Abstract
We propose a probabilistic method to compute the near mid-air collision risk as a function of predicted flight trajectory. The computations use state estimate and covariance from a target tracking filter based on angle-only sensors such as digital video cameras. The majority of existing work is focused on risk estimation at a certain time instant. Here we derive an expression for the integrated risk over the critical time horizon. This is possible using probability for level-crossing, and the expression applies to a three-dimensional piecewise straight flight trajectory. The Monte Carlo technique provides a method to compute the probability, but a huge number of simulations is needed to get sufficient reliability for the small risks that the applications require. Instead we propose a method which through sound geometric and numerical approximations yield a solution suitable for real-time implementations. The algorithm is applied to realistic angle-only tracking data, and shows promising results when compared to the Monte Carlo solution.

Keywords: Probability, near mid-air collision, avoidance.
Probabilistic Near Mid-Air Collision Avoidance

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Index Terms—Probability, collision, avoidance.

I. INTRODUCTION

To maintain a safe distance between each other, manned aircraft flying in controlled airspace use the service provided by an Air Traffic Control (ATC). ATC informs and orders human pilots to perform maneuvers in order to avoid Near Mid-Air Collisions (NMAC). A NMAC between two aircraft occurs if the relative distance between them becomes less than a predefined distance [1], [2]. The last decade semi-automatic systems such as TCAS (Traffic Collision Avoidance System) [3] have been implemented that essentially move this responsibility from ATC to the pilot. The TCAS system, however, assumes that both aircraft exchange data on speed, height and bearing over a data link and that both systems cooperate. When operating small UAVs this assumption is often no longer valid. A typical UAV operates at altitudes where small intruding aircraft are often present that do not carry transponders.

This paper presents a method for detecting and avoiding hazardous situations based on uncertain sensor readings. There are many proposals for solving this problem, see e.g. [4] for a review. Here we consider data from a passive angle-only sensor. A challenge with angle-only measuring sensors is how to deal with the significant uncertainty obtained in estimated distance and relative speed. One approach to increase accuracy in the distance estimate is to perform own platform maneuvers [5]. The method in this paper does not rely on accurate distance estimates. The proposed method is based on computing the probability of NMAC for a predicted trajectory. The majority of existing methods are based on instantaneous probability of NMAC [6], [7], [8]. Instantaneous probability corresponds to the probability that the relative position is within a predefined volume at a certain time instant. It is not straightforward how to interpret instantaneous probability of NMAC with respect to an entire future trajectory.

We present an approximate solution to the in general computationally intractable problem of computing the probability of NMAC for a predicted trajectory. Although Monte-Carlo methods are known to be able to approximate probabilities arbitrarily well [9], [10], they are also known to be computationally intensive particularly when the underlying probabilities are small. Here we do not rely on Monte-Carlo methods, but instead we make use of theory for stochastic processes and level-crossings as in [11]. The event corresponding to NMAC can be seen as the crossing of a safety sphere surrounding the vehicle. The concept of avoiding the safety sphere is also adopted in [12] but confined to deterministic trajectories. We derive expressions for the probability that the relative trajectory will cross the safety sphere, both for straight and piecewise straight trajectories. By appropriate approximations of the safety zone the probability of crossing the boundary becomes computationally tractable. The essence of the proposed method is to consider the crossing of a disc instead of the crossing of the sphere. The results for two dimensions given by [11] are extended to three dimensions. The main difference between two and three dimensions is that we need to compute a probability over a circular disc instead of a line. In contrast to the two-dimensional case, there does not exist any method to analytically compute the probability over the circular disc. An alternative approach is then needed, and we use the fact that the characteristic function exists as an analytical expression, and then compute the probability by numerically inverting the characteristic function. The result is extended to cover the important case with piecewise linear trajectories, which provides a way forward to deal with curved paths in general and avoidance maneuvers in particular [13]. We consider time horizons up to a couple of minutes, and therefore neglect effects from disturbances on the predicted path. When longer periods of time are considered the effects of for example wind disturbances can be significant [14].

