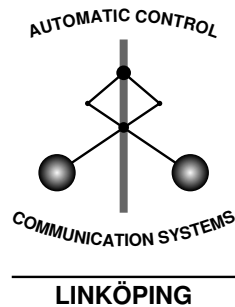


# Modelling of Laser Radar Systems

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## **Abstract**

In this report background material for some papers are stored. It concerns modelling of laser radar systems for hard targeting. The (returned) laser radar data is used for parameters estimation, where error-in-variables is assumed. The impulse response for some common (target) shapes are derived. The coordinate systems in output data is discussed. The system performance is analyzed using the Cramer-Rao lower bound. The models are developed for a scanning, monostatic system, but some are general enough for other type of systems.

**Keywords:** Laser radar, measurement error regression, Cramer-Rao, system model



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# 1 Introduction

In this report we study the properties of data from a generic scanning laser radar system. Data from the system is modelled with an error-in-variables model (also called measurement error model). The performance of a least squares fitting method, the total least squares (TLS) method, is also evaluated. Data from these type of systems generate detailed 3D maps of the terrain and can be used for rapid terrain mapping, scene reconstruction and target recognition. This report contains background data to a few papers:

- [5] Christina Grönwall, Thomas Carlsson, and Fredrik Gustafsson, “A Cramer-Rao lower bound expression for a scanning laser radar system”, Proc. Reglermötet 2002, Linköpings Universitet, Linköping, Sweden, May 29-30, 2002.
- [6] Christina Grönwall, Tomas Carlsson, and Fredrik Gustafsson, “Performance analysis of measurement error regression in direct-detection laser radar imaging”, Proc. ICASSP, vol. VI, pp. 545-548, Hong Kong, China, 6-10 April 2003.
- [7] Christina Grönwall, Ove Steinvall, Fredrik Gustafsson, and Tomas Chevalier, “Influence of Laser Radar Sensor Parameters on Range Measurements and Shape Fitting Uncertainties”, Submitted to Optical Engineering, September, 2006.

When measuring hard targets with a laser radar, the shape of the returning pulse depends on the target’s geometric properties. The impulse responses for some simple geometric shapes are derived in Section 2 and reported in [7]. The returning pulse is used for estimation of time-of-flight, which gives the target range estimate. The output from the measurement system and the post-processing, including fusion with GPS data, is discussed in Section 3. This is background data to [6]. In Section 4, the measurement model for estimation of tilted 1D surface is derived and in Section 5 the Cramer-Rao Lower Bound (CRLB) for line estimation is derived. These results are reported in [5, 6]. The CRLB results are expressed as functions of system parameters in Section 6, this is reported in [5].

The results in Section 3 is applicable for systems scanning in a zigzag pattern. The other results can be applied to generic scanning systems and can be applicable also for staring systems without too much effort.

## 2 Impulse responses for some common geometric shapes

### 2.1 Impulse response for a flat surface, 1D

This subsection contains calculations earlier performed in [2, 13]. They will be repeated here because the angle  $\alpha$  is defined differently and some errors in [2, 13]

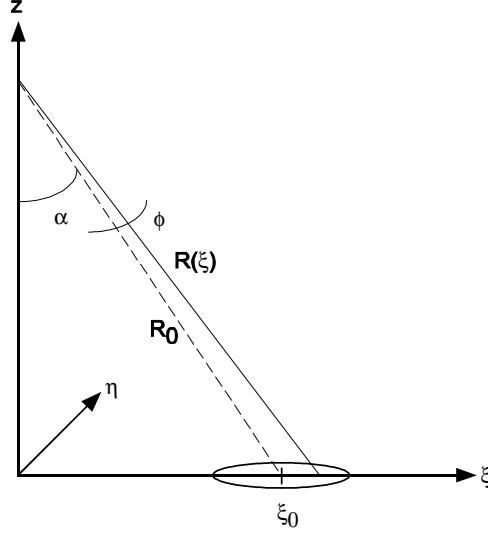


Figure 1: Geometry for the impulse response in 1D, flat surface.

will be corrected. In this case we assume that the spread of the laser beam is much smaller in the  $\eta$  dimension compared to the  $\xi$  dimension, i.e.  $\eta$  can be neglected, see Figure 1.

The distance between the sender/receiver and the target is

$$R(\xi) = \sqrt{R^2(\xi_0) + (\xi - \xi_0)^2}. \quad (1)$$

Using the assumptions that

$$(\xi - \xi_0) \ll R(\xi_0)$$

and

$$\sin \alpha = \xi_0 / R(\xi_0).$$

we can simplify (1) to

$$R(\xi) = \frac{\xi}{\sin \alpha}.$$

We set

$$k = \frac{1}{\sin \alpha} \quad (2)$$

and get

$$R(\xi) = k\xi.$$

The pulse response is given in [13], eq. (45), in as

$$h(t) = \iint_{-\infty}^{\infty} \exp\left(-2\frac{(\xi - \xi_0)^2 + \eta^2}{w^2}\right) \delta\left(t - \frac{2R(\xi)}{c}\right) d\xi d\eta, \quad (3)$$



where  $w$  is the laser beam's radius on the target (the footprint),  $c$  the speed of light,  $\delta(t)$  the Dirac function and  $t$  time. The radius of the laser beam at the target,  $w$ , at distance  $R$  is given by

$$w = w_0 \sqrt{\Omega_0^2 + \Omega^2},$$

where

$$\begin{aligned} w_0 &= \frac{2\lambda}{\pi\phi}, \\ \Omega_0 &= 1 - \frac{R}{R_{\text{focus}}}, \\ \Omega &= \frac{\lambda R}{\pi w_0^2}, \end{aligned}$$

where  $w_0$  is the beam radius at the laser source (the sender),  $\lambda$  is the wavelength of the laser source,  $\phi$  is the beam divergence and  $R_{\text{focus}}$  is the focusing range of the laser source (a very large number for a collimated beam). For a collimated beam we have  $\Omega_0 \approx 1$ .

A Dirac function has the following properties:

$$\begin{aligned} \delta(-t) &= \delta(t) \\ \delta(at) &= \frac{1}{a} \delta(t), \quad a > 0 \\ \int_{-\infty}^{\infty} f(t) \delta(t-a) dt &= f(a), \quad a > 0 \text{ and } f(t) \text{ is continuous} \end{aligned}$$

Inserting these properties in (3) we get

$$h(t) = \int_{-\infty}^{\infty} \exp\left(-2\frac{\eta^2}{w^2}\right) \int_{-\infty}^{\infty} \underbrace{\exp\left(-2\frac{(\xi - \xi_0)^2}{w^2}\right)}_{f(\xi)} \delta\left(t - \frac{2k\xi}{c}\right) d\xi d\eta,$$

where the second integral is

$$\begin{aligned} \int_{-\infty}^{\infty} f(\xi) \delta\left(t - \frac{2k\xi}{c}\right) d\xi &= \int_{-\infty}^{\infty} f(\xi) \delta\left(\frac{2k\xi}{c} - t\right) d\xi = \\ \int_{-\infty}^{\infty} f(\xi) \delta\left(\frac{2k\xi}{c} - t\right) d\xi &= \int_{-\infty}^{\infty} f(\xi) \delta\left(\frac{2k}{c} \left(\xi - t\frac{c}{2k}\right)\right) d\xi = \\ \frac{c}{2k} \int_{-\infty}^{\infty} f(\xi) \delta\left(\xi - t\frac{c}{2k}\right) d\xi &= \frac{c}{2k} f\left(t\frac{c}{2k}\right). \end{aligned}$$

The first integral is a symmetric function (symmetric around  $\eta = 0$ ) that can be solved by

$$\int_{-\infty}^{\infty} \exp\left(-2\frac{\eta^2}{w^2}\right) d\eta = 2 \int_0^{\infty} \exp\left(-\frac{\eta^2}{(w/\sqrt{2})^2}\right) d\eta = 2\sqrt{\pi} \left(w/\sqrt{2}\right) = \sqrt{2\pi}w.$$

Finally, the impulse response can be expressed as

$$\begin{aligned} h(t) &= \sqrt{2\pi}w \frac{c}{2k} \exp - \left( \frac{t\frac{c}{2k} - \xi_0}{w/\sqrt{2}} \right)^2 \\ &= \sqrt{\frac{\pi}{2}} \frac{cw}{k} \exp - \left( \frac{t\frac{c}{2k} - \xi_0}{w/\sqrt{2}} \right)^2 \end{aligned}$$

or

$$h(t) = \sqrt{\frac{\pi}{2}} \frac{cw}{k} \exp - \left( \frac{t - t'}{\tau_0} \right)^2, \quad (4a)$$

$$t' = \frac{2k\xi_0}{c} = \frac{2\xi_0}{c \sin \alpha}, \quad (4b)$$

$$\tau_0 = \frac{2kw}{c\sqrt{2}} = \frac{\sqrt{2}kw}{c} = \frac{\sqrt{2}w}{c \sin \alpha}. \quad (4c)$$

In [2, 13] we have

$$\begin{aligned} t' &= \frac{2\xi_0}{c \sin \alpha}, \\ \tau_0 &= \frac{\sqrt{2}w}{\sin \alpha}. \end{aligned}$$

After a discussion with the authors of [2, 13], we agreed that both  $t'$  and  $\tau_0$  must have  $c$  in the denominator, to get correct units. Further, simulation shows that using  $t'$  and  $\tau_0$  as in [2, 13] does not get correct models of the pulse responses, while the expressions given in (4) do.

## 2.2 Impulse response for a flat surface, 2D

The description of the pulse response in the previous section is quite straightforward but it is sometimes a too simple description. Usually, both  $\xi$  and  $\eta$  dimensions must be considered. The distance between the sender/receiver and position  $(\xi, \eta)$  on the target is

$$R(\xi, \eta) = \sqrt{R^2(\xi_0, \eta_0) + (\xi - \xi_0)^2 + (\eta - \eta_0)^2}. \quad (5)$$

Applying the assumptions

$$\begin{aligned} (\xi - \xi_0) &\ll R(\xi_0, \eta_0), \\ (\eta - \eta_0) &\ll R(\xi_0, \eta_0), \end{aligned}$$

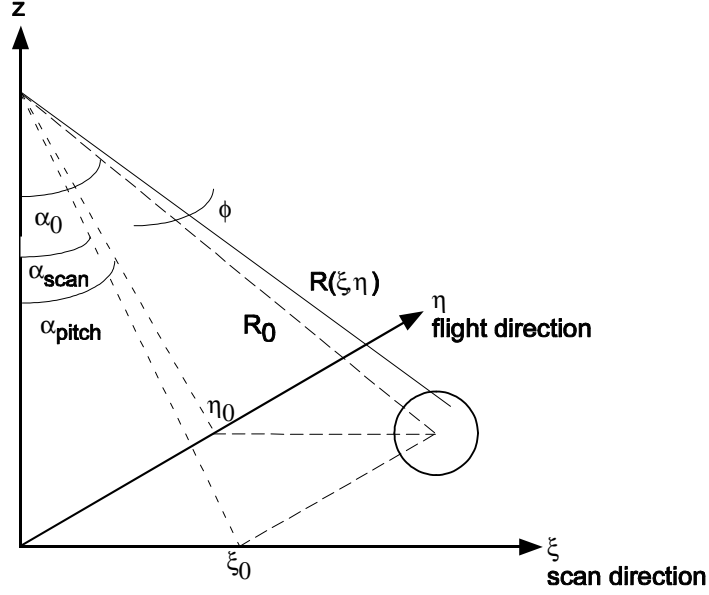


Figure 2: Geometry of the impulse responses in 2D, flat surface.

and

$$R(\xi_0, \eta_0) = z \sqrt{1 + \tan^2 \alpha_{\text{scan}} + \tan^2 \alpha_{\text{pitch}}},$$

where

$$z = \frac{\xi_0}{\tan \alpha_{\text{scan}}} = \frac{\eta_0}{\tan \alpha_{\text{pitch}}},$$

we can simplify (5) to

$$R(\xi, \eta) = \frac{\xi}{\tan \alpha_{\text{scan}}} \sqrt{1 + \tan^2 \alpha_{\text{scan}} + \tan^2 \alpha_{\text{pitch}}}.$$

We define or

$$k = \frac{1}{\tan \alpha_{\text{scan}}} \sqrt{1 + \tan^2 \alpha_{\text{scan}} + \tan^2 \alpha_{\text{pitch}}} \quad (6)$$

and, again, we have

$$R(\xi, \eta) = k\xi.$$

The time and range dependent impulse response can now be written

$$h(\xi, \eta, t) = \exp\left(-2 \frac{(\xi - \xi_0)^2 + (\eta - \eta_0)^2}{w^2}\right) \delta\left(t - \frac{2k\xi}{c}\right).$$

The time-only dependent impulse response is retrieved from

$$\begin{aligned}
h(t) &= \iint_{-\infty}^{\infty} \exp\left(-2\frac{(\xi - \xi_0)^2 + (\eta - \eta_0)^2}{w^2}\right) \delta\left(t - \frac{2R(\xi, \eta)}{c}\right) d\xi d\eta, \\
&= \iint_{-\infty}^{\infty} \exp\left(-2\frac{(\eta - \eta_0)^2}{w^2}\right) \exp\left(-2\frac{(\xi - \xi_0)^2}{w^2}\right) \delta\left(t - \frac{2k\xi}{c}\right) d\xi d\eta \\
&= \int_{-\infty}^{\infty} \exp\left(-2\frac{(\eta - \eta_0)^2}{w^2}\right) \int_{-\infty}^{\infty} \exp\left(-2\frac{(\xi - \xi_0)^2}{w^2}\right) \delta\left(t - \frac{2k\xi}{c}\right) d\xi d\eta.
\end{aligned}$$

The second integral has solution (see previous section)

$$\begin{aligned}
\int_{-\infty}^{\infty} \exp\left(-2\frac{(\xi - \xi_0)^2}{w^2}\right) \delta\left(t - \frac{2k\xi}{c}\right) d\xi d\eta &= \frac{c}{2k} \exp - \left(\frac{t - t'}{\tau_0}\right)^2, \\
t' &= \frac{2k\xi_0}{c}, \\
\tau_0 &= \frac{\sqrt{2}kw}{c}.
\end{aligned}$$

The first integral is formulated as a standard integral (see some mathematics handbook):

$$\begin{aligned}
&\int_{-\infty}^{\infty} \exp\left(-\frac{(\eta - \eta_0)^2}{(w/\sqrt{2})^2}\right) d\eta = \int_{-\infty}^{\infty} \exp\left(-\frac{\eta^2 - 2\eta_0\eta - \eta_0^2}{(w/\sqrt{2})^2}\right) d\eta \\
&\exp\left(\frac{\eta_0^2}{(w/\sqrt{2})^2}\right) \int_{-\infty}^{\infty} \exp\left(\frac{2\eta_0\eta - \eta^2}{(w/\sqrt{2})^2}\right) d\eta = \\
&\exp\left(\frac{\eta_0^2}{(w/\sqrt{2})^2}\right) \sqrt{\frac{\pi}{1}} \exp(\eta_0^2/1) = \sqrt{\pi} \exp\left(\frac{\eta_0^2}{(w/\sqrt{2})^2} + \eta_0^2\right) \\
&\sqrt{\pi} \exp\left(\eta_0^2 \frac{1 + (w/\sqrt{2})^2}{(w/\sqrt{2})^2}\right) = \sqrt{\pi} \exp\left(\eta_0^2 \frac{2 + w^2}{w^2}\right).
\end{aligned}$$

Finally, we get

$$h(t) = \frac{\sqrt{\pi}c}{2k} \exp\left(\eta_0^2 \frac{2 + w^2}{w^2}\right) \exp - \left(\frac{t - t'}{\tau_0}\right)^2, \quad (7a)$$

$$t' = \frac{2k\xi_0}{c}, \quad (7b)$$

$$\tau_0 = \frac{\sqrt{2}wk}{c}. \quad (7c)$$

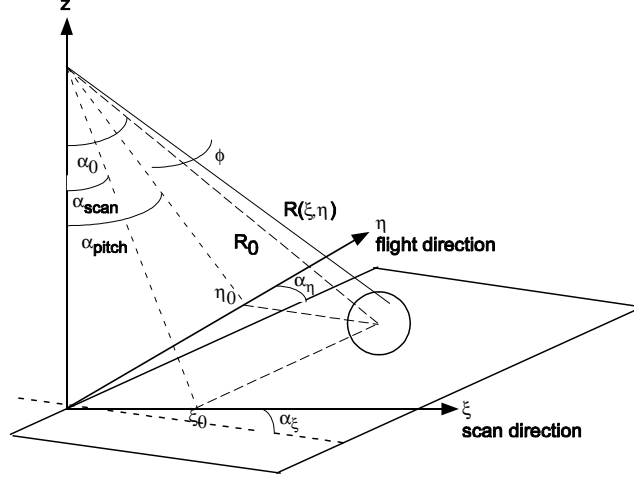


Figure 3: Geometry for the impulse response in 2D, flat and tilted surface.

