1, u)$ will be given by

$$\varphi^{[1]}(t, x_1, u) = -A_{22}^{-1}(t)A_{21}(t)x_1 - A_{22}^{-1}(t)B_2(t)u \quad (20)$$

since all other terms have degrees higher than one. Furthermore, since the lowest degree of the terms in $F_{2h}(t, x_1, x_2, u)$ is two, we know that

$$\begin{aligned} F_{2h}^{[m]}(t, x_1, \varphi(t, x_1, u), u) &= \\ F_{2h}^{[m]}(t, x_1, \varphi^{[1]}(t, x_1, u) + \dots + \varphi^{[m-1]}(t, x_1, u), u) \end{aligned} \quad (21)$$

This makes it possible to derive a recursive expression for a general degree term of $\varphi(t, x_1, u)$ as

$$\varphi^{[m]}(t, x_1, u) = -A_{22}^{-1}(t)F_{2h}^{[m]}(t, x_1, \varphi(t, x_1, u), u) \quad (22)$$

For later convenience the first order approximation of $\varphi(t, x_1, u)$, *i.e.*, (20), is used to define the variable transformation

$$\begin{aligned} \begin{pmatrix} x_1 \\ \varphi^{[1]}(t, x_1, u) \\ u \end{pmatrix} &= \Pi(t) \begin{pmatrix} x_1 \\ u \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ -A_{22}^{-1}(t)A_{21}(t) & -A_{22}^{-1}(t)B_2(t) \\ 0 & I \end{pmatrix} \begin{pmatrix} x_1 \\ u \end{pmatrix} \end{aligned} \quad (23)$$

The variable transformation will mainly be used compute the second order term of the reduced cost function $\hat{L}(t, x_1, u)$.

To compute the series expansion of $\hat{F}_1(t, x_1, u)$, (18) is inserted into (5a). The system (17a) can then be written as

$$\dot{x}_1 = \hat{A}(t)x_1 + \hat{B}(t)u + \hat{F}_{1h}(t, x_1, u) \quad (24a)$$

where

$$\begin{aligned} \hat{A}(t) &= A_{11}(t) - A_{12}(t)A_{22}^{-1}(t)A_{21}(t) \\ \hat{B}(t) &= B_1(t) - A_{12}(t)A_{22}^{-1}(t)B_2(t) \end{aligned} \quad (24b)$$

and

$$\begin{aligned} \hat{F}_{1h}(t, x_1, u) &= \\ F_{1h}(t, x_1, \varphi(t, x_1, u), u) &+ A_{12}(t)\varphi_h(t, x_1, u) \end{aligned} \quad (24c)$$

In the same manner, the series expansion of (17b) is obtained as

$$\begin{aligned} \hat{L}(t, x_1, u) &= \\ &= \begin{pmatrix} x_1 \\ u \end{pmatrix}^T \Pi^T(t) \begin{pmatrix} Q(t) & S(t) \\ S^T(t) & R(t) \end{pmatrix} \Pi(t) \begin{pmatrix} x_1 \\ u \end{pmatrix} + \hat{L}_h(t, x_1, u) \\ &= \begin{pmatrix} x_1 \\ u \end{pmatrix}^T \begin{pmatrix} \hat{Q}(t) & \hat{S}(t) \\ \hat{S}^T(t) & \hat{R}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ u \end{pmatrix} + \hat{L}_h(t, x_1, u) \end{aligned} \quad (25)$$

where

$$\begin{aligned} \hat{L}_h(t, x_1, u) &= \\ &= L_h(t, x_1, \varphi(t, x_1, u), u) + 2x_1^T Q_{12}(t)\varphi_h(t, x_1, u) \\ &\quad + 2\varphi^{[1]}(t, x_1, u)Q_{22}(t)\varphi_h(t, x_1, u) + 2u^T S_2(t)\varphi_h(t, x_1, u) \\ &\quad + \varphi_h(x_1, u)^T Q_{22}(t)\varphi_h(t, x_1, u) \end{aligned} \quad (26)$$

