Hamilton-Jacobi Equations for Nonlinear Descriptor Systems

Torkel Glad, Johan Sjöberg

Division of Automatic Control
Department of Electrical Engineering
Linköpings universitet, SE-581 83 Linköping, Sweden
WWW: <http://www.control.isy.liu.se>
E-mail: torkel@isy.liu.se, johans@isy.liu.se

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Abstract

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Keywords: Optimal Control, Descriptor Systems

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Torkel Glad

Division of Automatic Control
Department of Electrical Engineering
Linköpings universitet, SE-581 83 Linköping, SWEDEN
torkel@isy.liu.se

Johan Sjöberg

Division of Automatic Control
Department of Electrical Engineering
Linköpings universitet, SE-581 83 Linköping, SWEDEN
johans@isy.liu.se

Abstract—Optimal control problems for nonlinear descriptor systems are considered. An approach where the descriptor system is conceptually reduced to a state space form is compared to an approach where the Hamilton-Jacobi equation is directly formulated for the descriptor system. The two approaches are shown to give essentially the same systems of equations to be solved. A certain unknown function is present only in the second approach but is shown to be computable from the quantities common to both approaches.

I. INTRODUCTION

During the last decades, descriptor systems have been extensively studied, see for example the surveys [1, 2] and references therein. One reason is that many applications can be formulated more naturally in this way. The growing use of objected-oriented modeling languages such as MODELICA also increases the usage of descriptor system formulations, since most often this is the output of such tools.

Optimal feedback design for linear descriptor systems is thoroughly investigated area, see for example [3–5]. Also for nonlinear descriptor systems some references about optimal control can be found. Most of the references use methods based on variational calculus, *e.g.*, [6, 7], but some use the dynamic programming approach, *e.g.*, [8, 9]. The purpose of the present paper is to study the connection between the different approaches for solving the optimal control. In the approach by [8, 9] the Hamilton-Jacobi equation is formulated directly for the DAE model but it is also possible to reduce the DAE to a state space model and use the standard formulation of the Hamilton-Jacobi equation. We show that the two approaches are equivalent under certain regularity conditions. In this context we also give an interpretation of the W -function occurring in [8, 9].

II. SYSTEM DESCRIPTION

We will consider systems on descriptor or differential-algebraic equation (DAE) form

$$\dot{x}_1 = F_1(x_1, x_2, u) \quad (1a)$$

$$0 = F_2(x_1, x_2, u) \quad (1b)$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $u \in \mathbb{R}^m$. In the Appendix we will discuss how more general forms of DAEs can be reduced to this form.

For our analysis of optimal control problems we make the following assumptions.

Assumption 1. The system (1) has an equilibrium. There is no restriction in assuming the equilibrium to be at $x_1 = 0$, $x_2 = 0$, $u = 0$, i.e. $F_1(0, 0, 0) = 0$ and $F_2(0, 0, 0) = 0$.

If the Jacobian F_{2,x_2} is nonsingular the implicit function theorem implies that it is locally possible to solve for x_2 in (1b)

$$x_2 = \varphi(x_1, u) \quad (2)$$

In order to be able to use this relation in our derivations we make the following assumption.

Assumption 2. There is an open set $\Omega \subset \mathbb{R}^{n_1}$ containing the origin such that for all $x_1 \in \Omega$ and all u , (1b) can be solved to give (2). Also F_{2,x_2} is assumed nonsingular for all $x_1 \in \Omega$, x_2 and u solving (1b).

The set

$$\mathcal{N} = \{(x_1, x_2) \mid x_1 \in \Omega, x_2 = \varphi(x_1, u), u \in \mathbb{R}^m\}$$

will be denoted the set of consistent states.

It is possible to calculate φ_u . Using that

$$F_2(x_1, \varphi(x_1, u), u) = 0$$

identically in u , differentiation w.r.t. u gives

$$F_{2,x_2}(x_1, \varphi(x_1, u), u) \varphi_u(x_1, u) + F_{2,u}(x_1, \varphi(x_1, u), u) = 0$$

which can be solved for $\varphi_u(x_1, u)$ as

$$\varphi_u(x_1, u) = -F_{2,x_2}(x_1, \varphi(x_1, u), u)^{-1} F_{2,u}(x_1, \varphi(x_1, u), u) \quad (3)$$

A concept that is commonly used in the theory of descriptor systems is the (differential) index. The index is, loosely speaking, the minimum number of differentiations needed to obtain an equivalent system of ordinary differential equations, see [10]. Our assumption 2 above implies that we look at index one problems. In the Appendix we briefly discuss how higher order index problems are reduced.