To validate the expression, if we set  $\alpha_{\text{pitch}} = 0$ , and the results is equal to (4):

$$\begin{aligned}
 k &= \frac{1}{\tan \alpha_{\text{scan}}} \sqrt{1 + \tan^2 \alpha_{\text{scan}}} \\
 &= \frac{1}{\tan \alpha_{\text{scan}}} \sqrt{1 / \cos^2 \alpha_{\text{scan}}} \\
 &= \frac{1}{\tan \alpha_{\text{scan}} \cos \alpha_{\text{scan}}} \\
 &= \frac{1}{\sin \alpha_{\text{scan}}}.
 \end{aligned}$$

For  $\alpha_{\text{pitch}} = 0$  we have  $\eta_0 = 0$ , and thus,  $\exp\left(\eta_0^2 \frac{2+w^2}{w^2}\right) = 1$ .

### 2.3 Impulse response for a tilted, flat surface, 2D

We now allow the target's plane to tilt in 2D. We have measurement angles  $(\alpha_{\text{scan}}, \alpha_{\text{pitch}})$  and the tilt of the plane is denoted  $(\alpha_{\beta}, \alpha_{\eta})$ . The angles  $(\alpha_{\text{scan}}, \alpha_{\beta})$  are parallel with the  $\beta$  axis and  $(\alpha_{\text{pitch}}, \alpha_{\eta})$  are parallel with the  $\eta$  axis, see Figure 3. The distance between the sender/receiver and position  $(\xi, \eta)$  on the target is

$$R(\xi, \eta) = \sqrt{R^2(\xi_0, \eta_0) + (\xi - \xi_0)^2 + (\eta - \eta_0)^2}. \quad (8)$$

We apply the assumptions

$$\begin{aligned}
 (\xi - \xi_0) &\ll R(\xi_0, \eta_0), \\
 (\eta - \eta_0) &\ll R(\xi_0, \eta_0),
 \end{aligned}$$

and

$$R(\xi_0, \eta_0) = z \sqrt{1 + \frac{\tan^2 \alpha_{\text{scan}}}{1 - \tan \alpha_{\text{scan}} \tan \alpha_\xi} + \frac{\tan^2 \alpha_{\text{pitch}}}{1 - \tan \alpha_{\text{pitch}} \tan \alpha_\eta}},$$

where

$$z = \frac{\xi_0}{\tan \alpha_{\text{scan}}} = \frac{\eta_0}{\tan \alpha_{\text{pitch}}}.$$

We can now define

$$k = \frac{1}{\tan \alpha_{\text{scan}}} \sqrt{1 + \frac{\tan^2 \alpha_{\text{scan}}}{1 - \tan \alpha_{\text{scan}} \tan \alpha_\xi} + \frac{\tan^2 \alpha_{\text{pitch}}}{1 - \tan \alpha_{\text{pitch}} \tan \alpha_\eta}} \quad (9)$$

and, again, we can simplify (8) to

$$R(\xi, \eta) = k\xi.$$

The pulse response is expressed as

$$h(t) = \iint_{-\infty}^{\infty} \exp\left(-2\frac{(\xi - \xi_0)^2 + (\eta - \eta_0)^2}{w^2}\right) \delta\left(t - \frac{2R(\xi, \eta)}{c}\right) d\xi d\eta$$

and has solution (see previous section)

$$h(t) = \frac{\sqrt{\pi}c}{2k} \exp\left(\eta_0^2 \frac{2+w^2}{w^2}\right) \exp\left(-\frac{(t-t')^2}{\tau_0}\right), \quad (10)$$

$$t' = \frac{2k\xi_0}{c}, \quad (11)$$

$$\tau_0 = \frac{\sqrt{2}wk}{c}. \quad (12)$$

To validate the expression, we set  $\alpha_\beta = \alpha_\eta = 0$  and the result is equal to (6):

$$\begin{aligned} k &= \frac{1}{\tan \alpha_{\text{scan}}} \sqrt{1 + \frac{\tan^2 \alpha_{\text{scan}}}{1 - \tan \alpha_{\text{scan}} \tan \alpha_\xi} + \frac{\tan^2 \alpha_{\text{pitch}}}{1 - \tan \alpha_{\text{pitch}} \tan \alpha_\eta}} \\ &= \frac{1}{\tan \alpha_{\text{scan}}} \sqrt{1 + \tan^2 \alpha_{\text{scan}} + \tan^2 \alpha_{\text{pitch}}}. \end{aligned}$$

## 2.4 Impulse response for a cone

If we assume that the laser beam axis and the  $z$  axis coincide, i.e.,  $\alpha = 0$  compare to above, and that we have rotation-symmetric objects we can use polar coordinates  $(r, \omega)$ . The impulse response can now be written

$$\begin{aligned} h(r, \omega, t) &= 4\pi \exp\left(-\frac{r^2}{(w/\sqrt{2})^2}\right) \rho_b(\theta) \delta(t - 2z(r, \omega)/c), \\ 0 &\leq r \leq w = r_{\text{beam}}, \quad 0 \leq \omega \leq 2\pi. \end{aligned}$$

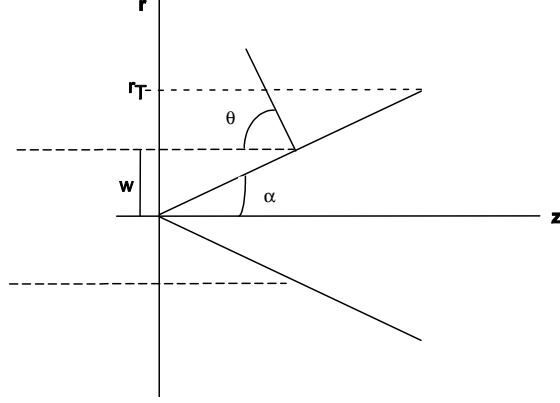


Figure 4: Geometry for calculation of the cone's impulse response.

For a cone with half-angle  $\alpha$ , where  $\alpha = \pi/2 - \theta$ , we have  $z(r, \alpha) = r / \tan \alpha$  or, equivalently,  $z(r, \theta) = r / \tan(\pi/2 - \theta)$ , see Figure 4. By setting  $\alpha = \pi/2$  we get the laser radar cross section for a **flat plate**. If we describe the impulse response as a function of  $\alpha$  instead of  $\theta$ , we have

$$h_{\text{cone}}(r, t) = 4\pi \exp\left(-2\frac{r^2}{w^2}\right) \rho_b(\pi/2 - \alpha) \delta\left(t - \frac{2r}{c \tan \alpha}\right).$$

An impulse response that only is time-dependent is given by

$$h_{\text{cone}}(t) = 4\pi \rho_b(\pi/2 - \alpha) \int_{-w}^w \exp\left(-2\frac{r^2}{w^2}\right) \delta\left(t - \frac{2r}{c \tan \alpha}\right) dr$$

Using the properties of Dirac functions, we get

$$\begin{aligned} & \int_{-w}^w \exp\left(-2\frac{r^2}{w^2}\right) \delta\left(t - \frac{2r}{c \tan \alpha}\right) dr = \\ & \int_{-w}^w \exp\left(-2\frac{r^2}{w^2}\right) \delta\left(\frac{2}{c \tan \alpha} \left(\frac{c \tan \alpha}{2} t - r\right)\right) dr = \\ & \frac{c \tan \alpha}{2} \int_{-w}^w \exp\left(-2\frac{r^2}{w^2}\right) \delta\left(r - \frac{c \tan \alpha}{2} t\right) dr = \\ & \frac{c \tan \alpha}{2} \exp\left(-2\frac{\left(\frac{ct \tan \alpha}{2}\right)^2}{w^2}\right) = \\ & \frac{c \tan \alpha}{2} \exp\left(-\frac{(ct \tan \alpha)^2}{(\sqrt{2}w)^2}\right). \end{aligned}$$

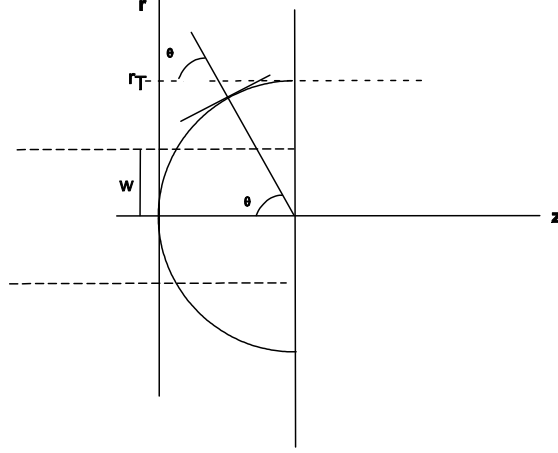


Figure 5: Geometry for calculation of the sphere's impulse response.

The time-dependent impulse response is

$$h_{\text{cone}}(t) = 2\pi\rho_b(\pi/2 - \alpha)c \tan \alpha \exp\left(-\frac{(ct \tan \alpha)^2}{(\sqrt{2}w)^2}\right),$$

where the tan-function implies that  $0 \leq \alpha < \pi/2$ .

## 2.5 Impulse response for a sphere

For a sphere we have  $\omega = \theta$ . We define  $r_T$  as the sphere's radius. Using the geometry in Figure 5, we have

$$\begin{aligned} z(2r_T - z) &= r^2 \\ z^2 - 2r_T z + r^2 &= 0 \\ z &= -\frac{1}{2}(-2r_T) \pm \frac{1}{2}\sqrt{(2r_T)^2 - 4r^2} \\ z &= r_T - \sqrt{r_T^2 - r^2} = r_T \left(1 - \sqrt{1 - r^2/r_T^2}\right), \quad |r| \leq r_T \end{aligned}$$

We approximate the surface to a Lambertian surface, i.e.,

$$\rho_b(\theta) = B \cos \theta = B\sqrt{1 - \sin^2 \theta},$$

where  $\sin \theta = r/r_T$ . The time and radius dependent impulse response can now be written

$$h_{\text{sphere}}(r, t) = 4\pi \exp\left(-2\frac{r^2}{w^2}\right) B\sqrt{1 - r^2/r_T^2} \delta\left(t - \frac{2r_T}{c} \left(1 - \sqrt{1 - r^2/r_T^2}\right)\right).$$



We define

$$x = \sqrt{1 - r^2/r_T^2}$$

and derive

$$\begin{aligned} r &= r_T \sqrt{1 - x^2} \\ \frac{dr}{dx} &= r_T \frac{1}{2} \frac{-2x}{\sqrt{1 - x^2}} = \frac{-r_T x}{\sqrt{1 - x^2}} \\ r = 0 &\Rightarrow x = 1 \\ r = r_T &\Rightarrow x = 0. \end{aligned}$$

We now have

$$h_{\text{sphere}}(x, t) = 4\pi B x \exp\left(-2 \frac{(r_T \sqrt{1 - x^2})^2}{w^2}\right) \delta\left(t - \frac{2r_T}{c}(1 - x)\right).$$

The time dependent impulse response is

$$h_{\text{sphere}}(t) = -4\pi B \int_0^1 \frac{-r_T x^2}{\sqrt{1 - x^2}} \exp\left(-2 \frac{(r_T \sqrt{1 - x^2})^2}{w^2}\right) \delta\left(t - \frac{2r_T}{c}(1 - x)\right) dx.$$

To get the time dependent expression we apply some properties of the Dirac function,

$$\begin{aligned} \delta\left(t - \frac{2r_T}{c}(1 - x)\right) &= \delta\left(\frac{2r_T}{c} \left(\frac{ct}{2r_T} - (1 - x)\right)\right) \\ \frac{c}{2r_T} \delta\left(\frac{ct}{2r_T} - 1 + x\right) &= \frac{c}{2r_T} \delta\left(x - \left(1 - \frac{ct}{2r_T}\right)\right). \end{aligned}$$

We rewrite the impulse response

$$\begin{aligned} h_{\text{sphere}}(t) &= 4\pi B \frac{c}{2r_T} r_T \int_0^1 f(x) \delta\left(x - \left(1 - \frac{ct}{2r_T}\right)\right) dx \\ &= 2\pi B c \int_0^1 f(x) \delta\left(x - \left(1 - \frac{ct}{2r_T}\right)\right) dx \\ f(x) &= \frac{x^2}{\sqrt{1 - x^2}} \exp\left(-\frac{(r_T \sqrt{1 - x^2})^2}{(w/\sqrt{2})^2}\right) \end{aligned}$$

Using the properties of Dirac functions, we get

$$h_{\text{sphere}}(t) = 2\pi B c f\left(x = 1 - \frac{ct}{2r_T}\right).$$

Let us study the quadratic term:

$$x^2 = \left(1 - \frac{ct}{2r_T}\right)^2 = 1 - \frac{ct}{r_T} + \left(\frac{ct}{2r_T}\right)^2$$

and

$$1 - x^2 = 1 - \left(1 - \frac{ct}{2r_T}\right)^2 = \frac{ct}{r_T} - \left(\frac{ct}{2r_T}\right)^2 = \frac{ct}{r_T} \left(1 - \frac{ct}{4r_T}\right).$$

This gives

$$\begin{aligned} f\left(x = 1 - \frac{ct}{2r_T}\right) &= \frac{\left(1 - \frac{ct}{2r_T}\right)^2}{\sqrt{1 - \left(1 - \frac{ct}{2r_T}\right)^2}} \exp\left(-2 \frac{\left(r_T \sqrt{1 - \left(1 - \frac{ct}{2r_T}\right)^2}\right)^2}{w^2}\right) \\ &= \frac{\left(1 - \frac{ct}{2r_T}\right)^2}{\sqrt{\frac{ct}{r_T}} \sqrt{\left(1 - \frac{ct}{4r_T}\right)}} \exp\left(-2 \frac{r_T^2 \left(1 - \left(1 - \frac{ct}{2r_T}\right)^2\right)}{w^2}\right) \\ &= \frac{(2r_T)^2}{\sqrt{(2r_T)^4}} \frac{\left(1 - \frac{ct}{2r_T}\right)^2}{\sqrt{\frac{ct}{r_T}} \sqrt{\left(1 - \frac{ct}{4r_T}\right)}} \exp\left(-2 \frac{r_T^2 \frac{ct}{r_T} \left(1 - \frac{ct}{4r_T}\right)}{w^2}\right) \\ &= \frac{(2r_T - ct)^2}{\sqrt{16r_T^4 \frac{ct}{r_T} \left(1 - \frac{ct}{4r_T}\right)}} \exp\left(-\frac{ct(4r_T - ct)}{2w^2}\right) \\ &= \frac{(2r_T - ct)^2}{\sqrt{4r_T^2 ct (4r_T - ct)}} \exp\left(-\frac{ct(4r_T - ct)}{2w^2}\right) \\ &= \frac{(2r_T - ct)^2}{2r_T \sqrt{ct} \sqrt{4r_T - ct}} \exp\left(-\frac{ct(4r_T - ct)}{2w^2}\right). \end{aligned}$$

Insert this in the impulse response

$$\begin{aligned} h_{\text{sphere}}(t) &= 2\pi Bc \frac{(2r_T - ct)^2}{2r_T \sqrt{ct} \sqrt{4r_T - ct}} \exp\left(-\frac{ct(4r_T - ct)}{2w^2}\right) \\ &= \frac{\pi B}{2r_T} \sqrt{\frac{c}{t}} \frac{(2r_T - ct)^2}{\sqrt{4r_T - ct}} \exp\left(-\frac{ct(4r_T - ct)}{2w^2}\right). \end{aligned}$$

This formula puts some demands on the simulations. First,  $\sqrt{c/t}$  implies that  $t > 0$ . Second,  $1/\sqrt{4r_T - ct}$  implies that  $4r_T - ct > 0$ , the maximum number of  $t$  for various values on  $r_T$  are shown in Table 1. In the table we define  $k = r_T/w$ .

	$w = 0.05$	$w = 0.5$	$w = 1$
$k$	$t_{\max}$	$t_{\max}$	$t_{\max}$
0.1	$6.73 \cdot 10^{-11}$	$6.73 \cdot 10^{-10}$	$1.33 \cdot 10^{-9}$
1	$6.73 \cdot 10^{-10}$	$6.73 \cdot 10^{-9}$	$1.33 \cdot 10^{-8}$
5	$3.34 \cdot 10^{-9}$	$3.34 \cdot 10^{-8}$	$6.67 \cdot 10^{-8}$
10	$6.73 \cdot 10^{-9}$	$6.73 \cdot 10^{-8}$	$1.33 \cdot 10^{-7}$

Table 1: Limiting values for the sphere expression.

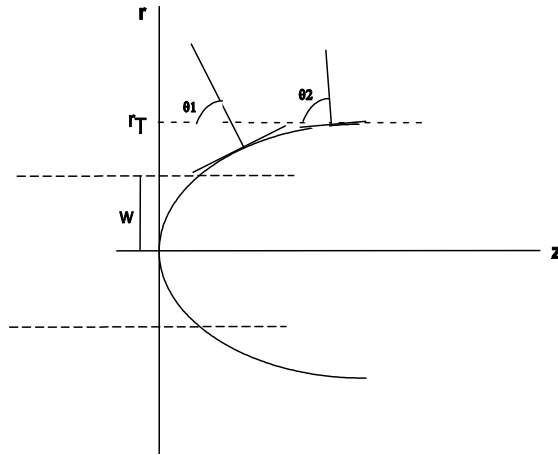


Figure 6: Geometry for calculation of the elliptic parabola's impulse response.