Using the first order terms of the series expansions (24a) and (25), the RDE (12a)–(12b) and the expression for the first order term in the feedback (12c) for the DAE formulation will become

$$\begin{aligned} 0 &= \dot{P}(t) + P(t)\tilde{\hat{A}}(t) + \tilde{\hat{A}}^T(t)P(t) \\ &\quad - P(t)\hat{B}(t)\hat{R}^{-1}(t)\hat{B}^T(t)P(t) + \tilde{\hat{Q}} \end{aligned} \quad (27a)$$

$$0 = P(T) - M \quad (27b)$$

and

$$0 = D_*(t) + \hat{R}^{-1}(t) \left(\hat{S}^T(t) + \hat{B}^T(t)P(t) \right) \quad (27c)$$

where

$$\begin{aligned} \tilde{\hat{A}}(t) &= \hat{A}(t) - \hat{B}(t)\hat{R}^{-1}(t)\hat{S}^T(t) \\ \tilde{\hat{Q}} &= \hat{Q}(t) - \hat{S}(t)\hat{R}^{-1}(t)\hat{S}^T(t) \end{aligned}$$

The higher order terms of $V(t, x_1)$ and $u_*(t, x_1)$ are obtained from (13). In (13) only the series expansion coefficients of the different functions are included and it is therefore possible to replace these functions with the series expansion coefficients of $\hat{F}_{1h}(t, x_1, u)$ and $\hat{L}_h(t, x_1, u)$, i.e.,

$$\begin{aligned} V_{x_1}^{[m]}(t, x_1) \hat{A}_c(t) x_1 + V_t^{[m]}(t, x_1) &= \\ - \sum_{k=3}^{m-1} V_{x_1}^{[k]}(t, x_1) \hat{B}(t) u_*^{[m-k+1]}(t, x_1) & \\ - \sum_{k=2}^{m-1} V_{x_1}^{[k]}(t, x_1) \hat{F}_{1h}^{[m-k+1]}(t, x_1, u_*) & \\ - 2 \sum_{k=2}^{\lfloor \frac{m-1}{2} \rfloor} u_*^{[k]}(t, x_1)^T \hat{R}(t) u_*^{[m-k]}(t, x_1) & \\ - u_*^{[m/2]}(t, x_1)^T \hat{R}(t) u_*^{[m/2]}(t, x_1) - \hat{L}_h^{[m]}(t, x_1, u_*) & \quad (28a) \\ V^{[m]}(T, x) = G^{[m]}(x) & \quad (28b) \end{aligned}$$

where $m = 3, 4, \dots$, $\hat{A}_c(t) = \hat{A}(t) + \hat{B}(t)D_*(t)$, and the terms $u_*^{[m/2]}$ are to be omitted if m is odd. The corresponding expression for the series expansion of the feedback law is

$$\begin{aligned} u_*^{[k]}(t, x_1) &= -\frac{1}{2} \hat{R}^{-1}(t) \left\{ V_{x_1}^{[k+1]}(t, x_1) \hat{B}(t) \right. \\ &+ \left. \sum_{i=1}^{k-1} V_{x_1}^{[k-i+1]}(t, x_1) \hat{F}_{1h}^{[i]}(t, x_1, u_*) + \hat{L}_{h,u}^{[k]}(t, x_1, u_*) \right\} \quad (28c) \end{aligned}$$

where for $k = 2, 3, \dots$. In (28a), the terms $\hat{F}_{1h}^{[i]}(t, x_1, u_*)$ and $\hat{L}_h^{[i]}(t, x_1, u_*)$ are given by the corresponding terms in (24c) and (26) and can therefore be expressed in terms of the series expansions of the original functions as

$$\hat{F}_{1h}^{[i]}(t, x_1, u_*) = F_{1h}^{[i]}(t, x_1, \varphi_*, u_*) + A_{12}(t) \varphi_{h,*}^{[i]} \quad (29a)$$

and

$$\begin{aligned} \hat{L}_h^{[i]}(t, x_1, u_*) &= L_h^{[i]}(t, x_1, \varphi_*, u_*) + 2x_1^T Q_{12}(t) \varphi_{h,*}^{[i-1]} \\ &+ 2(\varphi_*^{[1]})^T Q_{22}(t) \varphi_{h,*}^{[i-1]} + 2 \sum_{k=2}^{[(i-1)/2]} (\varphi_{h,*}^{[k]})^T Q_{22}(t) \varphi_{h,*}^{[i-k]} \\ &+ (\varphi_{h,*}^{[i/2]})^T Q_{22}(t) \varphi_{h,*}^{[i/2]} + 2 \sum_{k=1}^{i-2} (u_*^{[k]})^T S_2(t) \varphi_{h,*}^{[i-k]} \quad (29b) \end{aligned}$$

where $\varphi_* = \varphi(x_1, u_*)$ and $\varphi_{h,*} = \varphi_h(x_1, u_*)$. In (28), the series expansion coefficients of the functions $\hat{F}_{1h}$ and $\hat{L}_h$ were needed. These were easily obtained as (29a) and

