Frequency-Domain Identification of Continuous-Time Output Error Models from Non-Uniformly Sampled Data

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Abstract
This paper treats the identification of continuous-time output error (OE) models based on sampled data. The exact method for doing this is well known both for data given in the time domains, but this approach becomes somewhat complex, especially for non-uniformly sampled data. In this paper we assume that the system is fed with a zero order hold input signal and that the sampling rate is so high that the sampling rate is basically integrative. The conclusion is that if the system has relative degree \( \ell \) then the output should be interpolated using an \( \ell \) order polynomial spline.

Keywords: continuous-time systems; parameter estimation; continuous-time OE; splines
Abstract

This paper treats the identification of continuous-time output error (OE) models based on sampled data. The exact method for doing this is well known both for data given in the time domains, but this approach becomes somewhat complex, especially for non-uniformly sampled data. In this paper we assume that the system is fed with a zero order hold input signal and that the sampling rate is so high that the sampling rate is basically integrative. The conclusion is that if the system has relative degree \( \ell \) then the output should be interpolated using an \( \ell \) order polynomial spline.

1 Introduction

In this contribution we shall discuss identification of possibly grey-box structured linear continuous-time models from discrete-time measurements of inputs and outputs. This as such is a well known problem and discussed, e.g. in (Ljung, 1999). Several techniques for identification of continuous time models are also discussed in, among many references, (Rao and Garnier, 2002), (Unbehauen and Rao, 1990), (Mensler, 1999).

The “optimal” solution is well known as a Maximum-likelihood (ML) formulation. It consist of computing the Kalman filter predictions of the output at the sampling instants by sampling the continuous time model over the sampling instants. These predictions are functions of the parameters in the continuous time model and by minimizing the sum of squared prediction errors with respect to the parameters, the Maximum likelihood estimate is obtained in case of Gaussian disturbances. For equidistantly sampled data, this method is also implemented in the MATLAB® System Identification Toolbox, (Ljung, 2003).

No method can be better, in theory, asymptotically as the number of data tends to infinity, than this maximum likelihood method. However, it may encounter numerical problems at fast sampling, and it may be computationally demanding for irregularly sampled data.

In this paper we investigate an approximate route to the identification of continuous-time output error models from non-uniformly sampled data which is based on spline interpolation.
2 Outline

The outline of the paper will be the following. First in Section 3 the frequency domain continuous-time OE (COE) modelling and identification approach will be described. Here the method for estimating the continuous-time Fourier transform introduced by the authors in (Gillberg and Ljung, 2005) is also presented. In Section 4 the relationship between the previous approach and that of interpolation by polynomial splines discussed for the case of uniform sampling. Eventually in Section 5 the conclusions drawn from the case of uniform sampling are numerically illustrated for non-uniform sampling.

3 Introduction

The problem is to estimate the parameters $\theta$ in a continuous time transfer function

$$y_u(t) = G_c(p, \theta)u(t)$$  \hspace{1cm} (1)

The output is observed at sampling instances $t_k$ with some measurement noise

$$y(t_k) = y_u(t_k) + e(k)$$  \hspace{1cm} (2)

The output noise term $e$ is assumed to be Gaussian white noise. For the input $u$ and the output $y$ we define the continuous time Fourier transforms, restricted to an observation interval $[0 \ T]$:

$$Y_c(i \omega) = \int_0^T y(t)e^{-i \omega t}dt$$  \hspace{1cm} (3)

and analogously for $U_c(i \omega)$.

If $Y_c(i \omega_k)$ and $U_c(i \omega_k), k = 1, 2, \ldots, N_\omega$ of the continuous-time Fourier transforms (3) are available, the ML-procedure for estimating the parameters is

$$\hat{\theta} = \arg \min_{\theta} V_c(\theta)$$  \hspace{1cm} (4)

$$V_c(\theta) \triangleq \sum_{k=1}^{N_\omega} |Y_c(i \omega_k) - G_c(i \omega_k, \theta)U_c(i \omega_k)|^2.$$  \hspace{1cm} (5)

See, e.g. page 230 in (Ljung, 1999). Independence of the Fourier transforms at different frequencies is discussed in detail in e.g. (Brillinger, 1981), Chapter 5 and in (Gillberg, 2004), Chapter 3. The bottom line is that the frequencies should be separated by an interval that is $2\pi/T$.

The main crux is of course getting $Y_c$ from discrete time measurements. In a previous paper by the authors, a method was devised for equidistantly sampled data.