## 2.6 Impulse response for a paraboloid

For an elliptic paraboloid with coefficient  $k$  we have  $z(r) = kr^2$ . For a Lambertian surface we have  $\rho_b(\theta) = B \cos \theta$ , where

$$\begin{aligned}\tan \theta &= z/r = kr \\ \cos \theta &= 1/\sqrt{1 + \tan^2 \theta} = 1/\sqrt{1 + (kr)^2}.\end{aligned}$$

The time and radius dependent impulse response is

$$h_{\text{paraboloid}}(r, t) = 4\pi B \frac{1}{\sqrt{1 + (kr)^2}} \exp\left(-2\frac{r^2}{w^2}\right) \delta\left(t - \frac{2kr^2}{c}\right)$$

or

$$\begin{aligned}h_{\text{paraboloid}}(r, t) &= 4\pi B f(r) \delta\left(t - \frac{2kr^2}{c}\right) \\ f(r) &= \frac{1}{\sqrt{1 + (kr)^2}} \exp\left(-2\frac{r^2}{w^2}\right)\end{aligned}$$

We define

$$x = r^2$$

and derive

$$\begin{aligned}r &= \sqrt{x} \\ \frac{dr}{dx} &= \frac{1}{2\sqrt{x}} \\ r = 0 &\Rightarrow x = 0 \\ r = w &\Rightarrow x = \sqrt{w}.\end{aligned}$$

The Dirac function is now written

$$\delta\left(t - \frac{2kr^2}{c}\right) = \delta\left(t - \frac{2kx}{c}\right) = \frac{c}{2k} \delta\left(\frac{ct}{2k} - x\right) = \frac{c}{2k} \delta\left(x - \frac{ct}{2k}\right).$$

The time-dependent impulse response function is formulated

$$\begin{aligned}h_{\text{paraboloid}}(t) &= 4\pi B \int_{-w}^w f(r) \delta\left(t - \frac{2kr^2}{c}\right) dr. \\ &= 4\pi B \frac{c}{2k} \int_{-\sqrt{w}}^{\sqrt{w}} f(x) \delta\left(x - \frac{ct}{2k}\right) dx, \\ f(x) &= \frac{1}{2\sqrt{x}} \frac{1}{\sqrt{1 + k^2x}} \exp\left(-2\frac{x}{w^2}\right).\end{aligned}$$

The properties of the Dirac function gives

$$h_{\text{paraboloid}}(t) = 4\pi B \frac{c}{2k} f\left(x = \frac{ct}{2k}\right),$$

where

$$\begin{aligned} f\left(\frac{ct}{2k}\right) &= \frac{1}{2\sqrt{\frac{ct}{2k}}} \frac{1}{\sqrt{1 + k^2 \frac{ct}{2k}}} \exp\left(-2\frac{ct}{2kw^2}\right) \\ &= \frac{1}{2} \frac{1}{\sqrt{\frac{ct}{2k} \left(1 + k\frac{ct}{2}\right)}} \exp\left(-\frac{ct}{kw^2}\right) \\ &= \frac{1}{2} \frac{1}{\sqrt{\frac{ct}{2k} + \left(\frac{ct}{2}\right)^2}} \exp\left(-\frac{ct}{kw^2}\right). \end{aligned}$$

The time-dependent impulse response function is now formulated

$$\begin{aligned} h_{\text{paraboloid}}(t) &= 4\pi B \frac{c}{2k} \frac{1}{2} \frac{1}{\sqrt{\left(\frac{ct}{2k} + \left(\frac{ct}{2}\right)^2\right)}} \exp\left(-\frac{ct}{kw^2}\right) \\ &= \pi B \frac{1}{\sqrt{\frac{k^2}{c^2} \left(\frac{ct}{2k} + \left(\frac{ct}{2}\right)^2\right)}} \exp\left(-\frac{ct}{kw^2}\right) \\ &= \pi B \frac{1}{\sqrt{\left(\frac{kt}{2c} + \left(\frac{kt}{2}\right)^2\right)}} \exp\left(-\frac{ct}{kw^2}\right). \end{aligned}$$

The inverse root implies that  $t > 0$ .

### 3 The laser radar system's output

Scanning laser radar systems were originally point scanning, where a laser pointer was swept in 2D. The TopEye<sup>1</sup>, the Riegl<sup>2</sup> and the ILRIS<sup>3</sup> systems are examples of point scanning systems. There are also line scanning systems, like the LOCAAS<sup>4</sup> and the Tomahawk<sup>5</sup> systems. The TopEye system scans the object in a zigzag pattern, while the other systems scan in a regular grid (with some uncertainty). Some subjects in this section are only valid for zigzag scanning and usually not a problem for systems using regular grid.

<sup>1</sup> [www.topeye.com](http://www.topeye.com)

<sup>2</sup> [www.riegl.com](http://www.riegl.com)

<sup>3</sup> [www.optech.ca](http://www.optech.ca)

<sup>4</sup> [www.missilesandfirecontrol.com/our\\_products/strikeweapons/LOCAAS/product-locaas.html](http://www.missilesandfirecontrol.com/our_products/strikeweapons/LOCAAS/product-locaas.html)

<sup>5</sup> [www.designation-systems.net/dusrm/m-109.html](http://www.designation-systems.net/dusrm/m-109.html)

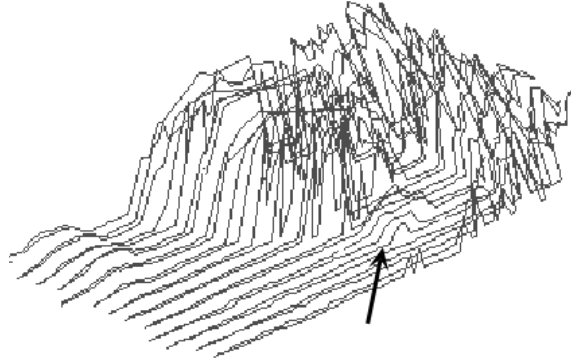


Figure 7: Example of (raw) data from a helicopter-carried scanning laser radar system. The arrow marks a vehicle placed in an opening of the forest. This data set comes from the TopEye system.

From the estimation of the range to the target,  $\hat{R}(\xi, \eta)$ , the height value  $z$  is calculated using information from the inertia navigation system (INS), mainly

$$z \sim f\left(\hat{R}(\xi, \eta), \alpha_{scan}, \alpha_{pitch}, \langle \text{INS parameters} \rangle\right).$$

Usually, a GPS system is used during the data collection and the laser samples are fused with the GPS coordinates. Each data point contains  $(x, y, z, I)$ , where  $(x, y)$  is the position in north-south and east-west, respectively,  $z$  is the altitude, and  $I$  is the intensity in the returning pulse. This means that the set of data/image points corresponds to a 3D mapping of the terrain. An data example is shown in Figure 7. The scanning is usually performed in another direction than  $(x, y)$ . Let us call the direction of the scanning  $\xi$  and the perpendicular direction for  $\eta$  (for TopEye  $\eta$  is the helicopter's flight direction). Thus, the data collection is performed in the coordinate system  $(\xi, \eta)$ , see Figure 8.

### 3.1 The relation between $\xi$ and $\eta$

Let us evaluate for which conditions that  $\xi$  can be considered perpendicular to  $\eta$ . In Figure 9, two consecutive samples in a scan are shown, for down-looking sampling. The difference in scan angle between two samples,  $\Delta\phi$ , is a function of the pulse repetition frequency (PRF),  $f_{PRF}$ , of the laser and the change of angle per second of the scanner mirror,  $\dot{\alpha}_{scan}$ .  $\xi$  is a function of  $\Delta\phi$  and the flight height,  $h$ , of the helicopter.  $\eta$  is a function of the speed of the helicopter,  $v$ , and the PRF. The difference in scan angle between two samples is

$$\Delta\phi = \frac{\dot{\alpha}_{scan}}{f_{PRF}}$$

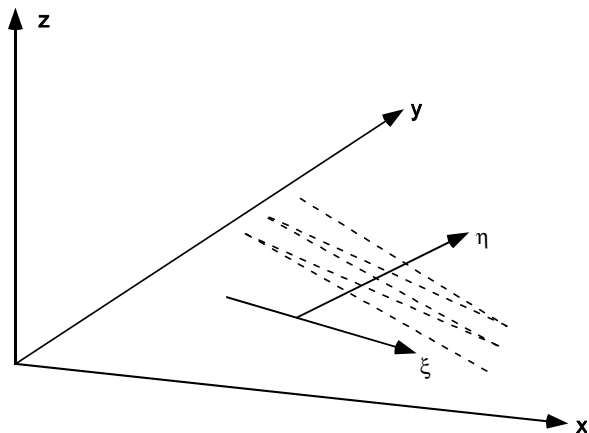


Figure 8: A view of the dimensions of data.  $(x, y)$  is the position in north-south and east-west, respectively, and  $z$  is the altitude. The direction of the scanning gives us a second coordinate system,  $(\xi, \eta)$ .

and from that we get the change in  $\xi$  between two samples as (see Figure 9 and assuming that  $\beta$  is small)

$$\Delta\xi = 2h \cos(\beta) \tan\left(\frac{\Delta\phi}{2}\right) \approx 2h \tan\left(\frac{\dot{\alpha}_{scan}}{2 \cdot f_{PRF}}\right).$$

The change in  $\eta$  between two samples is

$$\Delta\eta = \frac{v}{f_{PRF}}.$$

In a typical setting of the TopEye system we have  $\dot{\alpha}_{scan} = 700^\circ/s$ ,  $f_{PRF} = 7kHz$ ,  $h = 60m$  and  $v = 10m/s$ . From this setting we get  $\Delta\xi \approx 0,105m$ ,  $\Delta\eta \approx 1,43 \cdot 10^{-3}m$  and  $\beta \approx 0,78^\circ$ . For this case it is reasonable to approximate  $\xi$  to be perpendicular to  $\eta$ .

Let us also identify the maximum angle for the perpendicular approximation, say that a reasonable limit is  $\Delta\eta/\Delta\xi = 0,1$  (equals  $\beta \approx 5,7^\circ$ ).  $\Delta\eta/\Delta\xi = 0,1$  corresponds to a helicopter speed of  $v = 73m/s$ . Thus, for moderate speeds  $\xi$  can be considered perpendicular to  $\eta$ . This approximation is valid also for forward-looking measurements.

## 4 The measurement error model

In Figure 10 some typical results from scans over objects are shown. The first case (slope) may be a scan of a piece of terrain or a part of a man-made object (vehicle, building etc.). The second case (step) and the third case (combination of slope and step) are typical results of scans over man-made objects.

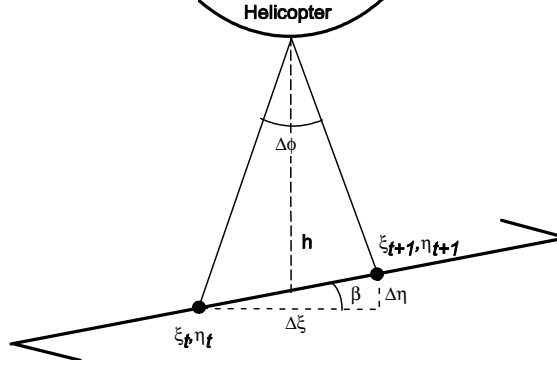


Figure 9: The distance between two consecutive samples.

Let us derive a model for a scan over a slope (case I in Figure 10). In each sample  $m$  we retrieve

$$\begin{aligned}\xi_m &= \xi^0 + e_\xi \\ z_m &= z^0 + e_z,\end{aligned}$$

where  $(\xi_m, z_m)$  is the measured coordinate,  $(\xi^0, z^0)$  is the unobservable, true coordinate and  $(e_\xi, e_z)$  is the noise in  $(\xi, z)$ , respectively.  $e_\xi$  and  $e_z$  are assumed to be independently and identically distributed (i.i.d.) with zero mean and variance  $\sigma_{e_\xi}^2$  and  $\sigma_{e_z}^2$ , respectively. Note that we have error in both coordinates and thus, we have a measurement error (ME) regression problem. For the estimation of a slope we have the following system

$$\mathcal{S} : n_1 \xi^0 + n_2 z^0 + c = 0 \quad (13a)$$

$$n_1^2 + n_2^2 = 1 \quad (13b)$$

and the model for instant  $i$  is

$$\mathcal{M} : n_1 (\xi_{m,i} + e_{\xi,i}) + n_2 (z_{m,i} + e_{z,i}) + c = 0. \quad (14)$$

The (total) estimation error for instant  $i$  can now be defined as

$$\varepsilon_i = n_1 e_{\xi,i} + n_2 e_{z,i} = n_1 \xi_{m,i} + n_2 z_{m,i} + c. \quad (15)$$

Including the constraint (13b), the parameters that will be estimated are

$$\theta = (n_1, c)^T.$$

The goal is to estimate the parameters with as small uncertainty as possible. The estimated parameters are denoted  $\hat{\theta} = (\hat{n}_1, \hat{c})^T$  and the true parameters of the system are denoted  $\theta^0 = (n_1^0, c^0)^T$ . Using model (14), the equations for the shapes in Figure 10 can be written as in Tables 2-4.



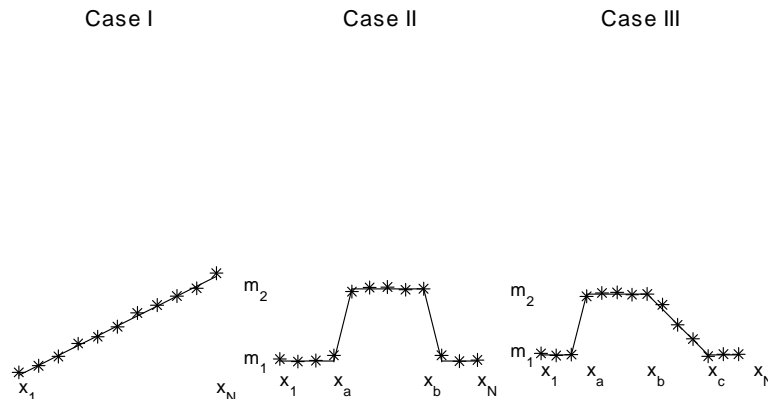


Figure 10: Three typical scans over some arbitrary objects. Case I: slope, Case II: step, case III: combination of step and slope.

	Slope
Linear error model ( $e_z = 0$ )	$n_1 (\xi_m - e_\xi) + n_2 z_m + c = 0$ $\xi_1 \leq \xi \leq \xi_N$
Measurement error model	$n_1 (\xi_m - e_\xi) + n_2 (z_m - e_z) + c = 0$ $\xi_1 \leq \xi \leq \xi_N$

Table 2: Linear regression and measurement error models for a slope.

## 5 The CRLB for measurement error regression

In this section we derive the Cramer-Rao lower bound (CRLB) of the used ME regression method, for a scan over a slope, see Figure 10 and Table 2. The CRLB gives us a quality measure of the method. The Cramér-Rao Lower Bound describes the lower limit achievable for an optimal estimator from a measurement vector  $x = (x_1, x_2, \dots, x_N)$ . The optimality is valid if the estimator is defined as minimum variance and unbiased [11, 12]. The derivation follows [3]. Numerical illustrations are shown in Section 5.2 and validations in Section 5.3.

### 5.1 The CRLB expression

Consider the measurement error model (14). The total error for instant  $i$  is defined in (15) as

$$\varepsilon_i = n_1 e_{\xi,i} + n_2 e_{z,i} = n_1 \xi_i + n_2 z_i + c$$

	Step
Linear error model ( $e_z = 0$ )	$n_2 z_m + c_1 = 0 \quad \xi_1 \leq \xi_m - e_\xi \leq \xi_a$
	$n_2 z_m + c_2 = 0 \quad \xi_a \leq \xi_m - e_\xi \leq \xi_b$
	$n_2 z_m + c_3 = 0 \quad \xi_b \leq \xi_m - e_\xi \leq \xi_N$
Measurement error model	$n_2 (z_m - e_z) + c_1 = 0 \quad \xi_1 \leq \xi_m - e_\xi \leq \xi_a$
	$n_2 (z_m - e_z) + c_2 = 0 \quad \xi_a \leq \xi_m - e_\xi \leq \xi_b$
	$n_2 (z_m - e_z) + c_3 = 0 \quad \xi_b \leq \xi_m - e_\xi \leq \xi_N$

Table 3: Linear regression and measurement error models for a step.