In Section II we formulate the problem mathematically and state the prerequisites to be able to provide a solution. Section III details the exact conditions for a NMAC to occur. The conditions are based on the minimum relative distance for a given predicted trajectory. If the minimum relative distance is less than a predefined threshold that particular trajectory will lead to a NMAC. We start by giving the solution for a straight path in Section III-A and then continue to the case with piecewise straight paths in Section III-B. Section IV provides
an approximate solution to compute the probability of NMAC. This solution is based on sampling methods and yields an arbitrarily small approximation error but is computationally demanding. In Section VI we present a novel solution based on an approximation of the NMAC geometry. A NMAC will occur if the predicted relative position ever crosses the surface of a predefined sphere. If we instead of a sphere consider the crossing of a circular disc with certain properties the problem becomes computationally tractable. The probability of crossing the circle can be computed using the distribution of the distance perpendicular to line-of-sight weighted with the probability density for time-to-go (tig) and then integrated over a time interval. In Section VII we give some results comparing the sampling solution with the geometric solution and finally in Section VIII we draw conclusions.

II. PROBLEM FORMULATION

The probability of near-midair collision (NMAC) between two aerial vehicles for a given time period \((0, T)\) is defined as

\[
P(\text{NMAC}(0, T)) = P\left( \min_{0 < t < T} |s(t)| < R \right),
\]

where \(s(t)\) represents the relative position between the two vehicles at time \(t \geq 0\) and \(t\) is the prediction time. \(R\) is the radius of a safety zone, which we assume has the shape of a sphere, and \(R = 150\) m, see Figure 1.

Fig. 1. A NMAC occurs if the relative position crosses the safety sphere.

The definition according to (1) means that if the distance

\[
|s(t)| = \sqrt{s_x^2(t) + s_y^2(t) + s_z^2(t)}
\]

for any \(0 < t < T\) falls below \(R\), no matter for how long, we have a NMAC.

Typically, an estimate of relative position is provided by an angle-only tracking filter [15]. Target tracking will not be pursued here in detail, we simply state that based on measurements from an angle measurement unit e.g. an electro-optical sensor, the tracking filter estimates three-dimensional relative position \(s(0)\) and velocity \(v(0)\) in cartesian coordinates together with their covariances. To simplify the problem formulation we assume the angle measurement unit is accurate and the coordinate system is rotated such that the \(x-\)axis is aligned with line of sight. This means that

\[
s_y(0) \equiv s_z(0) \equiv 0
\]

and the estimated state vector used for the probability computations is

\[
\hat{x}(0) = \begin{bmatrix} \hat{s}_x(0) & \hat{v}_x(0) & \hat{s}_y(0) & \hat{v}_x(0) \end{bmatrix}^T.
\]

The corresponding estimated covariance matrix is, using \(\text{var}(s_y) = \text{var}(s_z) = 0\) from (2),

\[
P(0) = \begin{bmatrix} P_{sx} & C \rho_{sx} \sigma_{vy} \rho_{sy} \sigma_{sy} \rho_{sy} \sigma_{vy} & \rho_{sy} \sigma_{vy} \sigma_{vy} \end{bmatrix},
\]

and similar for \(P_{vy}\). We assume the tracking filter output is normally distributed, i.e.

\[
\hat{x}(0) \sim \mathcal{N}(x(0), P(0)).
\]

Note that we will only deal with a relative time scale, represented by \(t = 0\) as the current time on an absolute time scale. At each new time instant on the absolute time scale the tracking filter provides updated estimates of \(x(0)\) and \(P(0)\).

We seek a method capable of accurately computing probability of NMAC when the underlying probability of NMAC is 0.01 or larger. The figure 0.01 comes from the performance of TCAS, which based on simulation studies has a failure rate of around 0.1 [16], [3]. Taking into account that the sensor and tracking filter have a limited intruder detection capability makes 0.01 reasonable. We must be able to detect a collision scenario with better accuracy than 0.1, due to e.g. sensor and tracking limitations, to achieve an overall system accuracy of 0.1. The computation accuracy should be 10% or better, i.e. if the probability is 0.01 then the method should provide a result which does not deviate more than 10% from 0.01. The method must be computationally tractable for real-time processing.