III. OPTIMAL CONTROL

The optimal control problem is defined by a performance criterion, the dynamics and some boundary conditions. The performance criterion in this case is

$$J = \int_0^\infty L(x_1, x_2, u) dt \quad (4)$$

with an infinite time horizon. The optimal control problem is

$$V(x_1(0)) = \min_{u(\cdot)} J \quad (5)$$

subject to the dynamics (1) and the boundary conditions

$$\begin{aligned} x_1(0) &= x_{1,0} \in \Omega \\ \lim_{t \rightarrow \infty} x_1(t) &= 0 \end{aligned}$$

A control u expressed as a feedback law from x_1, x_2 may change the index of the closed loop system [1]. However, with our assumptions we will show below that the index is automatically preserved, *i.e.*, the closed loop system is also of index one. Then, x_1 will be the free variables and x_2 is chosen consistently, *i.e.*, such that $F_2(x_1, x_2, u(x_1, x_2)) = 0$.

Obviously, the performance criterion has to converge for the optimal feedback law. This is guaranteed if the closed loop system is such that $x_1(\infty) = 0$, *i.e.*, asymptotically stable.

Remark 1: In some articles about optimal control for descriptor systems, *e.g.*, [3, 6, 8], the possibility of changing the index of a system description is used. They require the feedback law to be such that the closed loop system is of index one even if the system was of higher index.

For a system on state-space form (*i.e.* x_2 and F_2 are absent and we use the notation $x = x_1$), the optimal control problem is solved by the following theorem.

Theorem 1: Suppose there exist a continuously differentiable function $J(x)$ satisfying $J(0) = 0$ and

$$0 = \min_u (L(x, u) + J_x(x)F(x, u))$$

for $x \in \Omega$. Let

$$\mu(x) = \operatorname{argmin}_u (L(x, u) + J_x(x)F(x, u))$$

and let $\tilde{\Omega} \subset \Omega$ denote the initial states for which the trajectories, using $u(t) = \mu(x(t))$, remain in Ω and converge to the origin. Then

$$V(x) = J(x), \quad x \in \tilde{\Omega}$$

and $\mu(x)$ is an optimal feedback control law.

Proof: This is a standard result in optimal control. See [11] or [12]. ■

IV. THE HAMILTON-JACOBI EQUATION FOR THE REDUCED PROBLEM

Our Assumption 2 makes it possible to reduce the optimal control problem for the descriptor system to an optimal control problem for a state-space system. Assuming $x_1 \in \Omega$, substitution of (2) into (1a) and (4) yields the performance criterion

$$J = \int_0^\infty L(x_1, \varphi(x_1, u), u) dt$$

and the differential equation

$$\dot{x}_1 = F_1(x_1, \varphi(x_1, u), u)$$

The Hamilton-Jacobi equation solving this optimal control problem is

$$\begin{aligned} 0 &= \min_u (L(x_1, \varphi(x_1, u), u) + V_{x_1}(x_1)F_1(x_1, \varphi(x_1, u), u)) \\ x_1 &\in \Omega \end{aligned} \quad (6)$$

where $V(x_1)$ is the optimal return function (5).

The first-order necessary condition for optimality of (6) yields the set of equations

$$\begin{aligned} 0 &= L_u + V_{x_1}F_{1,u} + (L_{x_2} + V_{x_1}F_{1,x_2})\varphi_u \\ 0 &= L + V_{x_1}F_1 \end{aligned}$$

where the quantities in the right hand sides are evaluated at $(x_1, \varphi(x_1, u), u)$. Since $x_2 = \varphi(x_1, u)$ is the unique solution of (1b), it is also possible to write these equations according to

$$0 = L_u + V_{x_1}F_{1,u} - (L_{x_2} + V_{x_1}F_{1,x_2})F_{2,x_2}^{-1}F_{2,u} \quad (8a)$$

$$0 = L + V_{x_1}F_1 \quad (8b)$$

$$0 = F_2 \quad (8c)$$

where (3) is used and the right hand sides are evaluated at x_1, x_2, u . One way of looking at (8) is to regard (8a) and (8c) as $m + n_2$ equations from which one tries to solve for u and x_2 as functions of x_1 and V_{x_1} . When these quantities are substituted into (8b) the result is a first order partial differential equation for V as a function of x_1 . When this partial differential equation is solved the result can be substituted back into the expression for u to give the feedback law.