(29b) respectively. However, in the expression for the optimal control signal (28c), the derivatives of $\hat{F}_1(t, x_1, u)$ and $\hat{L}(t, x_1, u)$ with respect to u are needed. These expressions become

$$\begin{aligned} \hat{F}_{1h,u}^{[i]}(t, x_1, u_*) &= F_{1h,u}^{[i]}(t, x_1, \varphi_*, u_*)^{[i]} \\ &+ \sum_{j=1}^i F_{1h;x_2}^{[j]}(t, x_1, \varphi_*, u_*) \varphi_{u,*}^{[i-j]} + A_{12}(t) \varphi_{h,u,*}^{[i]} \quad (30a) \end{aligned}$$

where $\varphi_{h,u,*}^{[i]} = \varphi_{h,u}^{[i]}(t, x_1, u_*)$, $\varphi_{u,*}^{[i]} = \varphi_u^{[i]}(t, x_1, u_*)$ and

$$\begin{aligned} \hat{L}_{h,u}^{[k]}(t, x_1, u_*) &= L_{h,u}^{[k]}(t, x_1, \varphi_*, u_*) \\ &+ \sum_{j=2}^k L_{h;x_2}^{[j]}(t, x_1, \varphi_*, u_*) \varphi_{u,*}^{[k-j]} + 2x_1^T Q_{12}(t) \varphi_{h,u,*}^{[k-1]} \\ &+ 2\varphi_*^{[1]T} Q_{22}(t) \varphi_{h,u,*}^{[k-1]} - 2B_2^T(t) A_{22}^T(t) Q_{22}(t) \varphi_{h,*}^{[k]} \\ &+ \sum_{j=1}^{k-2} (\varphi_{h,u,*}^{[j]})^T Q_{22}(t) \varphi_{h,*}^{[k-j]} + \sum_{j=1}^{k-2} (\varphi_{h,*}^{[j]})^T Q_{22}(t) \varphi_{h,u,*}^{[j]} \\ &+ 2S_2(t) \varphi_{h,*}^{[k]} + 2 \sum_{j=1}^{k-1} u_*^{[j]T} S_2(t) \varphi_{h,u,*}^{[k-j]} \quad (30b) \end{aligned}$$

Since $F_{1h}(t, x, u)$, $\varphi_h(t, x_1, u)$ and $L_h(t, x, u)$ are power series of degree two, two and three respectively, and

$$\varphi^{[m]}(t, x_1, u_*) = \varphi(t, x_1, u_*^{[1]} + u_*^{[2]} + \dots + u_*^{[m]})$$

it can be concluded, as for the state-space case, that the right-hand side of (28a) only depends on the sequence (15) while the right-hand side of (28c) only depends on (16). So by consecutively calculating the terms of the series

$$V^{[2]}(t, x_1), u_*^{[1]}(t, x_1), \varphi^{[1]}(t, x_1, u_*^{[1]}), \dots$$

it is possible to generate the power series for $V(t, x_1)$, $u_*(t, x_1)$ and $\varphi(t, x_1, u_*(t, x_1))$.

In the sequence above, it can be seen that it is unnecessary to calculate orders of $\varphi(t, x_1, u)$ higher than the desired order of the approximation of $u_*(t, x_1)$. However, as can be seen in (20) and (22), it is possible to compute arbitrarily high orders of $\varphi(t, x_1, u)$ in advance.

Summarizing this section we have the following result.

Theorem 2: Consider the optimal control problem given by (1) and (2). The optimal feedback control $u_*(t, x)$ exists and is the unique solution for small $|x|$ and $t \in [T_0, T]$ of

$$0 = L_u + V_{x_1} F_{1,u} - (L_{x_2} + V_{x_1} F_{1;x_2}) F_{2;x_2}^{-1} F_{2,u} \quad (31a)$$

$$0 = L + V_{x_1} F_1 \quad (31b)$$

$$0 = F_2 \quad (31c)$$

where the right-hand sides are evaluated at (t, x_1, x_2, u) .

Furthermore, this feedback control can be calculated expressed as a power series from (27) and (28).

Proof: Using Theorem 1 on (17) gives

$$0 = L + V_t + V_{x_1} F_1$$

$$0 = L_u + V_{x_1} F_{1,u} + (L_{x_2} + V_{x_1} F_{1;x_2}) \varphi_u$$

where the functions are evaluated at $(t, x_1, \varphi(t, x_1, u), u)$. Differentiation of $F_2(t, x_1, \varphi(t, x_1, u), u)$ with respect to u gives

$$F_{2;x_2}(t, x_1, \varphi(t, x_1, u), u) \varphi_u(t, x_1, u) + F_{2;u}(t, x_1, \varphi(t, x_1, u), u) = 0$$

which can be solved for $\varphi_u(t, x_1, u)$ because $F_{2;x_2}$ is nonsingular locally around the origin for all $t \in [T_0, T]$. Substituting into (32) gives the desired result (31).