III. CROSSING OF THE SAFETY ZONE

A. Constant velocity

Let us first consider the event \(\text{NMAC}(0, T)\) assuming a constant velocity

\[
\begin{align*}
\dot{s}(t) &= v(0), \quad 0 < t < T, \quad (6a) \\
v(t) &= v(0), \quad (6b)
\end{align*}
\]

The definition of \(\text{NMAC}(0, T)\) according to (1) can also be written as

\[
P(\text{NMAC}(0, T)) = P\left( \min_{0 < t < T} |s(t)| < R \cap |s(0)| > R \right) + P\left(|s(0)| < R\right).
\]

The definition means that we use two mutually exclusive events for \(|s(0)|\) to split the probability, and at the same time noting that if \(|s(0)| < R\) we automatically have \(\min_{0 < t < T} |s(t)| < R\). The first probability term in (7) corresponds to a down-crossing of the surface of the sphere. The situation of a down-crossing is depicted in Figure 2. The minimum relative distance \(\min_{t > 0} |s(t)|\) is attained when \(s(t)\) is orthogonal to \(v\). This point is called closest point of approach and denoted by cpa. The time until cpa is reached, \(t_{\text{cpa}}\), can be computed using the equation

\[
v^T (s(0) + vt_{\text{cpa}}) = 0,
\]

\[
t_{\text{cpa}} = \frac{v^T s(0)}{v^T v}.
\]
which yields

\[ t_{\text{cpa}} = \frac{v^T s(0)}{|v|^2}. \tag{9} \]

The main condition for NMAC\(_{(0,T)}\) then becomes

\[ \min_{0 < t < T} |s(t)| = |s(t_{\text{cpa}})| = |s(0) + vt_{\text{cpa}}| < R, \tag{10} \]

which we denote

\[ C_1 = I([s(0) + vt_{\text{cpa}}| < R]). \tag{11} \]

A finite end time \( t < T \) means that \( t_{\text{cpa}} \) as computed by (9) could yield \( t_{\text{cpa}} > T \). As long as \( t_{\text{cpa}} < T \), condition \( C_1 \) applies, but in case \( t_{\text{cpa}} > T \) we can still have a NMAC situation if

\[ C_4 = |s(0) + vT| < R. \tag{12} \]

The two remaining conditions needed for a complete description of a down-crossing are \( t_{\text{cpa}} > 0 \) and \( |s(0)| > R \). The first one corresponds to the two vehicles approaching each other, and the second one comes from the definition of a down-crossing. To summarize, the conditions for NMAC\(_{(0,T)}\) are

\[ \text{NMAC}_{(0,T)} = (C_1 \cap C_2 \cup C_4) \cap C_3 \cup \bar{C}_3, \tag{13} \]

where \( \cap \) denotes intersection, \( \cup \) union, \( \bar{C} \) complement of \( C \) and

\[ C_1 = |s(0) + vt_{\text{cpa}}| < R, \]
\[ C_2 = 0 < t_{\text{cpa}} < T, \]
\[ C_3 = |s(0)| > R, \]
\[ C_4 = |s(0) + vT| < R. \tag{14} \]

**B. Piecewise constant velocity**

From now on we assume the relative position follows a piecewise straight path given by

\[ \dot{s}(t) = v^{(j)}, \quad T_j < t < T_{j+1}, \tag{15a} \]
\[ v^{(j)} = v(0) + \sum_{i=1}^{j} \Delta v^{(i)}, \tag{15b} \]

where all \( \Delta v^{(i)} \) are known. Note that the model in (15) is fairly general because we can approximate any continuous curved path arbitrarily well as long as we use a large enough number of straight segments.

The event NMAC\(_{(0,T)}\) can always be expressed according to

\[ \text{NMAC}_{(0,T)} = \bigcup_{j=0}^{J-1} \text{NMAC}_{(T_j, T_{j+1})}, \tag{16} \]

i.e. the union of events for each segment \( T_j < t < T_{j+1} \) with \( T_0 = 0 \) and \( T_J = T \). A geometric interpretation of NMAC\(_{(0,T)}\) for two segments is given by Figure 3.

The change to be incorporated for NMAC\(_{(T_j, T_{j+1})}\) compared to the conditions for the constant velocity case is caused by considering \( t = T_j \) instead of \( t = 0 \) as the starting time. This yields changing variables to \( s(T_j) \) and \( v^{(j)} \). It is possible to derive conditions for NMAC\(_{(T_j, T_{j+1})}\) using \( s(T_j) \). However, a more efficient way is to consider the distance obtained by extrapolating from \( s(T_j) \) backwards in time \( T_j \) seconds using the current velocity \( v^{(j)} \). This yields the distance at \( t = 0 \) which would give \( s(T_j) \) after \( T_j \) seconds using a constant velocity \( v^{(j)} \), compare with Figure 3. This means that we are back to starting time \( t = 0 \) and it enables us to reuse results from the case with constant velocity. Denote the distance obtained from extrapolation by \( s^{(j)} \), and we have

\[ s^{(j)} = s(T_j) - T_j v^{(j)}. \tag{17} \]