V. OPTIMALITY CONDITIONS FOR DESCRIPTOR SYSTEMS

According to Theorem 3.1 in [8], the optimal control problem can also be solved by solving the Hamilton-Jacobi-like equation

$$\begin{aligned} 0 &= \min_u (L(x_1, x_2, u) + W_1(x_1)F_1(x_1, x_2, u) \\ &\quad + W_2(x_1, x_2)F_2(x_1, x_2, u)) \end{aligned} \quad (9)$$

for some continuous functions $W_1(x_1)$ and $W_2(x_1, x_2)$ such that $W_1(x_1)$ is a gradient of some continuously differentiable function $V(x_1)$. (We have changed the notation of [8] slightly so that it is consistent with ours.)

This $V(x_1)$ is then the optimal cost (5). Using the first-order optimality conditions, the control is then defined by the following set of equations

$$0 = L_u + W_1F_{1,u} + W_2F_{2,u} \quad (10a)$$

$$0 = L + W_1F_1 + W_2F_2 \quad (10b)$$

From these equations it is not immediately obvious how one can obtain relation from which W_1 can be computed. We can obtain equations similar to (8) by restricting (10) to points satisfying $F_2 = 0$. We then get the following system

$$0 = L_u + W_1F_{1,u} + W_2F_{2,u} \quad (11a)$$

$$0 = L + W_1F_1 \quad (11b)$$

$$0 = F_2 \quad (11c)$$

It should be stressed that in (9), x_2 is considered to be independent of u when differentiating with respect to u .

VI. RELATIONSHIPS AMONG THE SOLUTIONS

The reduced Hamilton-Jacobi equation (6) and the Hamilton-Jacobi-like equation (9) solve the same underlying optimal control problem. Below we will investigate the relations between the functions V , W_1 and W_2 .

Lemma 2: Suppose there is a function $V(x_1)$ and a feedback $u = k(x_1)$ solving (6) on Ω . Then $W_1(x_1) = V_{x_1}(x_1)$, $u = k(x_1)$ solve (9) under the constraint $F_2(x_1, x_2, u) = 0$. Moreover, with the choice

$$W_1 = V_{x_1}, \quad W_2 = -(L_{x_2} + V_{x_1} F_{1,x_2}) F_{2,x_2}^{-1} \quad (12)$$

the necessary conditions (11) are satisfied for $u = k(x_1)$, $x_2 = \varphi(x_1, k(x_1))$.

Proof: When $F_2 = 0$ the right hand sides of (9) and (6) coincide. Comparing (8) and (11) shows that (11) is satisfied for $u = k(x_1)$, $x_2 = \varphi(x_1, k(x_1))$. ■

The converse relation is given by the following lemma.

Lemma 3: Assume that for $x_1 \in \Omega^1$ it holds that:

- (9) has a solution $u = \psi(x_1, x_2)$
- $F_2(x_1, x_2, \psi(x_1, x_2)) = 0$ has a solution $x_2 = \eta(x_1)$
- $W_1(x_1) = V_{x_1}(x_1)$ for some function $V(x_1)$

Then $V_{x_1}(x_1)$ and $u = k(x_1) = \psi(x_1, \eta(x_1))$ solve (6) for $x_1 \in \Omega$. Moreover $W_2(x_1, x_2)$ in (11) is given by

$$W_2 = -(L_{x_2} + W_1 F_{1,x_2}) F_{2,x_2}^{-1} \quad (13)$$

for all u and all x_1, x_2 in $\mathcal{N}$ satisfying (11).

Proof: We have

$$\begin{aligned} & L(x_1, \eta(x_1), \psi(x_1, \eta(x_1))) \\ & + V_{x_1}(x_1) F_1(x_1, \eta(x_1), \psi(x_1, \eta(x_1))) \\ & = L(x_1, \varphi(x_1, k(x_1)), k(x_1)) \\ & + V_{x_1}(x_1) F_1(x_1, \varphi(x_1, k(x_1)), k(x_1)) = 0 \end{aligned}$$

since the minimal value in (9) is attained for $u = \psi(x_1, x_2)$ for all $x_1 \in \Omega$ and $x_2 \in \mathbb{R}^{n_2}$, and then particularly for $x_2 = \eta(x_1)$. Since $x_1 \in \Omega$ it is also known that $\eta(x_1) = \varphi(x_1, k(x_1))$. According to (9) we have