The expressions for the series expansion follow from the derivation given by (18) – (30). ■

V. EXTENSION

For notational convenience is assumed in the preceding sections that the system is in semi-explicit form (2). As a result of this assumption we only need to compute the series expansion of φ in order to find the series expansions of the reduced state-space system and solve the optimal control problem. However, if this assumption is not satisfied it is shown in [16] that, possibly after introduction of an integrator chain as seen in [17], a rather general DAE

$$F(\dot{x}, x, u, t) = 0$$

can be rewritten in the form

$$\hat{F}_1(\dot{x}_1, x_1, x_2, u, t) = 0 \quad (33a)$$

$$\hat{F}_2(x_1, x_2, u, t) = 0 \quad (33b)$$

In this section we will use another assumption than Assumption 2 for the functions describing the system.

Assumption 4: The functions $\hat{F}_1, \hat{F}_2$ in (33) are analytical in some neighborhood of $(\dot{x}_1, x_1, x_2, u) = 0$ for all $t \in [T_0, T]$.

As described earlier this assumption makes it possible to write these functions as power series

$$\hat{F}_1(\dot{x}_1, x_1, x_2, u, t) = -\dot{x}_1 + A_{11}(t)x_1 + A_{12}(t)x_2 + B_1(t)u + \hat{F}_{1h}(\dot{x}_1, x_1, x_2, u, t) \quad (34a)$$

$$\hat{F}_2(x_1, x_2, u, t) = A_{21}(t)x_1 + A_{22}(t)x_2 + B_2(t)u + \hat{F}_{2h}(x_1, x_2, u, t) \quad (34b)$$

which are convergent around the origin $(\dot{x}_1, x_1, x_2, u) = 0$, uniformly for all $t \in [T_0, T]$. The functions $\hat{F}_{1h}$ and $\hat{F}_{2h}$ contain terms of degree two and higher.

If x_2 in (33a) is replaced by $\varphi(t, x_1, u)$ the result is

$$\hat{F}_1(\dot{x}_1, x_1, \varphi(t, x_1, u), u, t) = 0 \quad (35)$$

Here we will make one further assumption, similar to Assumption 1.

Assumption 5: Equation (35) can be solved for $\dot{x}_1$ to yield

$$\dot{x}_1 = \xi(t, x_1, u) \quad (36)$$

for all $t \in [T_0, T]$ and all x_1 and u in some open set U containing the origin.

As described after Assumption 1, the functions ξ and φ do not need to be explicitly expressible. However, to find the optimal feedback law it will be sufficient to have their power series expansions.

The series expansion of $\xi(t, x_1, u)$ can be computed recursively, in the same manner as $\varphi(t, x_1, u)$. Let

$$\dot{x}_1 = \xi(t, x_1, u) = \xi^{[1]}(t, x_1, u) + \xi_h(t, x_1, u) \quad (37)$$

where $\xi_h(t, x_1, u)$ is continuous in t and contains terms in x_1 and u beginning with degree two.

If (34a) is combined with (18) and (37) it leads to the equation

$$0 = -\xi^{[1]}(t, x_1, u) - \xi_h(t, x_1, u) + A_{11}(t)x_1 + A_{12}(t)(\varphi^{[1]}(t, x_1, u) + \varphi_h(t, x_1, u)) + B_1(t)u + \hat{F}_{1h}(\xi^{[1]}(t, x_1, u) + \xi_h(t, x_1, u), x_1, \varphi^{[1]}(t, x_1, u) + \varphi_h(t, x_1, u), u, t) \quad (38)$$

and the first term in $\xi(t, x_1, u)$ is obtained as

$$\begin{aligned} \xi^{[1]} &= A_{11}(t)x_1 + A_{12}(t)\varphi^{[1]}(t, x_1, u) + B_1(t)u \\ &= \hat{A}(t)x_1 + \hat{B}(t)u \end{aligned}$$

where (20) is used to obtain the second equality. Since $\hat{F}_{1h}(\dot{x}_1, x_1, x_2, u, t)$ contains terms of at least order two it follows that

$$\begin{aligned} \hat{F}_{1h}^{[m]}(\xi(t, x_1, u), x_1, \varphi(t, x_1, u), u, t) &= \\ = \hat{F}_{1h}^{[m]}(\xi(t, x_1, u) + \dots + \xi^{[m-1]}(t, x_1, u), x_1, \\ \varphi^{[1]}(t, x_1, u) + \varphi^{[m-1]}(t, x_1, u), u, t) \end{aligned}$$