	Combination
Linear error model ( $e_z = 0$ )	$n_2 z_m + c_1 = 0 \quad \xi_1 \leq \xi_m - e_\xi \leq \xi_a$
	$n_2 z_m + c_2 = 0 \quad \xi_a \leq \xi_m - e_\xi \leq \xi_b$
	$n_1 \xi_m + n_2 z_m + c = 0 \quad \xi_b \leq \xi_m - e_\xi \leq \xi_c$
	$n_2 z_m + c_1 = 0 \quad \xi_c \leq \xi_m - e_\xi \leq \xi_N$
Measurement error model	$n_2 (z_m - e_z) + c_1 = 0 \quad \xi_1 \leq \xi_m - e_\xi \leq \xi_a$
	$n_2 (z_m - e_z) + c_2 = 0 \quad \xi_a \leq \xi_m - e_\xi \leq \xi_b$
	$n_1 (\xi_m - e_\xi) + n_2 (z_m - e_z) + c = 0 \quad \xi_b \leq \xi_m - e_\xi \leq \xi_c$
	$n_2 (z_m - e_z) + c_1 = 0 \quad \xi_c \leq \xi_m - e_\xi \leq \xi_N$

Table 4: Linear regression and measurement error model for a combination of a slope and a step.

where  $\varepsilon_i, i = 1, \dots, N$ , is normal distributed with zero mean and variance

$$\sigma_\varepsilon^2 = n_1^2 \sigma_{e_\xi}^2 + n_2^2 \sigma_{e_z}^2.$$

Including the constraint  $n_1^2 + n_2^2 = 1$  we get

$$\varepsilon_i = n_1 e_{\xi,i} + \sqrt{1 - n_1^2} e_{z,i} = n_1 \xi_i + \sqrt{1 - n_1^2} z_i + c,$$

and

$$\sigma_\varepsilon^2 = n_1^2 \sigma_{e_\xi}^2 + (1 - n_1^2) \sigma_{e_z}^2.$$

The joint conditional density for  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)$  is [11, 12]

$$f(\varepsilon | \theta) = \frac{1}{(2\pi)^{N/2} \sigma_\varepsilon^N} \exp\left(-\frac{1}{2\sigma_\varepsilon^2} \sum_{i=1}^N \varepsilon_i^2\right)$$

and the logarithm is

$$\bar{f}(\varepsilon | \theta) = \log f(\varepsilon | \theta) = -\frac{N}{2} \log(2\pi) - N \log \sigma_\varepsilon - \frac{1}{2\sigma_\varepsilon^2} \sum_{i=1}^N \varepsilon_i^2,$$

or explicitly,

$$\begin{aligned} \bar{f}(\varepsilon | \theta) &= \log f(\varepsilon | \theta) = -\frac{N}{2} \log(2\pi) - N \log \sqrt{n_1^2 (\sigma_{e_\xi}^2 - \sigma_{e_z}^2) + \sigma_{e_z}^2} \\ &\quad - \frac{1}{2(n_1^2 (\sigma_{e_\xi}^2 - \sigma_{e_z}^2) + \sigma_{e_z}^2)} \sum_{i=1}^N \left( n_1 \xi_i + \sqrt{1 - n_1^2} z_i + c \right)^2. \end{aligned}$$

The Fisher matrix is defined as

$$J = E \left( \frac{\partial}{\partial \theta} \bar{f}(\varepsilon | \theta) \left[ \frac{\partial}{\partial \theta} \bar{f}(\varepsilon | \theta) \right]^T \right), \quad (16)$$

which is a rather messy expression, see Appendix A.2. The CRLB is given by

$$E \left( (\theta - \hat{\theta}) (\theta - \hat{\theta})^T \right) \geq J^{-1},$$

where

$$J = \frac{N}{\sigma_\varepsilon^2} \begin{pmatrix} 2n_1^2 \frac{(\sigma_{e_\xi}^2 - \sigma_{e_z}^2)^2}{\sigma_\varepsilon^2} + \frac{1}{N} \sum_{i=1}^N \left( \xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i \right)^2 & \frac{1}{N} \sum_{i=1}^N \left( \xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i \right) \\ \frac{1}{N} \sum_{i=1}^N \left( \xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i \right) & 1 \end{pmatrix}. \quad (17)$$

The inverse of such a matrix is

$$J^{-1} = \begin{pmatrix} a & b \\ b & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a-b^2} & -\frac{b}{a-b^2} \\ -\frac{b}{a-b^2} & \frac{a}{a-b^2} \end{pmatrix}.$$

The variance of  $\theta$  will then be

$$\begin{aligned} \text{Var}(n_1^0 - \hat{n}_1) &= \frac{\sigma_\varepsilon^2}{N} \frac{1}{\det(J)} \\ \text{Var}(c^0 - \hat{c}) &= \frac{\sigma_\varepsilon^2}{N} \frac{1}{\det(J)} \left( 2n_1^2 \frac{(\sigma_{e_\xi}^2 - \sigma_{e_z}^2)^2}{\sigma_\varepsilon^2} + \frac{1}{N} \sum_{i=1}^N \left( \xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i \right)^2 \right) \\ \det(J) &= 2n_1^2 \frac{(\sigma_{e_\xi}^2 - \sigma_{e_z}^2)^2}{\sigma_\varepsilon^2} + \frac{1}{N} \sum_{i=1}^N \left( \xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i \right)^2 - \left( \frac{1}{N} \sum_{i=1}^N \left( \xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i \right) \right)^2. \end{aligned}$$

For large  $N$  we can approximate  $\frac{1}{N} \sum_{i=1}^N \xi_i \approx \bar{E}(\xi)$  and  $\frac{1}{N} \sum_{i=1}^N \xi_i^2 \approx \bar{E}(\xi^2)$ . Further,  $\overline{Var}(\xi) = \bar{E}(\xi^2) - \bar{E}^2(\xi)$ . We now have

$$Var(n_1^0 - \hat{n}_1) = \frac{\sigma_\varepsilon^2}{N} \frac{1}{\det(J)} \quad (18)$$

$$Var(c^0 - \hat{c}) = \frac{\sigma_\varepsilon^2}{N} \frac{1}{\det(J)} \left( 2n_1^2 \frac{(\sigma_{e_\xi}^2 - \sigma_{e_z}^2)^2}{\sigma_\varepsilon^2} + \bar{E} \left( \xi - \frac{n_1}{\sqrt{1-n_1^2}} z \right)^2 \right) \quad (19)$$

$$\det(J) = 2n_1^2 \frac{(\sigma_{e_\xi}^2 - \sigma_{e_z}^2)^2}{\sigma_\varepsilon^2} + \overline{Var} \left( \xi - \frac{n_1}{\sqrt{1-n_1^2}} z \right) \quad (20)$$

$$\sigma_\varepsilon^2 = n_1^2 \sigma_{e_\xi}^2 + (1 - n_1^2) \sigma_{e_z}^2. \quad (21)$$

## 5.2 Numerical illustrations

The evaluations are performed in Matlab. Default values of the parameters are  $N = 100$ ,  $n_1 = 0.5$ ,  $n_2 = \sqrt{1 - n_1^2} \approx 0.87$  and  $c = 0$ , respectively, and  $\xi = -5 : 1/(N-1) : 5$ . From this,  $z$  is calculated. Random errors, normal distributed with zero mean and variance  $\sigma_{e_\xi}^2 = \sigma_{e_z}^2$  are added to the coordinates  $\xi$  and  $z$ , respectively. Default values of the variances are  $\sigma_{e_\xi}^2 = \sigma_{e_z}^2 = 0.01$ . Note the the constraint  $n_1^2 + n_2^2 = 1$  must be fulfilled to get a unique and unambiguous model. During a evaluation, the parameters were tested one at a time and afterwards they were changed back to their default value<sup>6</sup>.

The resulting CRLB function when  $N \rightarrow \infty$  is shown in Figure 11. The larger number of sample, the lower CRLB values. The resulting CRLB functions when  $\sigma_{e_\xi}^2 \rightarrow 1$  and  $\sigma_{e_z}^2 \rightarrow 1$  are shown in Figure 12 and 13, respectively. The larger noise variance, the larger CRLB value. The graphs for  $Var(c^0 - \hat{c})$  are quite similar when  $\sigma_{e_z}^2 \rightarrow 1$  compared to when  $\sigma_{e_\xi}^2 \rightarrow 1$ .

In Figure 14 CRLB as a function of  $n_1$  when  $\sigma_{e_z}^2 = k\sigma_{e_\xi}^2$  is shown. It is quite interesting that the CRLB function depends on the values of  $\sigma_{e_\xi}^2$  and  $\sigma_{e_z}^2$ . When  $k$  is large,  $\sigma_{e_\xi}^2$  is small and we have the linear regression case. When  $k$  is large and  $n_1 = \pm 1$ , we have a noise free system, which gives the "dips" in the CRLB graph.

## 5.3 Validation

The performance of the estimation method will be investigated in some Monte Carlo simulations. The performance will be evaluated in terms of correctness in estimates of  $\theta = (n_1, c)^T$ .

<sup>6</sup>File: ~stina/crb/crb\_simulation.m

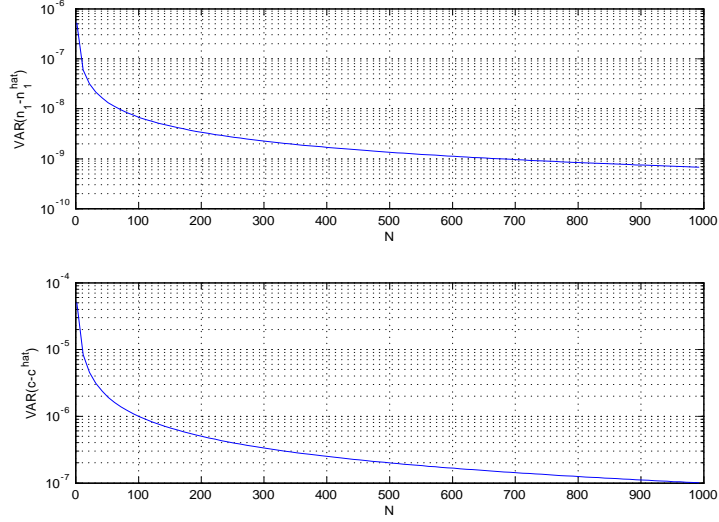


Figure 11: CRLB as a function of the number of samples,  $N$ . ME regression model.

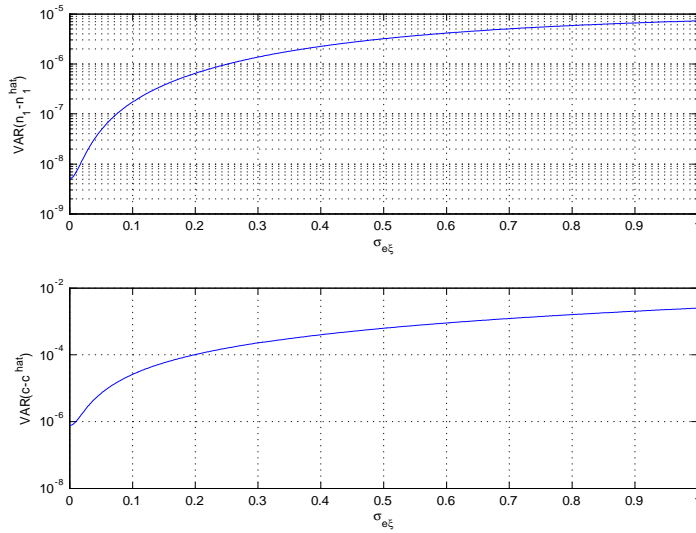


Figure 12: CRLB as a function of  $\sigma_{e_\xi}$ , ME regression model.

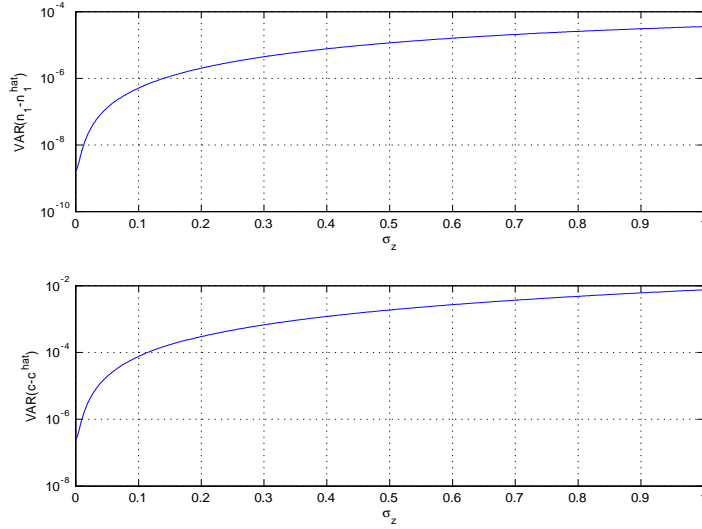


Figure 13: CRLB as a function of  $\sigma_{e_z}$ , ME regression model.

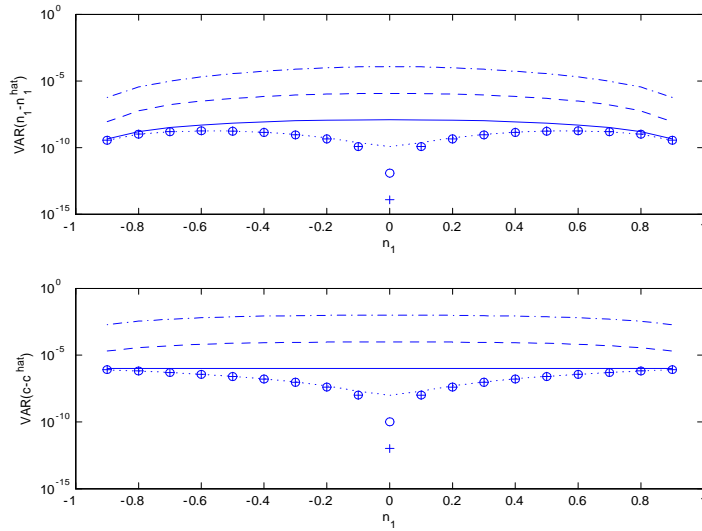


Figure 14: CRLB as a function of  $n_1$  when  $\sigma_{e_z} = k\sigma_{e_\xi}$ , ME regression model. Plus:  $k=0.001$ , circles:  $k=0.01$ , dots:  $k=0.1$ , solid:  $k=1$ , dashed:  $k=10$ , dash-dotted:  $k=100$ .

We start with an exact (error free) description of a straight line, see (13a). True values of  $n_1, n_2$  are  $n_1 = 0.5$  and  $n_2 = \sqrt{1 - n_1^2} \approx 0.87$ , respectively, and  $\xi = -5 : 1/(N - 1) : 5$ . From this,  $z$  is calculated. Random errors, normal distributed with zero mean and variance  $\sigma_{e_\xi}^2 = \sigma_{e_z}^2$  are added to the coordinates  $\xi$  and  $z$ , respectively. The noise is generated separately for the  $\xi$  and  $z$  coordinates. Then the parameters are estimated using TLS [4, 9] on the perturbed data set. The simulations are repeated for increasing number of samples  $N$ . Further, the simulations are repeated for  $\sigma_{e_\xi}^2 = \sigma_{e_z}^2 = 0.01$ ,  $\sigma_{e_\xi}^2 = \sigma_{e_z}^2 = 1$  and for both cases with  $c = 0$  and  $c = 100$ .

The statistical properties of the estimates are studied by the mean square error (MSE), defined as (see [9], page 244)

$$MSE(\hat{\theta}) = E \left\{ (\hat{\theta} - \theta_0)^T (\hat{\theta} - \theta_0) \right\} \quad (22)$$

and estimated by

$$MSE(\hat{\theta}) = \frac{1}{J} \sum_{j=1}^J (\hat{\theta} - \theta_0)^T (\hat{\theta} - \theta_0).$$

The MSE is averaged over 100 sets (i.e.,  $J = 100$ ). The CRLB (theoretical limit) is plotted together with the MSE for each case. The test was performed in Matlab<sup>7</sup>.

### 5.3.1 What can we expect in the validation?

Let us study (18)-(19) under the circumstances that we have in the validation. First, when  $\sigma_{e_\xi}^2 = \sigma_{e_z}^2$  we have

$$Var(n_1^0 - \hat{n}_1) = \frac{\sigma_\varepsilon^2}{N} \frac{1}{\det(J)} \quad (23)$$

$$Var(c^0 - \hat{c}) = \frac{\sigma_\varepsilon^2}{N} \frac{1}{\det(J)} \overline{E} \left( \xi - \frac{n_1}{\sqrt{1 - n_1^2}} z \right)^2 \quad (24)$$

$$\det(J) = \overline{Var} \left( \xi - \frac{n_1}{\sqrt{1 - n_1^2}} z \right) \quad (25)$$

$$\sigma_\varepsilon^2 = \sigma_{e_z}^2. \quad (26)$$

Let us study this expression a bit further. Inserting

$$z = -\frac{n_1}{n_2} \xi - \frac{c}{n_2}$$

in the expression of the determinant gives

$$\det(J) = \overline{Var} \left( \xi \left( 1 + \frac{n_1^2}{n_2^2} \right) + \frac{n_1}{n_2} c \right) = \left( 1 + \frac{n_1^2}{n_2^2} \right)^2 \overline{Var}(\xi) = \frac{1}{n_2^4} \overline{Var}(\xi),$$

<sup>7</sup>File: ~stina/crb/crb\_validation.m

where  $\xi$  is a variable and  $c$  is a constant. Inserting  $z = -\frac{n_1}{n_2}\xi - \frac{c}{n_2}$  in  $\overline{E}\left(\xi - \frac{n_1}{\sqrt{1-n_1^2}}z\right)^2$  we get

$$\begin{aligned}\overline{E}\left(\xi - \frac{n_1}{\sqrt{1-n_1^2}}z\right)^2 &= \overline{Var}\left(\xi - \frac{n_1}{n_2}z\right) + \left(\overline{E}\left(\xi - \frac{n_1}{n_2}z\right)\right)^2 \\ &= \frac{1}{n_2^4}\overline{Var}(\xi) + \left(\overline{E}(\xi) - \frac{n_1}{n_2}\overline{E}(z)\right)^2 \\ &= \frac{1}{n_2^4}\overline{Var}(\xi) + \left(\overline{E}(\xi) + \frac{n_1^2}{n_2^2}\overline{E}(\xi) + \frac{n_1}{n_2}c\right)^2 \\ &= \frac{1}{n_2^4}\overline{Var}(\xi) + \left(\frac{1}{n_2^2}\overline{E}(\xi) + \frac{n_1}{n_2}c\right)^2.\end{aligned}$$

Inserting that  $\overline{E}(\xi_i) = 0$  in this validation, we now have

$$Var(n_1^0 - \hat{n}_1) = \frac{\sigma_{e_z}^2}{N} \frac{n_2^4}{\overline{Var}(\xi)} \quad (27)$$

$$Var(c^0 - \hat{c}) = \frac{\sigma_{e_z}^2}{N} \left(1 + \frac{(n_1 c)^2}{\overline{Var}(\xi)}\right) \quad (28)$$

$$\det(J) = \frac{1}{n_2^4} \overline{Var}(\xi). \quad (29)$$

From this we get that  $Var(n_1^0 - \hat{n}_1) \sim \frac{1}{N}$ ,  $Var(n_1^0 - \hat{n}_1) \sim \sigma_{e_z}^2$  and that  $Var(n_1^0 - \hat{n}_1)$  does not depend on the value of  $c$ . Further,  $Var(n_1^0 - \hat{n}_1) \sim \frac{1}{\overline{Var}(\xi)}$ , so we will have a smaller estimation error in  $n_1$  when  $\overline{Var}(\xi)$  is large. Concerning  $\sigma_{e_z}^2$ ,  $Var(n_1^0 - \hat{n}_1)$  will have a 100 times higher value when  $\sigma_{e_z}^2 = 1$  than when  $\sigma_{e_z}^2 = 0.01$ .