$$\begin{aligned} 0 \leq & L(x_1, x_2, u) + V_{x_1}(x_1) F_1(x_1, x_2, u) \\ & + W_2(x_1, x_2) F_2(x_1, x_2, u) \end{aligned}$$

for all $x_1 \in \Omega$, $x_2 \in \mathbb{R}^{n_2}$ and $u \in \mathbb{R}^m$. In particular we have

$$0 \leq L(x_1, \varphi(x_1, u), u) + V_{x_1}(x_1) F_1(x_1, \varphi(x_1, u), u)$$

and (6) is thus satisfied.

Since a u solving (11a) is given by $u = \psi(x_1, x_2)$, (11b) and (11c) give

$$\begin{aligned} 0 &= L(x_1, x_2, \psi(x_1, x_2)) + W_1(x_1) F_1(x_1, x_2, \psi(x_1, x_2)) \\ 0 &= F_2(x_1, x_2, \psi(x_1, x_2)) \end{aligned}$$

¹If this Ω is smaller than the Ω defined in Section II, we redefine Ω as the smaller region.

Differentiating these relations w.r.t. x_2 yield

$$0 = L_{x_2} + (L_u + W_1 F_{1,u}) \psi_{x_2} + W_1 F_{1,x_2} \quad (14a)$$

$$0 = F_{2,x_2} + F_{2,u} \psi_{x_2} \quad (14b)$$

If (11a) is multiplied from right with ψ_{x_2} and after (14) is inserted the result is that W_2 is given by

$$0 = W_1 F_{1,x_2} + L_{x_2} + W_2 F_{2,x_2}$$

Due to the fact that F_{2,x_2} is nonsingular for all u and all x_1, x_2 in $\mathcal{N}$, it follows that

$$W_2 = -(L_{x_2} + W_1 F_{1,x_2}) F_{2,x_2}^{-1} \quad (15)$$

Hence, for a system with index equal to one, we get one further necessary condition for the optimal solution, namely (15). ■

Remark 2: A special case which yields simple equations is

$$\dot{x}_1 = f_1(x_1, x_2) + g_1(x_1)u \quad (16a)$$

$$0 = f_2(x_1, x_2) + g_2(x_1)u \quad (16b)$$

and $L(x_1, x_2, u) = l(x_1) + \frac{1}{2}u^T u$. Then (11a) can be solved explicitly in u for all x_1, x_2 since (11a) will become

$$0 = u^T - W_1 g_1 - W_2 g_2 \quad (17)$$

and from Lemma 3, we have that W_2 is given by

$$W_2 = -W_1 f_{1,x_2} f_{2,x_2}^{-1} \quad (18)$$

Note that f_{2,x_2} is nonsingular for all x_1, x_2 since F_{2,x_2} is nonsingular for all x_1, x_2, u and then particularly for $u = 0$. Combining (17) and (18) yields

$$u = -\hat{g}(x_1, x_2)^T W_1(x_1)^T$$

and after some manipulation the necessary conditions can be rewritten as

$$0 = l(x_1) + W_1(x_1) \hat{f}(x_1, x_2) \quad (19a)$$

$$- \frac{1}{2} W_1(x_1) \hat{g}(x_1, x_2) \hat{g}(x_1, x_2)^T W_1(x_1)^T \quad (19b)$$

$$0 = f_2(x_1, x_2) + g_2(x_1) \hat{g}(x_1, x_2)^T W_1(x_1)^T \quad (19c)$$

where

$$\begin{aligned} \hat{f}(x_1, x_2) &= f_1(x_1, x_2) \\ &\quad - f_{1,x_2}(x_1, x_2) f_{2,x_2}(x_1, x_2)^{-1} f_2(x_1, x_2) \\ \hat{g}(x_1, x_2) &= g_1(x_1) - f_{1,x_2}(x_1, x_2) f_{2,x_2}(x_1, x_2)^{-1} g_2(x_1) \end{aligned}$$

VII. EXAMPLE

Consider the simple system

$$\dot{x}_1 = x_2 \quad (20)$$

$$0 = u - x_1^3 - x_2 \quad (21)$$

with optimality criterion

$$J = \int_0^\infty \left(\frac{x_1^2}{2} + \frac{u^2}{2} \right) dt \quad (22)$$