Hence, it is possible to compute $\xi(t, x_1, u)$ recursively. The series expansion obtained fits into earlier described results, and the computation order for the optimal control problem becomes

$$V^{[2]}(t, x_1), u_*^{[1]}(t, x_1), \varphi^{[1]}(t, x_1, u_*^{[1]}), \xi^{[1]}(t, x_1, u_*^{[1]}) \dots$$

As an result, it is possible to compute the optimal solution for systems which is given on the more general form (33), i.e., not being semi-explicit, with the method derived in this paper.

VI. EXAMPLES

In order to illustrate the method we will study a small example, namely an electrical circuit.

Fig. 1. Electrical circuit

The electrical circuit, which can be seen in Figure 1, consists of an ideal voltage source, an inductor with a ferromagnetic core, a capacitor and a resistor. Because of the ferromagnetic core, the flux of the inductor saturates for large currents. Furthermore, the flux is assumed to decrease with time and for large currents to model temperature dependence. The capacitor has a voltage dependent capacitance and the

resistor depends both linearly and cubically on the current. The complete model is can then be written as

$$\dot{u}_C = \frac{i}{1+10^{-2}u_C} \quad (39a)$$

$$\dot{\Phi} = u_L \quad (39b)$$

$$0 = \Phi - \frac{\arctan(i)}{1+10^{-1}i+10^{-2}i^2} \quad (39c)$$

$$0 = u_R - i - i^3 \quad (39d)$$

$$0 = u - u_R - u_C - u_L \quad (39e)$$

where u_C is the voltage over the capacitance, Φ is the flux, u_L is the voltage over the inductor, i is the current, u_R is the voltage over the resistor and u is the voltage over the voltage source. The dynamic variables is in this case chosen as $x_1 = (u_C, \Phi)$ and the algebraic variables is $x_2 = (i, u_L, u_R)$. The control signal is the voltage over the voltage source u . This model satisfies Assumptions 1, 2 and 3. We choose the cost functions as

$$L(u_C, \Phi, i, u_L, u_R) = i^2 + i^4 + \frac{1}{2}u^2$$

$$G(u_C, \Phi) = 0$$

and the final time as $T = 4$.

Applying the method described in the paper will then yield the third order approximation of the optimal cost function as

$$V(t, u_C, \Phi) \approx p_{11}(t)u_C^2 + 2p_{12}(t)u_C\Phi + p_{22}(t)\Phi^2$$

$$+ a_{30}(t)u_C^3 + a_{21}(t)u_C^2\Phi + a_{12}(t)u_C\Phi^2 + a_{03}(t)\Phi^3$$

where the coefficients can be found in Figure 2. The corresponding second order approximation of the optimal control feedback will be

$$u_*(t, u_C, \Phi) \approx d_1(t)u_C + d_2(t)\Phi$$

$$+ b_{20}(t)u_C^2 + b_{11}(t)u_C\Phi + b_{02}(t)\Phi^2$$

where the coefficients can be found in Figure 3.

Fig. 2. Left: The second order terms of V – solid: $p_{11}(t)$, dash-dotted: $p_{12}(t)$, dashed: $p_{22}(t)$, Right: The third order terms of V – star-marked: $a_{30}(t)$, dashed: $a_{21}(t)$, dash-dotted: $a_{12}(t)$, solid: $a_{03}(t)$

VII. CONCLUSIONS

In this paper a method has been presented which makes it possible to calculate a power series solution to the HJB equation for systems described by time-varying differential-algebraic equations. As long as the solution is not truncated the exact optimal solution is obtained. The first terms $u^{[1]}(t, x_1)$ and $V^{[2]}(t, x_1)$ are obtained by solving a Riccati

Fig. 3. Left: The first order terms of u_* – solid: $d_1(t)$, dashed: $d_2(t)$. Right: The second order terms of u_* – solid: $b_{20}(t)$, dash: $b_{11}(t)$, dash-dotted: $b_{02}(t)$

differential equation. The higher order terms of $u_*(t, x_1)$ and $V(t, x_1)$ are then calculated recursively using systems of linear differential equations. A fundamental limit is that it must be possible to, at least implicitly, solve $F_2(x_1, x_2, u) = 0$ for x_2 .

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