We also see that  $Var(c^0 - \hat{c}) \sim \frac{1}{N}$ ,  $Var(c^0 - \hat{c}) \sim \sigma_{e_z}^2$  and  $Var(c^0 - \hat{c}) \sim c^2$ . We see that  $Var(c^0 - \hat{c})$  too will have a 100 times higher value when  $\sigma_{e_z}^2 = 1$  than when  $\sigma_{e_z}^2 = 0.01$ . Concerning dependencies on  $\overline{Var}(\xi)$  and  $c$ , when  $c = 0$  or  $\overline{Var}(\xi) \ll (n_1 c)^2$ ,  $Var(c^0 - \hat{c})$  will have a  $10^4$  times higher value when  $c = 100$  than when  $c = 1$ . When  $c = 0$  or  $\overline{Var}(\xi) \gg (n_1 c)^2$   $Var(c^0 - \hat{c})$  will be independent on  $\overline{Var}(\xi)$ .

### 5.3.2 Results

The figures describing the test cases are listed in Table 5. Let us first study case I (Figure 15). We can see that the CRLB and MSE graphs follow each other, this means that the method performs as good as possible for this test case. If we compare case I (Figure 15) and case II (Figure 16), we see that the graphs in case II is 100 times higher. This comes from the fact that  $\sigma_{e_\xi}^2$  and  $\sigma_{e_z}^2$  are 100 times larger than in case I.



	$c = 0$	$c = 100$
$\sigma_{e_\xi} = \sigma_{e_z} = 0.1$	Case I: Figure 15	Case III: Figure 17
$\sigma_{e_\xi} = \sigma_{e_z} = 1$	Case II: Figure 16	Case IV: Figure 18

Table 5: List of figures for ME validation.

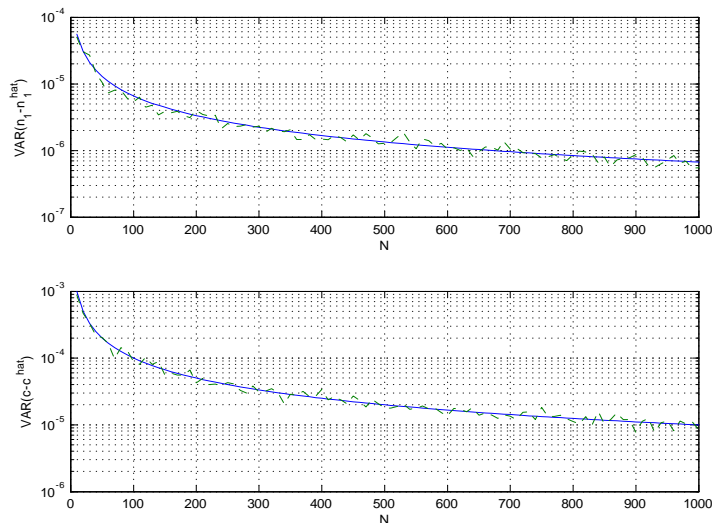


Figure 15: Case I: ME regression,  $\sigma_{e_\xi} = \sigma_{e_z} = 0.1, c = 0$ . Solid line: CRLB, dashed: MSE from Monte Carlo validation.

If we compare case I and case III (Figure 17), we see that  $Var(n_1^0 - \hat{n}_1)$  is not effected by that  $c = 100$ . This is in order, as  $Var(n_1^0 - \hat{n}_1)$  is independent of  $c$ , see (27). We also see that the graph of  $Var(c^0 - \hat{c})$  is about  $10^4$  times higher than in case I, this is because  $c^2 = 10^4$ . For both  $Var(n_1^0 - \hat{n}_1)$  and  $Var(c^0 - \hat{c})$  the CRLB and MSE graphs follow each other, this means that the method performs as good as possible for this test case.

Comparing case I and case IV (Figure 18), we see that the graph of  $Var(n_1^0 - \hat{n}_1)$  is about  $10^2$  times higher than in case I, this is because  $\sigma_{e_\xi}^2$  and  $\sigma_{e_z}^2$  is 100 times larger than in case I. The graph of  $Var(c^0 - \hat{c})$  is about  $10^6$  times higher than in case I, this is because  $\sigma_{e_z}^2 c^2$  is  $10^6$  times larger than in case I.

For all cases the MSE values from the Monte Carlo simulations follow the theoretical CRLB expressions. This means that this estimation method has the possibility, under good circumstances, to return the best estimates possible.

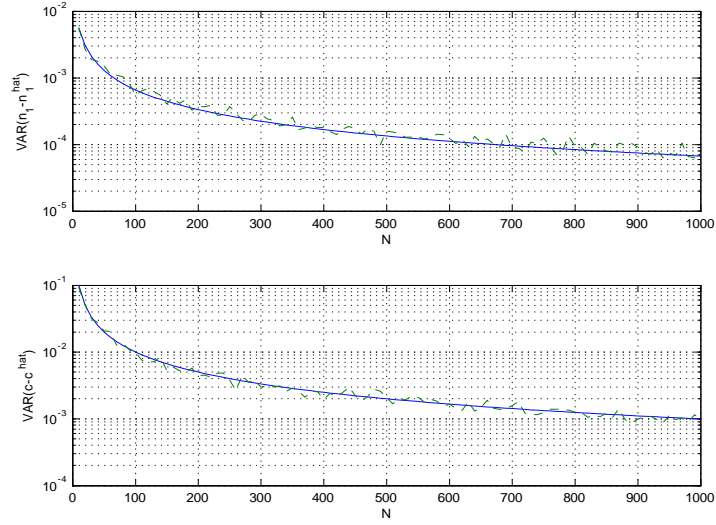


Figure 16: Case II: ME regression,  $\sigma_{e_\xi} = \sigma_{e_z} = 1, c = 0$ . Solid line: CRLB, dashed: MSE from Monte Carlo validation.

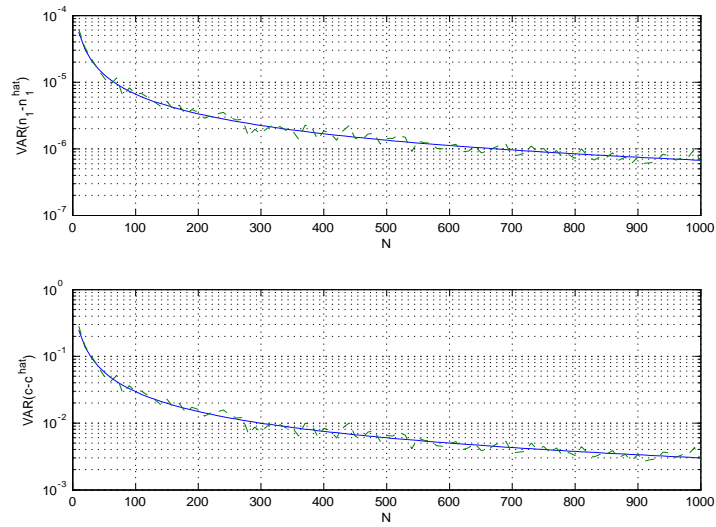


Figure 17: Case III: ME regression,  $\sigma_{e_\xi} = \sigma_{e_z} = 0.1, c = 100$ . Solid line: CRLB, dashed: MSE from Monte Carlo validation.

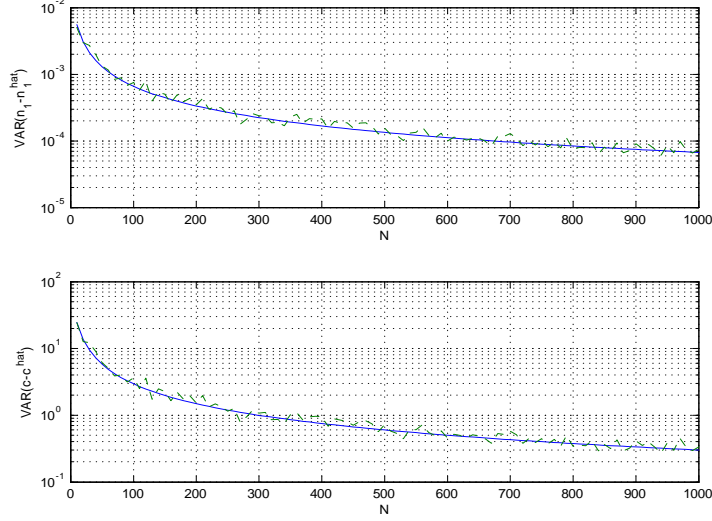


Figure 18: Case IV: ME regression,  $\sigma_{e_\xi} = \sigma_{e_z} = 1$ ,  $c = 100$ . Solid line: CRLB, dashed: MSE from Monte Carlo validation.

## 6 CRLB as a function of system parameters

Let us repeat the CRLB expression:

$$\text{Var} (n_1^0 - \hat{n}_1) = \frac{\sigma_\varepsilon^2}{N} \frac{1}{\det(J)} \quad (30)$$

$$\text{Var} (c^0 - \hat{c}) = \frac{\sigma_\varepsilon^2}{N} \frac{1}{\det(J)} \left( 2n_1^2 \frac{(\sigma_{e_\xi}^2 - \sigma_{e_z}^2)^2}{\sigma_\varepsilon^2} + \overline{E} \left( \xi - \frac{n_1}{\sqrt{1-n_1^2}} z \right)^2 \right) \quad (31)$$

$$\det(J) = 2n_1^2 \frac{(\sigma_{e_\xi}^2 - \sigma_{e_z}^2)^2}{\sigma_\varepsilon^2} + \overline{\text{Var}} \left( \xi - \frac{n_1}{\sqrt{1-n_1^2}} z \right) \quad (32)$$

$$\sigma_\varepsilon^2 = n_1^2 \sigma_{e_\xi}^2 + (1 - n_1^2) \sigma_{e_z}^2. \quad (33)$$

### 6.1 1D case

For the 1D case the expressions of  $\xi$  and  $z$  are (see Section 2.1):

$$\xi = \hat{R} \sin(\alpha),$$

$$z = \hat{R} \cos(\alpha),$$

where  $\hat{R}$  is the slant range (from laser to center of laser beam on target) estimated by the detector. Inserting these expressions in the CRLB equations we get

$$\begin{aligned} \text{Var}(n_1^0 - \hat{n}_1) &= \frac{\sigma_\varepsilon^2}{N} \frac{1}{\det(J)}, \\ \text{Var}(c^0 - \hat{c}) &= \frac{\sigma_\varepsilon^2}{N} \frac{1}{\det(J)} \left\{ 2n_1^2 \frac{(\sigma_{e_\xi}^2 - \sigma_{e_z}^2)^2}{\sigma_\varepsilon^2} + \bar{E} \left( \hat{R}^2 \left( \sin(\alpha) - \frac{n_1}{\sqrt{1-n_1^2}} \cos(\alpha) \right)^2 \right) \right\}, \\ \det(J) &= 2n_1^2 \frac{(\sigma_{e_\xi}^2 - \sigma_{e_z}^2)^2}{\sigma_\varepsilon^2} + \overline{\text{Var}} \left\{ \hat{R} \left( \sin(\alpha) - \frac{n_1}{\sqrt{1-n_1^2}} \cos(\alpha) \right) \right\}, \\ \sigma_\varepsilon^2 &= n_1^2 \sigma_{e_\xi}^2 + (1 - n_1^2) \sigma_{e_z}^2. \end{aligned}$$

Using GUM [10], expressions of  $\sigma_{e_\xi}^2$  and  $\sigma_{e_z}^2$  can be found by

$$\begin{aligned} \xi &= f_\xi(\hat{R}, \alpha), \quad z = f_z(\hat{R}, \alpha) \\ \sigma_{e_\xi}^2 &= \left( \frac{\partial}{\partial \hat{R}} f_\xi(\hat{R}, \alpha) \right)^2 \sigma_{e_{\hat{R}}}^2 + \left( \frac{\partial}{\partial \alpha} f_\xi(\hat{R}, \alpha) \right)^2 \sigma_{e_\alpha}^2, \\ \sigma_{e_z}^2 &= \left( \frac{\partial}{\partial \hat{R}} f_z(\hat{R}, \alpha) \right)^2 \sigma_{e_{\hat{R}}}^2 + \left( \frac{\partial}{\partial \alpha} f_z(\hat{R}, \alpha) \right)^2 \sigma_{e_\alpha}^2, \end{aligned}$$

or explicitly,

$$\begin{aligned} \sigma_{e_\xi}^2 &= \sin^2(\alpha) \sigma_{e_{\hat{R}}}^2 + \hat{R}^2 \cos^2(\alpha) \sigma_{e_\alpha}^2 \\ \sigma_{e_z}^2 &= \cos^2(\alpha) \sigma_{e_{\hat{R}}}^2 + \hat{R}^2 \sin^2(\alpha) \sigma_{e_\alpha}^2, \end{aligned}$$

where  $\sigma_{e_\alpha}^2$  is the variance of the error in  $\alpha$ ,  $\sigma_{e_{\hat{R}}}^2$  is the variance in the error in  $\hat{R}$ ,  $\hat{R} = R^0 + e_R$ . Thus, we have the total error variance

$$\begin{aligned} \sigma_\varepsilon^2 &= n_1^2 \left( \sin^2(\alpha) \sigma_{e_{\hat{R}}}^2 + \hat{R}^2 \cos^2(\alpha) \sigma_{e_\alpha}^2 \right) + (1 - n_1^2) \left( \cos^2(\alpha) \sigma_{e_{\hat{R}}}^2 + \hat{R}^2 \sin^2(\alpha) \sigma_{e_\alpha}^2 \right) \\ &= (n_1^2 \sin^2(\alpha) + (1 - n_1^2) \cos^2(\alpha)) \sigma_{e_{\hat{R}}}^2 + (n_1^2 \cos^2(\alpha) + (1 - n_1^2) \sin^2(\alpha)) \hat{R}^2 \sigma_{e_\alpha}^2. \end{aligned}$$

## 6.2 2D case

For a flat 2D surface the expressions for  $\xi$  and  $z$  are (see Section 2.2):

$$\xi = \hat{R}(\xi_0, \eta_0) \frac{\tan(\alpha_{scan})}{\sqrt{1 + \tan^2(\alpha_{scan}) + \tan^2(\alpha_{pitch})}} \quad (34)$$

$$z = \hat{R}(\xi_0, \eta_0) \frac{1}{\sqrt{1 + \tan^2(\alpha_{scan}) + \tan^2(\alpha_{pitch})}}, \quad (35)$$

where  $\hat{R}(\xi_0, \eta_0)$  is the slant range estimated by the detector. The variance of the errors in  $(\xi, z)$  measurements,  $(\sigma_{e_\xi}, \sigma_{e_z})$  can be expressed as functions of  $\hat{R}(\xi_0, \eta_0)$ ,  $\alpha_{scan}$  and  $\alpha_{pitch}$  using GUM [10]. More detailed calculations of the measurement uncertainties in the TopEye system are performed in [1].