The necessary conditions (8) give

$$\begin{aligned} 0 &= u + V_{x_1} \cdot 0 + V_{x_1} \cdot 1 \\ 0 &= \frac{x_1^2}{2} + \frac{u^2}{2} + V_{x_1} x_2 \\ 0 &= u - x_1^3 - x_2 \end{aligned}$$

Eliminating u and x_1 gives the following equation for V_{x_1}

$$V_{x_1}^2 + 2x_1^3 V_{x_1} - x_1^2 = 0$$

with the solution

$$V_{x_1} = x_1(\sqrt{1 + x_1^4} - x_1^2)$$

and with the optimal feedback law

$$u = -V_{x_1} = -x_1(\sqrt{1 + x_1^4} - x_1^2)$$

If the problem is instead solved using (10) we get the equations

$$\begin{aligned} 0 &= u + W_1 \cdot 0 + W_2 \cdot 1 \\ 0 &= \frac{x_1^2}{2} + \frac{u^2}{2} + W_1 x_2 + W_2(u - x_1^3 - x_2) \end{aligned}$$

Since the last equation holds identically in x_2 we get by differentiating

$$0 = W_1 + W_{2,x_2}(u - x_1^3 - x_2) - W_2$$

which shows that $W_1 = W_2$ when the equation $u - x_1^3 - x_2 = 0$ is satisfied. This gives the same equation for W_1 as for V_{x_1} so the solution is the same.

VIII. CONCLUSIONS

We have shown the connections between the Hamilton-Jacobi type equation (9) of an optimal control problem for descriptor systems and the Hamilton-Jacobi equation (9) which is obtained when the problem is reduced to a state space problem. As a result of these calculations we show that W_1 of (9) is the gradient of the optimal return function V of the reduced problem, while W_2 (restricted to the constraints of the problem, which is all that is needed) can be calculated from V . We have also shown that the sets of equations that have to be solved when actually computing a solution are essentially the same.

IX. APPENDIX. INDEX REDUCTION

Suppose we start with a DAE model of the form

$$\hat{F}(\dot{x}, x, u) = 0 \quad (23)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

In a series of papers, [13–15], it has been described how a rather general class of such systems can be written in the form

$$\bar{F}_1(\dot{x}_1, \dot{x}_2, x_1, x_2, u) = 0 \quad (24a)$$

$$\bar{F}_2(x_1, x_2, u, \dot{u}, \dots, u^{(\nu)}) = 0 \quad (24b)$$

Here the variables in the vector x have been grouped into two vectors $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$. The variables x_1

are described by differential equations and hence form the dynamical part, while x_2 is algebraically connected to x_1 and u . The functions $\bar{F}_1$ and $\bar{F}_2$ are formed from suitable linear combinations of the rows of F and (possibly) differentiations of F . An assumption that will be used throughout is that $\bar{F}_1$ and $\bar{F}_2$ are continuously differentiable. The Jacobian of $\bar{F}_2$ with respect to x_2 is nonsingular, so from the implicit function theorem x_2 can be solved from (24b), at least locally. Furthermore, in the equation

$$\begin{aligned} \bar{F}_1(\dot{x}_1, -\bar{F}_{2,x_2}^{-1}(\bar{F}_{2,x_1}\dot{x}_1 + \bar{F}_{2,u}\dot{u} + \dots \\ + \bar{F}_{2,u^{(\nu)}}u^{(\nu+1)}), x_1, x_2, u) = 0 \end{aligned} \quad (25)$$

obtained by substituting $\dot{x}_2$ from (24b), it is possible to solve for $\dot{x}_1$. Here we will make an assumption that is in practice not very restrictive, since in many applications $\bar{F}_1$ is actually affine in $\dot{x}_1$ and $\dot{x}_2$.

Assumption $\dot{x}_1$ can be solved from (25) to give

$$\dot{x}_1 = \tilde{F}_1(x_1, x_2, u, \dot{u}, \dots, u^{(\nu+1)}) \quad (26a)$$

$$0 = \tilde{F}_2(x_1, x_2, u, \dot{u}, \dots, u^{(\nu)}) \quad (26b)$$

One complication is the possible presence of derivatives of the control variable (originating from differentiations of the equations). For linear systems it is possible to make transformations removing the input derivatives from the differential equations. This might not be possible in the nonlinear case. In that case it could be necessary to redefine the control signal so that its highest derivative becomes a new control variable and the lower order derivatives become state variables. We assume that these operations have been performed giving a description of the form

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