By assuming that the input is ZOH and approximating the system by as set of integrators

$$G_c(s) \sim \frac{1}{s^\ell}$$  \hspace{1cm} (6)
where $\ell$ is the relative degree of the original system, a method

$$
\hat{\theta} = \arg \min_\theta V_c(\theta)
$$

(7)

$$
V_c(\theta) = \sum_{k=1}^{N_c} |F_c^{(\ell)}(i\omega_k)Y_d(e^{i\omega_k T_s})
- G_c(i\omega_k)H_{T_s}(i\omega_k)U_d(e^{i\omega_k T_s})|^2
$$

(8)

for parameter estimation was devised. Here

$$
H_{T_s}(i\omega) = \frac{1 - e^{-i\omega T_s}}{i\omega T_s}
$$

(9)

and for the system with property (6), the function $F_c^{(\ell)}$ takes the form:

$$
F_c^{(\ell)}(i\omega) = \left(\frac{e^{i\omega T_s} - 1}{i\omega T_s}B_\ell(e^{i\omega T_s})\right)^{\ell+1}
$$

(10)

where $B_\ell(z)$ are the Euler-Frobenius polynomials

$$
B_1(z) = 1 \quad (11a)
$$

$$
B_2(z) = z + 1 \quad (11b)
$$

$$
B_3(z) = z^2 + 4z + 1 \quad (11c)
$$

$$
B_4(z) = z^3 + 11z^2 + 11z + 1 \quad (11d)
$$

(see the references above)(Åström et al., 1984), (Wahlberg, 1988), (Weller et al., 2001). This approach was introduced in a previous paper by the authors (Gillberg and Ljung, 2005).

4 Interpretations in terms of Cardinal (Equidistant) Polynomial Splines

In this section we will show that the method in [7] is in fact equivalent to interpolating the output $y$ in terms of polynomial spline functions. The material presented here consists of selected pieces of theory on splines for signal processing that can be found in a number of excellent papers by M. Unser (Unser et al., 1993a)(Unser et al., 1993b) or in standard books such as the one by DeBoor (de Boor, 1978).

4.1 B-Splines

Cardinal polynomial splines of order $\ell$ are a representation of piecewise polynomial functions of degree $l$ such that

$$
\hat{y}(t) = \sum_{k=-\infty}^{\infty} c(k)\beta_c^{(\ell)}(t - T_sk)
$$

(12)
Figure 1: Cubic Spline function

where

$$
\beta_c^{(\ell)}(t) = \sum_{k=0}^{\ell+1} \frac{(-1)^k}{\ell!} \binom{\ell + 1}{k} (t - T_s k)^n H(t - T_s k)
$$

(13)

and $H(t)$ is the Heaviside step function. Also, we define the sampled version of the spline function as

$$
\beta_d^{(\ell)}(k) = \beta_c^{(\ell)}(T_s k) \quad \forall k \in \mathbb{Z}
$$

(14)

Assume that a sequence $\{y(kT_s), k = -\infty \ldots \infty\}$ of uniformly distributed samples of a continuous function $y(t)$ is available, and we wish to find a cardinal polynomial spline of order $\ell$ with the interpolation property

$$
y(T_s l) = \hat{y}(T_s l) \quad \forall l \in \mathbb{Z}
$$

$$
= \sum_{k=\infty}^\infty c(k) \beta_c^{(\ell)}(T_s l - T_s k) \quad \forall l \in \mathbb{Z}
$$

$$
= \sum_{k=\infty}^\infty c(k) \beta_d^{(\ell)}(l - k) \quad \forall l \in \mathbb{Z}
$$

$$
= \left( c(.) \ast \beta_d^{(\ell)}(.) \right)(T_s l) \quad \forall l \in \mathbb{Z}.
$$

By taking the z-transform of the equation above we will get

$$
Y_d(z) = C(z)B_d^{(\ell)}(z)
$$

(15)

and the coefficients in expression (12) can be computed as

$$
C(z) = \left[ B_d^{(\ell)}(z) \right]^{-1} Y_d(z).
$$

(16)

The transformed version of the spline basis function is here

$$
B_d^{(\ell)}(z) = T_s^{\ell+1} \sum_{k=0}^{\ell+1} \frac{(-1)^k}{\ell!} \binom{\ell + 1}{k} z^{-k} \sum_{n=0}^{\infty} n^\ell z^{-n}
$$

$$
= (1 - z^{-1})^{\ell+1} \sum_{n=0}^{\infty} n^\ell z^{-n}.
$$
For different values of $\ell$ this will be

$$B_d^{(1)}(z) = T_{s}^{\ell+1} \frac{1}{z}$$

$$B_d^{(2)}(z) = T_{s}^{\ell+1} \frac{1 + z^{-1}}{2z}$$

$$B_d^{(3)}(z) = T_{s}^{\ell+1} \frac{z + 4 + z^{-1}}{6z^2}$$

$$B_d^{(3)}(z) = T_{s}^{\ell+1} \frac{z + 11 + 11z^{-1} + z^{-2}}{24z^2}$$

which is apparently related to the Euler-Frobenius polynomials found in (11).