Let us study the variance expression in  $\det(J)$ :

$$\overline{Var} \left( \xi - \frac{n_1}{\sqrt{1-n_1^2}} z \right) = \overline{Var} \left( \frac{\hat{R}(\xi_0, \eta_0)}{\sqrt{1+\tan^2(\alpha_{scan})+\tan^2(\alpha_{pitch})}} \left( \tan(\alpha_{scan}) - \frac{n_1}{\sqrt{1-n_1^2}} \right) \right).$$

Using GUM [10], expressions of  $\sigma_{e_\xi}^2$  and  $\sigma_{e_z}^2$  can be found by

$$\begin{aligned} \xi &= f_\xi(\hat{R}, \alpha_{scan}, \alpha_{pitch}), & z &= f_z(\hat{R}, \alpha_{scan}, \alpha_{pitch}), \\ \sigma_{e_\xi}^2 &= \left( \frac{\partial}{\partial \hat{R}} f_\xi(\hat{R}, \alpha_{scan}, \alpha_{pitch}) \right)^2 \sigma_{e_{\hat{R}}}^2 + \left( \frac{\partial}{\partial \alpha_{scan}} f_\xi(\hat{R}, \alpha_{scan}, \alpha_{pitch}) \right)^2 \sigma_{e_{\alpha_{scan}}}^2 \\ &\quad + \left( \frac{\partial}{\partial \alpha_{pitch}} f_\xi(\hat{R}, \alpha_{scan}, \alpha_{pitch}) \right)^2 \sigma_{e_{\alpha_{pitch}}}^2, \\ \sigma_{e_z}^2 &= \left( \frac{\partial}{\partial \hat{R}} f_z(\hat{R}, \alpha_{scan}, \alpha_{pitch}) \right)^2 \sigma_{e_{\hat{R}}}^2 + \left( \frac{\partial}{\partial \alpha_{scan}} f_z(\hat{R}, \alpha_{scan}, \alpha_{pitch}) \right)^2 \sigma_{e_{\alpha_{scan}}}^2 \\ &\quad + \left( \frac{\partial}{\partial \alpha_{pitch}} f_z(\hat{R}, \alpha_{scan}, \alpha_{pitch}) \right)^2 \sigma_{e_{\alpha_{pitch}}}^2, \end{aligned}$$

Calculations give

$$\begin{aligned} \sigma_{e_\xi}^2 &= \sigma_{e_{\hat{R}}}^2 + \\ &\quad \left\{ \frac{1+\tan^2(\alpha_{scan})}{\sqrt{1+\tan^2(\alpha_{scan})+\tan^2(\alpha_{pitch})}} - \frac{\tan^2(\alpha_{scan})(1+\tan^2(\alpha_{scan}))}{(1+\tan^2(\alpha_{scan})+\tan^2(\alpha_{pitch}))^{3/2}} \right\}^2 \sigma_{e_{\alpha_{scan}}}^2 \\ &\quad + \left\{ -\frac{\tan(\alpha_{scan})\tan(\alpha_{pitch})(1+\tan^2(\alpha_{pitch}))}{(1+\tan^2(\alpha_{scan})+\tan^2(\alpha_{pitch}))^{3/2}} \right\}^2 \sigma_{e_{\alpha_{pitch}}}^2 \end{aligned}$$

and

$$\begin{aligned} \sigma_{e_z}^2 &= \sigma_{e_{\hat{R}}}^2 + \left\{ -\frac{\tan(\alpha_{scan})(1+\tan^2(\alpha_{scan}))}{(1+\tan^2(\alpha_{scan})+\tan^2(\alpha_{pitch}))^{3/2}} \right\}^2 \sigma_{e_{\alpha_{scan}}}^2 \\ &\quad + \left\{ -\frac{\tan(\alpha_{pitch})(1+\tan^2(\alpha_{pitch}))}{(1+\tan^2(\alpha_{scan})+\tan^2(\alpha_{pitch}))^{3/2}} \right\}^2 \sigma_{e_{\alpha_{pitch}}}^2. \end{aligned}$$

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## A Calculation of CRLB

### A.1 Various calculations

Some calculations that have been checked in Maple:

$$\begin{aligned}
& \frac{d}{dn_1} \left( -N \log \sqrt{n_1^2 (\sigma_{e_\xi}^2 - \sigma_{e_z}^2) + \sigma_{e_z}^2} \right) \\
& : -N n_1 \frac{\sigma_{e_\xi}^2 - \sigma_{e_z}^2}{n_1^2 \sigma_{e_\xi}^2 - n_1^2 \sigma_{e_z}^2 + \sigma_{e_z}^2} \\
& \frac{d}{dn_1} \left( -\frac{1}{2 (n_1^2 (\sigma_{e_\xi}^2 - \sigma_{e_z}^2) + \sigma_{e_z}^2)} \right) \\
& : n_1 \frac{\sigma_{e_\xi}^2 - \sigma_{e_z}^2}{(n_1^2 \sigma_{e_\xi}^2 - n_1^2 \sigma_{e_z}^2 + \sigma_{e_z}^2)^2} \\
& \frac{d}{dn_1} \left( n_1 \xi_i + \sqrt{1 - n_1^2} z_i + c \right)^2 \\
& : -2 \left( n_1 \xi_i + \sqrt{(-n_1^2 + 1)} z_i + c \right) \frac{-\xi_i \sqrt{(-n_1^2 + 1)} + z_i n_1}{\sqrt{(-n_1^2 + 1)}} \\
& \frac{d}{dc} \left( n_1 \xi_i + \sqrt{1 - n_1^2} z_i + c \right)^2 \\
& : 2 n_1 \xi_i + 2 \sqrt{(-n_1^2 + 1)} z_i + 2c \\
& \frac{d}{dn_1} \left( n_1 \xi_i + \sqrt{1 - n_1^2} z_i + c \right) \\
& : -\frac{-\xi_i \sqrt{(-n_1^2 + 1)} + z_i n_1}{\sqrt{(-n_1^2 + 1)}} \\
& \frac{d}{dc} \left( n_1 \xi_i + \sqrt{1 - n_1^2} z_i + c \right) \\
& : 1
\end{aligned}$$

### A.2 Calculation of the Fisher information matrix (J)

The Fisher information matrix is (see (16))

$$J = E \left( \frac{\partial}{\partial \theta} \bar{f}(\varepsilon | \theta) \left[ \frac{\partial}{\partial \theta} \bar{f}(\varepsilon | \theta) \right]^T \right),$$

where

$$\begin{aligned}\bar{f}(\varepsilon | \theta) &= -\frac{N}{2} \log(2\pi) - N \log \sqrt{n_1^2 (\sigma_{e_\xi}^2 - \sigma_{e_z}^2) + \sigma_{e_z}^2} \\ &\quad - \frac{1}{2(n_1^2 (\sigma_{e_\xi}^2 - \sigma_{e_z}^2) + \sigma_{e_z}^2)} \sum_{i=1}^N \left( n_1 \xi_i + \sqrt{1 - n_1^2} z_i + c \right)^2, \\ \theta &= (n_1, c)^T.\end{aligned}$$

Differentiation (on  $\theta$ ) gives

$$\begin{aligned}\frac{\partial}{\partial n_1} \bar{f}(\varepsilon | \theta) &= -N n_1 \frac{\sigma_{e_\xi}^2 - \sigma_{e_z}^2}{n_1^2 \sigma_{e_\xi}^2 - n_1^2 \sigma_{e_z}^2 + \sigma_{e_z}^2} + n_1 \frac{\sigma_{e_\xi}^2 - \sigma_{e_z}^2}{(n_1^2 \sigma_{e_\xi}^2 - n_1^2 \sigma_{e_z}^2 + \sigma_{e_z}^2)^2} \sum_{i=1}^N \left( n_1 \xi_i + \sqrt{1 - n_1^2} z_i + c \right)^2 \\ &\quad + \frac{1}{n_1^2 (\sigma_{e_\xi}^2 - \sigma_{e_z}^2) + \sigma_{e_z}^2} \sum_{i=1}^N \left( n_1 \xi_i + \sqrt{1 - n_1^2} z_i + c \right) \frac{-\xi_i \sqrt{1 - n_1^2} + z_i n_1}{\sqrt{1 - n_1^2}}, \\ \frac{\partial}{\partial c} \bar{f}(\varepsilon | \theta) &= -\frac{1}{n_1^2 (\sigma_{e_\xi}^2 - \sigma_{e_z}^2) + \sigma_{e_z}^2} \sum_{i=1}^N n_1 \xi_i + \sqrt{1 - n_1^2} z_i + c,\end{aligned}$$

which can be rewritten as

$$\begin{aligned}\frac{\partial}{\partial n_1} \bar{f}(\varepsilon | \theta) &= -N n_1 \frac{\sigma_{e_\xi}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^2} + n_1 \frac{\sigma_{e_\xi}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^4} \sum_{i=1}^N \varepsilon_i^2 - \frac{1}{\sigma_\varepsilon^2} \sum_{i=1}^N \varepsilon_i \left( \xi_i - \frac{n_1}{\sqrt{1 - n_1^2}} z_i \right), \\ \frac{\partial}{\partial c} \bar{f}(\varepsilon | \theta) &= -\frac{1}{\sigma_\varepsilon^2} \sum_{i=1}^N \varepsilon_i.\end{aligned}$$

The matrix multiplication in  $J$  can be expressed as

$$\frac{\partial}{\partial \theta} \bar{f}(\varepsilon | \theta) \left[ \frac{\partial}{\partial \theta} \bar{f}(\varepsilon | \theta) \right]^T = \begin{bmatrix} \frac{\partial}{\partial n_1} \bar{f}(\varepsilon | \theta) \\ \frac{\partial}{\partial c} \bar{f}(\varepsilon | \theta) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial n_1} \bar{f}(\varepsilon | \theta) & \frac{\partial}{\partial c} \bar{f}(\varepsilon | \theta) \end{bmatrix}, \quad (36)$$



where

$$\begin{aligned}
\frac{\partial^2}{\partial n_1^2} \bar{f}(\varepsilon | \theta) &= \left( -Nn_1 \frac{\sigma_{e_\varepsilon}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^2} + n_1 \frac{\sigma_{e_\varepsilon}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^4} \sum_{i=1}^N \varepsilon_i^2 - \frac{1}{\sigma_\varepsilon^2} \sum_{i=1}^N \varepsilon_i \left( \xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i \right) \right)^2 \\
&= \left( Nn_1 \frac{\sigma_{e_\varepsilon}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^2} \right)^2 + \left( n_1 \frac{\sigma_{e_\varepsilon}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^4} \right)^2 \left( \sum_{i=1}^N \varepsilon_i^2 \right)^2 + \frac{1}{\sigma_\varepsilon^4} \left( \sum_{i=1}^N \varepsilon_i \left( \xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i \right) \right)^2 \\
&\quad - 2N \left( n_1 \frac{\sigma_{e_\varepsilon}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^2} \right)^2 \frac{1}{\sigma_\varepsilon^2} \sum_{i=1}^N \varepsilon_i^2 + 2Nn_1 \frac{\sigma_{e_\varepsilon}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^4} \sum_{i=1}^N \varepsilon_i \left( \xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i \right) \\
&\quad - 2n_1 \frac{\sigma_{e_\varepsilon}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^6} \sum_{i=1}^N \varepsilon_i^2 \sum_{i=1}^N \varepsilon_i \left( \xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i \right), \\
\frac{\partial^2}{\partial c^2} \bar{f}(\varepsilon | \theta) &= \left( -\frac{1}{\sigma_\varepsilon^2} \sum_{i=1}^N \varepsilon_i \right)^2 = \frac{1}{\sigma_\varepsilon^4} \left( \sum_{i=1}^N \varepsilon_i \right)^2,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial n_1} \frac{\partial}{\partial c} \bar{f}(\varepsilon | \theta) &= \left( -Nn_1 \frac{\sigma_{e_\varepsilon}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^2} + n_1 \frac{\sigma_{e_\varepsilon}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^4} \sum_{i=1}^N \varepsilon_i^2 - \frac{1}{\sigma_\varepsilon^2} \sum_{i=1}^N \varepsilon_i \left( \xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i \right) \right) \\
&\quad \left( -\frac{1}{\sigma_\varepsilon^2} \sum_{i=1}^N \varepsilon_i \right) \\
&= Nn_1 \frac{\sigma_{e_\varepsilon}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^4} \sum_{i=1}^N \varepsilon_i - n_1 \frac{\sigma_{e_\varepsilon}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^6} \sum_{i=1}^N \varepsilon_i^2 \sum_{i=1}^N \varepsilon_i \\
&\quad + \frac{1}{\sigma_\varepsilon^4} \sum_{i=1}^N \varepsilon_i \sum_{i=1}^N \varepsilon_i \left( \xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i \right).
\end{aligned}$$

Using the calculations of moments and variances (see Section A.3) we can

calculate the elements in the Fisher information matrix  $J$ :

$$\begin{aligned}
E\left(\frac{\partial^2}{\partial n_1^2} \bar{f}(\varepsilon | \theta)\right) &= \left(Nn_1 \frac{\sigma_{e_\xi}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^2}\right)^2 + \left(n_1 \frac{\sigma_{e_\xi}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^4}\right)^2 E\left(\left(\sum_{i=1}^N \varepsilon_i\right)^2\right) \\
&+ \frac{1}{\sigma_\varepsilon^4} E\left(\left(\sum_{i=1}^N \varepsilon_i \left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right)\right)^2\right) - 2N \left(n_1 \frac{\sigma_{e_\xi}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^2}\right)^2 \frac{1}{\sigma_\varepsilon^2} E\left(\sum_{i=1}^N \varepsilon_i^2\right) \\
&+ 2Nn_1 \frac{\sigma_{e_\xi}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^4} E\left(\sum_{i=1}^N \varepsilon_i \left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right)\right) \\
&- 2n_1 \frac{\sigma_{e_\xi}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^6} E\left(\sum_{i=1}^N \varepsilon_i^2 \sum_{i=1}^N \varepsilon_i \left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right)\right) \\
&= \left(Nn_1 \frac{\sigma_{e_\xi}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^2}\right)^2 + \left(n_1 \frac{\sigma_{e_\xi}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^4}\right)^2 (N^2 \sigma_\varepsilon^4 + 2N \sigma_\varepsilon^4) \\
&+ \frac{1}{\sigma_\varepsilon^4} \sigma_\varepsilon^2 \sum_{i=1}^N \left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right)^2 - 2N \left(n_1 \frac{\sigma_{e_\xi}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^2}\right)^2 \frac{1}{\sigma_\varepsilon^2} N \sigma_\varepsilon^2 - 0 + 0 \\
&= \left(Nn_1 \frac{\sigma_{e_\xi}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^2}\right)^2 + \left(Nn_1 \frac{\sigma_{e_\xi}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^2}\right)^2 + 2N \left(n_1 \frac{\sigma_{e_\xi}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^2}\right)^2 \\
&- 2 \left(Nn_1 \frac{\sigma_{e_\xi}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^2}\right)^2 + \frac{1}{\sigma_\varepsilon^2} \sum_{i=1}^N \left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right)^2 \\
&= 2N \left(n_1 \frac{\sigma_{e_\xi}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^2}\right)^2 + \frac{1}{\sigma_\varepsilon^2} \sum_{i=1}^N \left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right)^2, \\
E\left(\frac{\partial^2}{\partial c^2} \bar{f}(\varepsilon | \theta)\right) &= \frac{1}{\sigma_\varepsilon^4} E\left(\left(\sum_{i=1}^N \varepsilon_i\right)^2\right) = \frac{1}{\sigma_\varepsilon^4} N \sigma_\varepsilon^2 = \frac{1}{\sigma_\varepsilon^2} N,
\end{aligned}$$

and

$$\begin{aligned}
E\left(\frac{\partial}{\partial n_1} \frac{\partial}{\partial c} \bar{f}(\varepsilon | \theta)\right) &= Nn_1 \frac{\sigma_{e_\xi}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^4} E\left(\sum_{i=1}^N \varepsilon_i\right) - n_1 \frac{\sigma_{e_\xi}^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^6} E\left(\sum_{i=1}^N \varepsilon_i^2 \sum_{i=1}^N \varepsilon_i\right) \\
&+ \frac{1}{\sigma_\varepsilon^4} E\left(\sum_{i=1}^N \varepsilon_i \sum_{i=1}^N \varepsilon_i \left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right)\right) \\
&= \frac{1}{\sigma_\varepsilon^4} \sigma_\varepsilon^2 \sum_{i=1}^N \left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right).
\end{aligned}$$

Thus, the Fisher matrix is

$$J = \frac{N}{\sigma_\varepsilon^2} \begin{pmatrix} 2 \left( n_1 \frac{\sigma_\varepsilon^2 - \sigma_{e_z}^2}{\sigma_\varepsilon^2} \right)^2 \sigma_\varepsilon^2 + \frac{1}{N} \sum_{i=1}^N \left( \xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i \right)^2 & \frac{1}{N} \sum_{i=1}^N \left( \xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i \right) \\ \frac{1}{N} \sum_{i=1}^N \left( \xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i \right) & 1 \end{pmatrix}.$$

Note that  $\frac{d}{dn_1} \varepsilon_i = \frac{d}{dn_1} \left( n_1 \xi_i + \sqrt{1-n_1^2} z_i + c \right) = \xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i$ .

### A.3 Calculation of moments for $\varepsilon$

Let us study the statistical properties of  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)$ . The moments will be calculated using the characteristic function. The characteristic function  $\varphi_X(x)$  for the normal distributed stochastic variable  $X$  is (see for example [8])

$$X \in N(\mu, \sigma^2), \quad \varphi_X(x) = \exp\left(j\mu x - \frac{1}{2}\sigma^2 x^2\right),$$

where  $j$  is indicating complex numbers. From the characteristic function the moments can be generated by differentiation:

$$\varphi_X^{(k)}(x) = j^k E(X^k).$$

We will also use some properties of expectation values and variances:

$$\begin{aligned} E(X^2) &= Var(X) + (E(X))^2, \\ Cov(X, X) &= Var(X), \\ E(X, Y) &= Cov(X, Y) + E(X)E(Y), \\ Var(X + Y) &= Var(X) + Var(Y) + 2Cov(X + Y). \end{aligned}$$

In our case we have the stochastic variable  $A \in N(0, \sigma_\varepsilon^2)$ ,  $\varepsilon \in A$  and thus, the characteristic function is  $\varphi_A(\varepsilon) = \exp\left(-\frac{1}{2}\sigma_\varepsilon^2 \varepsilon^2\right)$ . Differentiation of  $\varphi_A(\varepsilon)$  gives

$$\begin{aligned} \frac{d}{d\varepsilon} \varphi_A(\varepsilon) &= -\sigma_\varepsilon^2 \varepsilon \exp\left(-\frac{1}{2}\sigma_\varepsilon^2 \varepsilon^2\right), \\ \frac{d^2}{d\varepsilon^2} \varphi_A(\varepsilon) &= -\sigma_\varepsilon^2 \exp\left(-\frac{1}{2}\sigma_\varepsilon^2 \varepsilon^2\right) + \sigma_\varepsilon^4 \varepsilon^2 \exp\left(-\frac{1}{2}\sigma_\varepsilon^2 \varepsilon^2\right), \\ \frac{d^3}{d\varepsilon^3} \varphi_A(\varepsilon) &= 3\sigma_\varepsilon^4 \varepsilon \exp\left(-\frac{1}{2}\sigma_\varepsilon^2 \varepsilon^2\right) - \sigma_\varepsilon^6 \varepsilon^3 \exp\left(-\frac{1}{2}\sigma_\varepsilon^2 \varepsilon^2\right), \\ \frac{d^4}{d\varepsilon^4} \varphi_A(\varepsilon) &= 3\sigma_\varepsilon^4 \exp\left(-\frac{1}{2}\sigma_\varepsilon^2 \varepsilon^2\right) - 6\sigma_\varepsilon^6 \varepsilon^2 \exp\left(-\frac{1}{2}\sigma_\varepsilon^2 \varepsilon^2\right), \\ &\quad + \sigma_\varepsilon^8 \varepsilon^4 \exp\left(-\frac{1}{2}\sigma_\varepsilon^2 \varepsilon^2\right). \end{aligned}$$

Setting  $\varepsilon = 0$  we get the moments:

$$\begin{aligned} E(A) &= j \frac{d}{d\varepsilon} \varphi_A(0) = 0, \\ E(A^2) &= -\frac{d^2}{d\varepsilon^2} \varphi_A(0) = \sigma_\varepsilon^2, \\ E(A^3) &= -j \frac{d^3}{d\varepsilon^3} \varphi_A(0) = 0, \\ E(A^4) &= \frac{d^4}{d\varepsilon^4} \varphi_A(0) = 3\sigma_\varepsilon^4, \end{aligned}$$

and from the moments we get the variances by  $Var(X^k) = E(X^k) - (E(X))^k$ :

$$\begin{aligned} Var(A) &= E(A^2) - (E(A))^2 = \sigma_\varepsilon^2, \\ Var(A^2) &= E(A^4) - (E(A^2))^2 = 3\sigma_\varepsilon^4 - \sigma_\varepsilon^4 = 2\sigma_\varepsilon^4. \end{aligned}$$

To calculate the expectation values in  $J$  we assume that  $\varepsilon_i, \varepsilon_j$  are independent for  $j \neq i$  and further that  $\xi_i$  can be considered constant in the calculations below. The expectations of the different parts of  $J$  can now be calculated:

$$\begin{aligned} E\left(\sum_{i=1}^N \varepsilon_i\right) &= \sum_{i=1}^N E(\varepsilon_i) = NE(\varepsilon_i) = 0, \\ Var\left(\sum_{i=1}^N \varepsilon_i\right) &= \sum_{i=1}^N Var(\varepsilon_i) = NVar(\varepsilon_i) = N\sigma_\varepsilon^2, \\ E\left(\sum_{i=1}^N \varepsilon_i^2\right) &= \sum_{i=1}^N E(\varepsilon_i^2) = NE(\varepsilon_i^2) = N\sigma_\varepsilon^2, \\ Var\left(\sum_{i=1}^N \varepsilon_i^2\right) &= \sum_{i=1}^N Var(\varepsilon_i^2) = NVar(\varepsilon_i^2) = 2N\sigma_\varepsilon^4, \\ E\left(\sum_{i=1}^N \varepsilon_i \sum_{i=1}^N \varepsilon_i\right) &= Var\left(\sum_{i=1}^N \varepsilon_i\right) + \left(E\left(\sum_{i=1}^N \varepsilon_i\right)\right)^2 = N\sigma_\varepsilon^2, \\ E\left(\sum_{i=1}^N \varepsilon_i \sum_{i=1}^N \varepsilon_i^2\right) &= E\left(\sum_{i=1}^N \varepsilon_i\right) E\left(\sum_{i=1}^N \varepsilon_i^2\right) + Cov\left(\sum_{i=1}^N \varepsilon_i, \sum_{i=1}^N \varepsilon_i^2\right) \\ &= \sum_{i=1}^N Cov(\varepsilon_i, \varepsilon_i^2) = \sum_{i=1}^N (E(\varepsilon_i^3) - E(\varepsilon_i) E(\varepsilon_i^2)) \\ &= 0, \\ E\left(\sum_{i=1}^N \varepsilon_i^2 \sum_{i=1}^N \varepsilon_i^2\right) &= \left(E\left(\sum_{i=1}^N \varepsilon_i^2\right)\right)^2 + Var\left(\sum_{i=1}^N \varepsilon_i^2\right) \\ &= (N\sigma_\varepsilon^2)^2 + 2N\sigma_\varepsilon^4 \\ &= N^2\sigma_\varepsilon^4 + 2N\sigma_\varepsilon^4, \end{aligned}$$

and

$$\begin{aligned}
E\left(\sum_{i=1}^N \varepsilon_i \left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right)\right) &= \sum_{i=1}^N \left(\left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right) E(\varepsilon_i)\right) = 0, \\
E\left(\left(\sum_{i=1}^N \varepsilon_i \left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right)\right)^2\right) &= \left(E\left(\sum_{i=1}^N \varepsilon_i \left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right)\right)\right)^2 \\
&\quad + \text{Var}\left(\sum_{i=1}^N \varepsilon_i \left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right)\right) \\
&= \sum_{i=1}^N \text{Var}\left(\varepsilon_i \left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right)\right) \\
&= \sum_{i=1}^N \left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right)^2 \text{Var}(\varepsilon_i) \\
&= \sigma_\varepsilon^2 \sum_{i=1}^N \left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right)^2, \\
E\left(\sum_{i=1}^N \varepsilon_i \sum_{i=1}^N \varepsilon_i \left(-\xi_i + \frac{n_1}{\sqrt{1-n_1^2}} z_i\right)\right) &= E\left(\sum_{i=1}^N \varepsilon_i\right) E\left(\sum_{i=1}^N \varepsilon_i \left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right)\right) \\
&\quad + \text{Cov}\left(\sum_{i=1}^N \varepsilon_i, \sum_{i=1}^N \varepsilon_i \left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right)\right) \\
&= \sum_{i=1}^N \text{Cov}\left(\varepsilon_i, \varepsilon_i \left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right)\right) \\
&= \sum_{i=1}^N E\left(\varepsilon_i^2 \left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right)\right) \\
&\quad - \sum_{i=1}^N E(\varepsilon_i) E\left(\varepsilon_i \left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right)\right) \\
&= \sum_{i=1}^N \left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right) E(\varepsilon_i^2) \\
&= \sigma_\varepsilon^2 \sum_{i=1}^N \left(\xi_i - \frac{n_1}{\sqrt{1-n_1^2}} z_i\right).
\end{aligned}$$

Finally,

$$\begin{aligned}
E \left( \sum_{i=1}^N \varepsilon_i^2 \sum_{i=1}^N \varepsilon_i \left( -\xi_i + \frac{n_1}{\sqrt{1-n_1^2}} z_i \right) \right) &= E \left( \sum_{i=1}^N \varepsilon_i^2 \right) E \left( \sum_{i=1}^N \varepsilon_i \left( -\xi_i + \frac{n_1}{\sqrt{1-n_1^2}} z_i \right) \right) \\
&+ Cov \left( \sum_{i=1}^N \varepsilon_i^2, \sum_{i=1}^N \varepsilon_i \left( -\xi_i + \frac{n_1}{\sqrt{1-n_1^2}} z_i \right) \right) \\
&= \sum_{i=1}^N Cov \left( \varepsilon_i^2, \varepsilon_i \left( -\xi_i + \frac{n_1}{\sqrt{1-n_1^2}} z_i \right) \right) \\
&= \sum_{i=1}^N E \left( \varepsilon_i^3 \left( -\xi_i + \frac{n_1}{\sqrt{1-n_1^2}} z_i \right) \right) \\
&- \sum_{i=1}^N E \left( \varepsilon_i^2 \right) E \left( \varepsilon_i \left( -\xi_i + \frac{n_1}{\sqrt{1-n_1^2}} z_i \right) \right) \\
&= \sum_{i=1}^N \left( -\xi_i + \frac{n_1}{\sqrt{1-n_1^2}} z_i \right) E \left( \varepsilon_i^3 \right) \\
&= 0.
\end{aligned}$$

## B Calculation of (15) in [3]

In [3] we have the model

$$\mathcal{M} : q_x = (\bar{p} + \delta\bar{p}) K + \delta q,$$

where  $\bar{p}$  is a  $1 \times 3$  vector,  $\delta\bar{p} \in \mathcal{N}(0, \sigma^2 I)$  is the perturbation,  $\delta q \in \mathcal{N}(0, \sigma^2)$  is the observation error and  $\mathcal{N}(\cdot)$  denoted normal distribution.  $K \in \mathcal{R}^{3 \times 1}$  is the parameters that will be estimated. The total error for the  $i$ th observation is

$$\varepsilon_i = q_x - \bar{p}K = \delta\bar{p}K + \delta q,$$

where  $\varepsilon_i \in \mathcal{N}(0, B\sigma^2)$ ,  $B = 1 + K^T K = 1 + K_1^2 + K_2^2 + K_3^2$ . The general term in the Fisher information matrix is given in [3], eq. (15), as

$$E \left( \frac{\partial \ln f(\varepsilon | \mathbf{K})}{\partial K_m} \cdot \frac{\partial \ln f(\varepsilon | \mathbf{K})}{\partial K_n} \right) = \frac{1}{\sigma^2} \left( \frac{2K_m K_n}{B^2} N \sigma^2 + \frac{1}{B} \sum_{i=1}^N p_m p_n \right).$$

This results will be derived below.

$$\begin{aligned}
& E \left( \frac{\partial \ln f(\varepsilon | \mathbf{K})}{\partial K_m} \cdot \frac{\partial \ln f(\varepsilon | \mathbf{K})}{\partial K_n} \right) \\
&= E \left( \frac{K_m K_n}{B^2 \sigma^4} \left\{ -N\sigma^2 + \frac{1}{B} \sum_{i=1}^N \varepsilon_i^2 + \frac{1}{K_m} \sum_{i=1}^N p_m \varepsilon_i \right\} \cdot \left\{ -N\sigma^2 + \frac{1}{B} \sum_{i=1}^N \varepsilon_i^2 + \frac{1}{K_n} \sum_{i=1}^N p_n \varepsilon_i \right\} \right) \\
&= E \left( \frac{K_m K_n}{B^2 \sigma^4} \{-N\sigma^2\} \cdot \left\{ -N\sigma^2 + \frac{1}{B} \sum_{i=1}^N \varepsilon_i^2 + \frac{1}{K_n} \sum_{i=1}^N p_n \varepsilon_i \right\} \right) + \\
& E \left( \frac{K_m K_n}{B^2 \sigma^4} \left\{ \frac{1}{B} \sum_{i=1}^N \varepsilon_i^2 \right\} \cdot \left\{ -N\sigma^2 + \frac{1}{B} \sum_{i=1}^N \varepsilon_i^2 + \frac{1}{K_n} \sum_{i=1}^N p_n \varepsilon_i \right\} \right) + \\
& E \left( \frac{K_m K_n}{B^2 \sigma^4} \left\{ \frac{1}{K_m} \sum_{i=1}^N p_m \varepsilon_i \right\} \cdot \left\{ -N\sigma^2 + \frac{1}{B} \sum_{i=1}^N \varepsilon_i^2 + \frac{1}{K_n} \sum_{i=1}^N p_n \varepsilon_i \right\} \right) \\
&= \frac{K_m K_n}{B^2 \sigma^4} N^2 \sigma^4 - \frac{K_m K_n}{B^2 \sigma^4} \frac{N\sigma^2}{B} E \left( \sum_{i=1}^N \varepsilon_i^2 \right) - \frac{K_m K_n}{B^2 \sigma^4} \frac{N\sigma^2}{K_n} E \left( \sum_{i=1}^N p_n \varepsilon_i \right) \\
& - \frac{K_m K_n}{B^2 \sigma^4} \frac{N\sigma^2}{B} E \left( \sum_{i=1}^N \varepsilon_i^2 \right) + \frac{K_m K_n}{B^2 \sigma^4} \frac{1}{B^2} E \left( \sum_{i=1}^N \varepsilon_i^2 \sum_{i=1}^N \varepsilon_i^2 \right) \\
& + \frac{K_m K_n}{B^2 \sigma^4} \frac{1}{BK_n} E \left( \sum_{i=1}^N \varepsilon_i^2 \sum_{i=1}^N p_n \varepsilon_i \right) - \frac{K_m K_n}{B^2 \sigma^4} \frac{N\sigma^2}{K_m} E \left( \sum_{i=1}^N p_m \varepsilon_i \right) \\
& + \frac{K_m K_n}{B^2 \sigma^4} \frac{1}{BK_m} E \left( \sum_{i=1}^N p_m \varepsilon_i \sum_{i=1}^N \varepsilon_i^2 \right) + \frac{K_m K_n}{B^2 \sigma^4} \frac{1}{K_m K_n} E \left( \sum_{i=1}^N p_m \varepsilon_i \sum_{i=1}^N p_n \varepsilon_i \right),
\end{aligned}$$

where

$$\begin{aligned}
E\left(\sum_{i=1}^N \varepsilon_i^2\right) &= \sum_{i=1}^N E(\varepsilon_i^2) = \sum_{i=1}^N \text{Var}(\varepsilon_i) = NB\sigma^2, \\
E\left(\sum_{i=1}^N p_n \varepsilon_i\right) &= \sum_{i=1}^N p_n E(\varepsilon_i) = 0, \\
E\left(\sum_{i=1}^N \varepsilon_i^2 \sum_{i=1}^N \varepsilon_i^2\right) &= \text{Var}\left(\sum_{i=1}^N \varepsilon_i^2\right) + \left(E\left(\sum_{i=1}^N \varepsilon_i^2\right)\right)^2 \\
&= \sum_{i=1}^N \text{Var}(\varepsilon_i^2) + (NB\sigma^2)^2 \\
&= \sum_{i=1}^N 2B^2\sigma^4 + (N^2B\sigma^2)^2 \\
&= 2NB^2\sigma^4 + N^2B^2\sigma^4, \\
E\left(\sum_{i=1}^N \varepsilon_i^2 \sum_{i=1}^N p_n \varepsilon_i\right) &= 0 \text{ (see calculations above),} \\
E\left(\sum_{i=1}^N p_m \varepsilon_i \sum_{i=1}^N p_n \varepsilon_i\right) &= \text{Cov}\left(\sum_{i=1}^N p_m \varepsilon_i, \sum_{i=1}^N p_n \varepsilon_i\right) + E\left(\sum_{i=1}^N p_m \varepsilon_i\right) E\left(\sum_{i=1}^N p_n \varepsilon_i\right) \\
&= \text{Cov}\left((p_m \varepsilon_1 + p_m \varepsilon_2 + \dots + p_m \varepsilon_N), (p_n \varepsilon_1 + p_n \varepsilon_2 + \dots + p_n \varepsilon_N)\right) \\
&= \text{Cov}\left(\sum_{i=1}^N p_m \varepsilon_i p_n \varepsilon_i\right) = \sum_{i=1}^N p_m p_n \text{Cov}(\varepsilon_i \cdot \varepsilon_i) \\
&= \text{Var}(\varepsilon_i) \sum_{i=1}^N p_m p_n = B\sigma^2 \sum_{i=1}^N p_m p_n.
\end{aligned}$$