These expressions can now be used to estimate the continuous-time Fourier transform.

When a function is represented as (12) its transform will be

$$\hat{Y}_c(i\omega) \triangleq \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c(k) \beta^{(\ell)}(t - T_s k) e^{-i\omega t} dt$$

$$= \sum_{k=-\infty}^{\infty} c(k) \int_{-\infty}^{\infty} \beta^{(\ell)}(t - T_s k) e^{-i\omega t} dt$$

$$= \int_{-\infty}^{\infty} \beta^{(\ell)}(t) e^{-i\omega t} dt \sum_{k=-\infty}^{\infty} c(k) e^{-i\omega T_s k}. \quad (17)$$

If we use the relationship (Schoenberg, 1973)

$$B_c^{(\ell)}(i\omega) \triangleq \int_{-\infty}^{\infty} \beta^{(\ell)}(t) e^{-i\omega t} dt = \left( \frac{1 - e^{-i\omega T_s}}{i\omega} \right)^{\ell+1} \quad (18)$$

and insert this expression into (17) we get

$$\hat{Y}_c(i\omega) = F_c^{(\ell)}(i\omega) Y_d(e^{i\omega T_s})$$

$$= \frac{B_c^{(\ell)}(i\omega)}{B_d^{(\ell)}(e^{i\omega T_s})} Y_d(e^{i\omega T_s}) \quad (19)$$

which is an estimate of the continuous-time Fourier Transform.

### 4.2 Fundamental Cardinal Spline Function

An interesting consequence of the above line of reasoning is that if we interpret

$$F_c^{(\ell)}(i\omega) = \frac{B_c^{(\ell)}(e^{i\omega T_s})}{B_d^{(\ell)}(e^{i\omega T_s})} = \int_{-\infty}^{\infty} f^{(\ell)}(t) e^{-i\omega t} dt \quad (21)$$

then $f^{(\ell)}$ is the so called fundamental cardinal spline function of order $\ell$ (see for instance Lecture 4 in (Schoenberg, 1973)) which corresponds to the solution of the interpolation problem

$$\delta(l) = \sum_{k=-\infty}^{\infty} c(k) \beta^{(\ell)}(T_s l - T_s k) \quad (22)$$
where

\[ \delta(l) = \begin{cases} 1 & l = 0 \\ 0 & l \neq 0 \end{cases} \]  \quad (23) \]

is the Kronecker delta function. This means that we can also write our interpolation function in (12) as

\[ \hat{y}(t) = \sum_{k=-\infty}^{\infty} y(T_s k) f^{(l)}(t - T_s k) \]  \quad (24) \]

which is called the Lagrange form (de Boor, 1978) of the cardinal spline representation.

5 Non-Uniform Sampling and Polynomial Splines

The conclusion of the above discussion is that if we have an input which is zero-order hold and the system has relative degree \( \ell \) a reasonable way to interpolate the output \( y \) is by polynomial spline functions. For a more extensive empirical investigation we refer to the paper by Rolain (Rolain et al., 1998).

In the following numerical examples we will illustrate how to identify continuous-time output error models from non-uniformly sampled output data using spline interpolation. In the examples we have used the continuous-time model

\[ y(t) = \frac{1}{p^2 + a_1 p + a_2} e(t). \]  \quad (25) \]

where \( a_1 = 3 \) and \( a_2 = 2 \). The output \( y \) has been sampled at time instances subject to jitter such that

\[ t_k = k T_s + \delta_k, \quad k = 1 \ldots N - 1 \]

where

\[ \delta_k \in U[-\delta_0, \delta_0] \quad k = 1 \ldots N - 1. \]

For the sake of simplicity, the initial and endpoint time instances have been chosen such that \( t_0 = 0 \) and \( t_N = N T_s \). The input is piecewise constant with constant sampling time \( T_s \) where the amplitude of the segments have a Normal distribution.
5.1 Non-parametric Estimation

First we show a non-parametric estimate of the Fourier transform. Here the estimation procedure is as follows. The output $y(t_k), k = 1 \ldots N$ is interpolated using polynomial splines of order $\ell = 2$ (parabolic) on the non-uniform grid $t_k, k = 1 \ldots N$. This is done using the MATLAB® Spline Toolbox (de Boor, 2004). The now continuous interpolated output $\hat{y}$ is then sampled at the new denser grid points $t_k^u = T_s k, k = 0 \ldots N^u - 1$ where $N^u = MN$ and $T_s^u = T_s / M$. The variable $M$ denotes the upsampling which is $M = 10$. The discrete-time Fourier transform is then computed as an approximation of its continuous-time counterpart.

$$\hat{Y}_c(i\omega_k) \approx \frac{1}{\sqrt{N}} \sum_{l=0}^{N^u-1} \hat{y}(T_s^u l) e^{-i\omega_k T_s^u l}. \quad (26)$$

This is illustrated in Figure 3 where we have estimated the continuous-time Fourier Transform of the system found in (25). The dotted curve represents the average Fourier transform of $\hat{y}$ estimated from $N_{MC} = 250$ Monte-Carlo simulations. The solid line is the transform of the true signal $y(T_s^u k), k = 1 \ldots N^u$ at the upsampled grid points. In this example $T_s = 0.5$ and $N = 200$. As you can see the "true" and estimated spectrums coincide at low frequencies and differ only marginally at higher ones.

5.2 Parametric Estimation

In Table 1, 2 and 2 we have estimated the parameters of three of models with different relative degrees for a set of different nominal sampling frequencies. The criterion that have been used is

$$\hat{\theta} = \arg \min_{\theta} V_c(\theta) \quad (27)$$

$$V_c(\theta) \triangleq \sum_{k=1}^{N_c} \left| \hat{Y}_c(i\omega_k) - G_c(i\omega_k, \theta)U_c(i\omega_k) \right|^2 \quad (28)$$
where \( \hat{Y}(i\omega) \) is taken from (26). Only frequencies up to the nominal Nyquist frequency \( 2\pi/T_s \) have been used. From the tables one can see that the effect of the non-uniform sampling is quite limited for moderately high nominal sampling rates.

<table>
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<tr>
<th>True/Ts</th>
<th>0.02</th>
<th>0.1</th>
<th>0.5</th>
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<td></td>
<td></td>
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</tr>
<tr>
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<td>0.9584</td>
<td>0.8305</td>
<td>0.6825</td>
</tr>
<tr>
<td>0.5</td>
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</tr>
<tr>
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<td>2.0033</td>
<td>1.9511</td>
<td>1.8444</td>
<td>1.7791</td>
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<tr>
<td>3</td>
<td>3.0327</td>
<td>3.0471</td>
<td>2.8720</td>
<td>2.5359</td>
</tr>
</tbody>
</table>

Table 1: Results for the system \( \frac{p^{0.5}}{p^3 + 2p^2 + 3} \). This system has a pole excess of 1 and a bandwidth of 8.60 rad/s

<table>
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<tr>
<th>True/Ts</th>
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<th>0.1</th>
<th>0.5</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Appr 1</td>
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<td></td>
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</tr>
<tr>
<td>1</td>
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<td>0.9644</td>
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<td>3.0927</td>
</tr>
<tr>
<td>3</td>
<td>2.0218</td>
<td>2.0160</td>
<td>1.9893</td>
<td>2.4671</td>
</tr>
</tbody>
</table>

Table 2: Results for the system \( \frac{1}{p^4 + 3p^2 + 2} \). This system has a pole excess of 2 and a bandwidth of 0.8358 rad/s

<table>
<thead>
<tr>
<th>True/Ts</th>
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<th>0.1</th>
<th>0.5</th>
<th>1</th>
</tr>
</thead>
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<td></td>
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<tr>
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</tr>
</tbody>
</table>

Table 3: Results for the system \( \frac{1}{p^4 + 2p^3 + 3p + 4} \). This system has a pole excess of 3 and a bandwidth of 2.10 rad/s

### 6 Conclusions

In this paper we assume that a linear continuous-time system is feeded with a zero order hold input signal and that the sampling rate is so high that the system is approximately integrative in nature. The conclusion is that if the system has relative degree \( \ell \) then the output should be interpolated using an \( \ell \) order polynomial spline. This is illustrated by examples of both parametric and non-parametric estimation.

### References


This paper treats the identification of continuous-time output error (OE) models based on sampled data. The exact method for doing this is well known both for data given in the time domains, but this approach becomes somewhat complex, especially for non-uniformly sampled data. In this paper we assume that the system is fed with a zero order hold input signal and that the sampling rate is so high that the sampling rate is basically integrative. The conclusion is that if the system has relative degree $\ell$ then the output should be interpolated using an $\ell$ order polynomial spline.