The variance  $\text{Var}(\varepsilon_i^2)$  is calculated using the characteristic function;  $A \in N(0, B\sigma^2)$ ,  $\varepsilon \in A$  and thus, the characteristic function is  $\varphi_A(\varepsilon) = \exp\left(-\frac{1}{2}B\sigma^2\varepsilon^2\right)$ . Differentiation of  $\varphi_A(\varepsilon)$  gives

$$\begin{aligned}
\frac{d}{d\varepsilon} \varphi_A(\varepsilon) &= -B\sigma^2\varepsilon \exp\left(-\frac{1}{2}B\sigma^2\varepsilon^2\right), \\
\frac{d^2}{d\varepsilon^2} \varphi_A(\varepsilon) &= -B\sigma^2 \exp\left(-\frac{1}{2}B\sigma^2\varepsilon^2\right) + B^2\sigma^4\varepsilon^2 \exp\left(-\frac{1}{2}B\sigma^2\varepsilon^2\right), \\
\frac{d^3}{d\varepsilon^3} \varphi_A(\varepsilon) &= 3B^2\sigma^4\varepsilon \exp\left(-\frac{1}{2}B\sigma^2\varepsilon^2\right) - B^3\sigma^6\varepsilon^3 \exp\left(-\frac{1}{2}B\sigma^2\varepsilon^2\right), \\
\frac{d^4}{d\varepsilon^4} \varphi_A(\varepsilon) &= 3B^2\sigma^4 \exp\left(-\frac{1}{2}B\sigma^2\varepsilon^2\right) - 6B^3\sigma^6\varepsilon^2 \exp\left(-\frac{1}{2}B\sigma^2\varepsilon^2\right) \\
&\quad + B^4\sigma^8\varepsilon^4 \exp\left(-\frac{1}{2}B\sigma^2\varepsilon^2\right),
\end{aligned}$$



From this we can calculate the moments:

$$\begin{aligned}
E(A) &= j \frac{d}{d\varepsilon} \varphi_A(0) = 0, \\
E(A^2) &= -\frac{d^2}{d\varepsilon^2} \varphi_A(0) = B\sigma^2, \\
E(A^3) &= -j \frac{d^3}{d\varepsilon^3} \varphi_A(0) = 0, \\
E(A^4) &= \frac{d^4}{d\varepsilon^4} \varphi_A(0) = 3B^2\sigma^4,
\end{aligned}$$

and from the moments we get the variances by using  $Var(X^k) = E(X^k) - (E(X))^k$ :

$$\begin{aligned}
Var(A) &= E(A^2) - (E(A))^2 = B\sigma^2, \\
Var(A^2) &= E(A^4) - (E(A^2))^2 = 3B^2\sigma^4 - B^2\sigma^4 = 2B^2\sigma^4.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&E\left(\frac{\partial \ln f(\varepsilon | \mathbf{K})}{\partial K_m} \cdot \frac{\partial \ln f(\varepsilon | \mathbf{K})}{\partial K_n}\right) \\
&= \frac{K_m K_n}{B^2 \sigma^4} N^2 \sigma^4 - \frac{K_m K_n}{B^2 \sigma^4} \frac{N \sigma^2}{B} N B \sigma^2 - \frac{K_m K_n}{B^2 \sigma^4} \frac{N \sigma^2}{K_n} \cdot 0 \\
&\quad - \frac{K_m K_n}{B^2 \sigma^4} \frac{N \sigma^2}{B} N B \sigma^2 + \frac{K_m K_n}{B^2 \sigma^4} \frac{1}{B^2} (2N B^2 \sigma^4 + N^2 B^2 \sigma^4) + \frac{K_m K_n}{B^2 \sigma^4} \frac{1}{B K_n} \cdot 0 \\
&\quad - \frac{K_m K_n}{B^2 \sigma^4} \frac{N \sigma^2}{K_m} \cdot 0 + \frac{K_m K_n}{B^2 \sigma^4} \frac{1}{B K_m} \cdot 0 + \frac{K_m K_n}{B^2 \sigma^4} \frac{1}{K_m K_n} B \sigma^2 \sum_{i=1}^N p_m p_n \\
&= \frac{K_m K_n}{B^2} N^2 - \frac{K_m K_n}{B^2} N^2 \\
&\quad - \frac{K_m K_n}{B^2} N^2 + \frac{K_m K_n}{B^2} 2N + \frac{K_m K_n}{B^2} N^2 + \frac{K_m}{B^3 \sigma^4} \cdot 0 \\
&\quad + \frac{K_n}{B^3 \sigma^4} \cdot 0 + \frac{1}{B \sigma^2} \sum_{i=1}^N p_m p_n \\
&= \frac{2N K_m K_n}{B^2} + \frac{1}{B \sigma^2} \sum_{i=1}^N p_m p_n \\
&= \frac{1}{\sigma^2} \left( \frac{2K_m K_n}{B^2} N \sigma^2 + \frac{1}{B} \sum_{i=1}^N p_m p_n \right).
\end{aligned}$$

This expression can also be written

$$E\left(\frac{\partial \ln f(\varepsilon | \mathbf{K})}{\partial K_m} \cdot \frac{\partial \ln f(\varepsilon | \mathbf{K})}{\partial K_n}\right) = \frac{N}{B \sigma^2} \left( \frac{2\sigma^2}{B} K_m K_n + \frac{1}{N} \sum_{i=1}^N p_m p_n \right).$$

For a linear regression we have

$$E \left( \frac{\partial \ln f(\varepsilon | \mathbf{K})}{\partial K_m} \cdot \frac{\partial \ln f(\varepsilon | \mathbf{K})}{\partial K_n} \right) = \frac{N}{B\sigma^2} \left( \frac{1}{N} \sum_{i=1}^N p_m p_n \right).$$

## C CRLB for linear regression

### C.1 CRLB when $\sigma_{e_z} = 0$

When  $\sigma_{e_z}^2 = 0$ , equations the Fisher matrix (17) will be simplified to

$$J = \frac{N}{n_2^2 \sigma_{e_\xi}^2} \begin{pmatrix} 2\sigma_{e_\xi}^2 + \bar{E} \left( \left( \xi - \frac{n_1}{\sqrt{1-n_1^2}} z \right)^2 \right) & \bar{E} \left( \xi - \frac{n_1}{\sqrt{1-n_1^2}} z \right) \\ \bar{E} \left( \xi - \frac{n_1}{\sqrt{1-n_1^2}} z \right) & 1 \end{pmatrix} \quad (37)$$

$$\sigma_\varepsilon^2 = n_1^2 \sigma_{e_\xi}^2. \quad (38)$$

The variances of  $\theta$  will then be

$$\text{Var} (n_1^0 - \hat{n}_1) = \frac{n_1^2 \sigma_{e_\xi}^2}{N} \frac{1}{\det(J)} \quad (39)$$

$$\text{Var} (c^0 - \hat{c}) = \frac{n_1^2 \sigma_{e_\xi}^2}{N} \frac{1}{\det(J)} \left( 2\sigma_{e_\xi}^2 + \bar{E} \left( \xi - \frac{n_1}{\sqrt{1-n_1^2}} z \right)^2 \right) \quad (40)$$

$$\det(J) = 2\sigma_{e_\xi}^2 + \bar{\text{Var}} \left( \xi - \frac{n_1}{\sqrt{1-n_1^2}} z \right). \quad (41)$$

### C.2 Validation

In this section we will investigate the Cramer-Rao lower bound (CRLB) for linear regression, i.e., when  $e_z = 0$ . The calculations will follow [3]. Consider the measurement error model (14). The total error for instant  $i$  is defined as  $\varepsilon_i$ :

$$\begin{aligned} n_1 (\xi_i - e_{\xi,i}) + n_2 z_i + c &= 0, \\ \varepsilon_i &= n_1 e_{\xi,i} = n_1 \xi_i + n_2 z_i + c, \\ n_1^2 + n_2^2 &= 1. \end{aligned}$$

The total error  $\varepsilon_i, i = 1, \dots, N$ , is normal distributed with zero mean and variance

$$\sigma_\varepsilon^2 = n_1^2 \sigma_{e_\xi}^2.$$

The joint conditional density for  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)$  is

$$f(\varepsilon | \theta) = \frac{1}{(2\pi)^{N/2} \sigma_\varepsilon^N} \exp \left( -\frac{1}{2\sigma_\varepsilon^2} \sum_{i=1}^N \varepsilon_i^2 \right)$$

and the logarithm is

$$\bar{f}(\varepsilon | \theta) = \log f(\varepsilon | \theta) = -\frac{N}{2} \log(2\pi) - N \log \sigma_\varepsilon - \frac{1}{2\sigma_\varepsilon^2} \sum_{i=1}^N \varepsilon_i^2$$

or explicitly,

$$\bar{f}(\varepsilon | \theta) = \log f(\varepsilon | \theta) = -\frac{N}{2} \log(2\pi) - N \log(n_1 \sigma_{e_\varepsilon}) - \frac{1}{2n_1^2 \sigma_{e_\varepsilon}^2} \sum_{i=1}^N \left( n_1 \xi_i + \sqrt{1 - n_1^2} z_i + c \right)^2.$$

Applying the following differentiations:

$$\begin{aligned} \frac{\partial}{\partial n_1} - N \log(n_1 \sigma_{e_\varepsilon}) &= -\frac{N}{n_1}, \\ \frac{\partial}{\partial n_1} - \frac{1}{2n_1^2 \sigma_{e_\varepsilon}^2} &= \frac{1}{n_1^3 \sigma_{e_\varepsilon}^2}, \\ \frac{\partial}{\partial n_1} \left( n_1 \xi_i + \sqrt{1 - n_1^2} z_i + c \right)^2 &= -2 \left( n_1 \xi_i + \sqrt{(-n_1^2 + 1)} z_i + c \right) \frac{-\xi_i \sqrt{(-n_1^2 + 1)} + z_i n_1}{\sqrt{(-n_1^2 + 1)}}, \end{aligned}$$

we get

$$\begin{aligned} \frac{\partial}{\partial n_1} \bar{f}(\varepsilon | \theta) &= -\frac{N}{n_1} + \frac{1}{n_1^3 \sigma_{e_\varepsilon}^2} \sum_{i=1}^N \left( n_1 \xi_i + \sqrt{1 - n_1^2} z_i + c \right)^2 \\ &\quad - \frac{1}{n_1^2 \sigma_{e_\varepsilon}^2} \sum_{i=1}^N \left( n_1 \xi_i + \sqrt{1 - n_1^2} z_i + c \right) \left( \xi_i - \frac{n_1}{\sqrt{(-n_1^2 + 1)}} z_i \right), \\ \frac{\partial}{\partial c} \bar{f}(\varepsilon | \theta) &= -\frac{1}{n_1^2 \sigma_{e_\varepsilon}^2} \sum_{i=1}^N \left( n_1 \xi_i + \sqrt{1 - n_1^2} z_i + c \right), \end{aligned}$$

or

$$\begin{aligned} \frac{\partial}{\partial n_1} \bar{f}(\varepsilon | \theta) &= -\frac{N}{n_1} + \frac{1}{n_1 \sigma_\varepsilon^2} \sum_{i=1}^N \varepsilon_i^2 - \frac{1}{\sigma_\varepsilon^2} \sum_{i=1}^N \varepsilon_i \left( \xi_i - \frac{n_1}{\sqrt{(-n_1^2 + 1)}} z_i \right), \\ \frac{\partial}{\partial c} \bar{f}(\varepsilon | \theta) &= -\frac{1}{\sigma_\varepsilon^2} \sum_{i=1}^N \varepsilon_i. \end{aligned}$$

The Fisher matrix is

$$J = E \left( \frac{\partial}{\partial \theta} \bar{f}(\varepsilon | \theta) \left[ \frac{\partial}{\partial \theta} \bar{f}(\varepsilon | \theta) \right]^T \right),$$

where

$$\begin{aligned}
\frac{\partial^2}{\partial n_1^2} \bar{f}(\varepsilon | \theta) &= \left( -\frac{N}{n_1} + \frac{1}{n_1 \sigma_\varepsilon^2} \sum_{i=1}^N \varepsilon_i^2 - \frac{1}{\sigma_\varepsilon^2} \sum_{i=1}^N \varepsilon_i \left( \xi_i - \frac{n_1}{\sqrt{(-n_1^2 + 1)}} z_i \right) \right)^2 \\
&= \left( \frac{N}{n_1} \right)^2 + \left( \frac{1}{n_1 \sigma_\varepsilon^2} \right)^2 \left( \sum_{i=1}^N \varepsilon_i^2 \right)^2 + \frac{1}{\sigma_\varepsilon^4} \left( \sum_{i=1}^N \varepsilon_i \left( \xi_i - \frac{n_1}{\sqrt{(-n_1^2 + 1)}} z_i \right) \right)^2 \\
&\quad - 2 \frac{N}{n_1} \frac{1}{n_1 \sigma_\varepsilon^2} \sum_{i=1}^N \varepsilon_i^2 + \frac{N}{n_1} \frac{1}{\sigma_\varepsilon^2} \sum_{i=1}^N \varepsilon_i \left( \xi_i - \frac{n_1}{\sqrt{(-n_1^2 + 1)}} z_i \right) \\
&\quad - \frac{1}{n_1 \sigma_\varepsilon^4} \sum_{i=1}^N \varepsilon_i^2 \sum_{i=1}^N \varepsilon_i \left( \xi_i - \frac{n_1}{\sqrt{(-n_1^2 + 1)}} z_i \right), \\
\frac{\partial^2}{\partial c^2} \bar{f}(\varepsilon | \theta) &= \frac{1}{\sigma_\varepsilon^4} \left( \sum_{i=1}^N \varepsilon_i \right)^2,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial n_1} \frac{\partial}{\partial c} \bar{f}(\varepsilon | \theta) &= \left( -\frac{N}{n_1} + \frac{1}{n_1 \sigma_\varepsilon^2} \sum_{i=1}^N \varepsilon_i^2 - \frac{1}{\sigma_\varepsilon^2} \sum_{i=1}^N \varepsilon_i \left( \xi_i - \frac{n_1}{\sqrt{(-n_1^2 + 1)}} z_i \right) \right) \cdot \\
&\quad \left( -\frac{1}{\sigma_\varepsilon^2} \sum_{i=1}^N \varepsilon_i \right) \\
&= \frac{N}{n_1} \frac{1}{\sigma_\varepsilon^2} \sum_{i=1}^N \varepsilon_i - \frac{1}{n_1 \sigma_\varepsilon^4} \sum_{i=1}^N \varepsilon_i^2 \sum_{i=1}^N \varepsilon_i \\
&\quad - \frac{1}{\sigma_\varepsilon^4} \sum_{i=1}^N \varepsilon_i \left( \xi_i - \frac{n_1}{\sqrt{(-n_1^2 + 1)}} z_i \right) \sum_{i=1}^N \varepsilon_i.
\end{aligned}$$

Using the calculations of expectation values in Appendix A.2, but setting  $\sigma_\varepsilon^2 = n_1^2 \sigma_{e_\xi}^2$ , we get

$$\begin{aligned}
\frac{\partial^2}{\partial n_1^2} \bar{f}(\varepsilon | \theta) &= \left( \frac{N}{n_1} \right)^2 + \left( \frac{1}{n_1 \sigma_\varepsilon^2} \right)^2 (N^2 \sigma_\varepsilon^4 + 2N \sigma_\varepsilon^4) + \frac{1}{\sigma_\varepsilon^4} \sigma_\varepsilon^2 \sum_{i=1}^N \left( \xi_i - \frac{n_1}{\sqrt{(-n_1^2 + 1)}} z_i \right)^2 \\
&\quad - 2 \frac{N}{n_1} \frac{1}{n_1 \sigma_\varepsilon^2} N \sigma_\varepsilon^2 \\
&= 2 \frac{N}{n_1^2} + \frac{1}{n_1^2 \sigma_{e_\xi}^2} \sum_{i=1}^N \left( \xi_i - \frac{n_1}{\sqrt{(-n_1^2 + 1)}} z_i \right)^2, \\
\frac{\partial^2}{\partial c^2} \bar{f}(\varepsilon | \theta) &= \frac{N}{n_1^2 \sigma_{e_\xi}^2} \left( \sum_{i=1}^N \varepsilon_i \right)^2,
\end{aligned}$$

and

$$E \left( \frac{\partial}{\partial n_1} \frac{\partial}{\partial c} \bar{f}(\varepsilon | \theta) \right) = \frac{1}{n_1^2 \sigma_{e_\varepsilon}^2} \sum_{i=1}^N \left( \xi_i - \frac{n_1}{\sqrt{(-n_1^2 + 1)}} z_i \right).$$

The CRLB is given by

$$E \left( (\theta - \hat{\theta}) (\theta - \hat{\theta})^T \right) \geq J^{-1},$$

where

$$J = \frac{N}{n_1^2 \sigma_{e_\varepsilon}^2} \begin{pmatrix} 2\sigma_{e_\varepsilon}^2 + \bar{E} \left( \left( \xi - \frac{n_1}{\sqrt{(-n_1^2 + 1)}} z \right)^2 \right) & \bar{E} \left( \xi - \frac{n_1}{\sqrt{(-n_1^2 + 1)}} z \right) \\ \bar{E} \left( \xi - \frac{n_1}{\sqrt{(-n_1^2 + 1)}} z \right) & 1 \end{pmatrix}.$$

This equation equals (37).

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Title

Modelling of Laser Radar Systems

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Abstract

In this report background material for some papers are stored. It concerns modelling of laser radar systems for hard targetting. The (returned) laser radar data is used for parameters estimation, where error-in-variables is assumed. The impulse response for some common (target) shapes are derived. The coordinate systems in output data is discussed. The system performance is analyzed using the Cramer-Rao lower bound. The models are developed for a scanning, monostatic system, but some are general enough for other type of systems.

**Nyckelord**

Keywords

Laser radar, measurement error regression, Cramer-Rao, system model