An advantage using \( s^{(j)} \) instead of \( s(T_j) \) comes from the fact that the covariance of \( s^{(j)} \) is equal to the covariance of \( s(0) \) for all \( j = 0, \ldots, J - 1 \). This is clear from the expression [11]

\[ s^{(j)} = s(0) - \sum_{i=1}^{j} T_i \Delta v^{(i)}. \tag{18} \]

Inserting the new variables into the conditions from (14) yield

\[ \text{C}_1^{(j)} = |s^{(j)} + v^{(j)} t_{\text{cpa}}| < R, \]
\[ \text{C}_2^{(j)} = T_j < t_{\text{cpa}} < T_{j+1}, \]
\[ \text{C}_3^{(j)} = |s^{(j)} + T_j v^{(j)}| > R, \]
\[ \text{C}_4^{(j)} = |s^{(j)} + v^{(j)} T_{j+1}| < R, \tag{19} \]

where

\[ t_{\text{cpa}}^{(j)} = -\frac{(v^{(j)})^T s^{(j)}}{|v^{(j)}|^2}. \tag{20} \]
Recall that we are dealing with stochastic variables, which means that we compute
\[
P(N_{\text{MAC}}(0,T)) = P\left( \bigcup_{j=0}^{J-1} (C_1^{(i,j)} \cap C_2^{(i,j)} \cap C_3^{(i,j)}) \cap (\bar{C}_3^{(i,j)} \cup \bar{C}_4^{(i,j)}) \right). \tag{21}
\]

IV. MONTE-CARLO APPROXIMATION

The probability according to (21) is in general very difficult to compute. A straightforward approximate solution is to use a Monte-Carlo method, i.e. to draw \( N \) samples of \( x(0) \) from (5) and approximate the probability with the outcome of the sampling, i.e.
\[
P_{\text{MC}}(N_{\text{MAC}}(0,T)) = \frac{1}{N} \sum_{i=1}^{N} I \left( \bigcup_{j=0}^{J-1} (C_1^{(i,j)} \cap C_2^{(i,j)} \cap C_3^{(i,j)}) \cap (\bar{C}_3^{(i,j)} \cup \bar{C}_4^{(i,j)}) \right), \tag{22}
\]
where \( I(\cdot) \) is the indicator function. Denote the true value of the sought probability with \( p \). The set of samples is binomially distributed, Bin\((N,p)\), but for a large enough \( N \), usually \( Np(1-p) > 20 \) is sufficient, the probability is approximated well by [17]
\[
P_{\text{MC}}(N_{\text{MAC}}(0,T)) \sim \mathcal{N}(p, \sigma^2), \tag{23}
\]
For a relative mean square error \( \epsilon \leq \frac{\sigma}{p} \) we can write needed number of samples according to
\[
N \geq \frac{1-p}{\epsilon^2p} \approx \frac{1}{\epsilon^2p}, \tag{24}
\]
where the last approximation is valid for small \( p \). Assume \( p = 0.01 \) and \( \epsilon \leq 0.1 \), i.e a relative error smaller than 10% with probability 0.997. These values plugged into (24) suggests that we must use \( N \geq 90000 \). For many on-line applications this means a too high computational load.

V. APPROXIMATE THE SAFETY ZONE WITH A DISC

A. Constant velocity

Let us first consider the constant velocity case according to (6). An approximate near mid-air collision, denoted by \( N_{\text{MAC}}(0,T) \), is given by considering the crossing of a circular disc instead of a sphere. The disc can be seen as a cross-section of the sphere perpendicular to line-of-sight as illustrated in Figure 4. Note that the location and radius of the disc is a matter of choice. For example placing the disc at \( x = R \), i.e. in front of the safety sphere, and with radius \( R \) yields a conservative result for probability of \( N_{\text{MAC}}(0,T) \). Below we place the disc at \( x = 0 \) primarily for notational convenience.

Following the same principle as in [11] we define a stochastic time variable \( \tau \), representing time-to-go (ttg), according to
\[
\tau = \begin{cases} \frac{s_{\perp}}{v_x} & \text{if } s_{\perp} > 0 \cap v_x < 0, \\ \infty & \text{otherwise} \end{cases} \tag{25}
\]
Time-to-go is the ratio of distance and closing speed, and corresponds to time left before the relative position crosses the \( yz \)-plane. The probability density function for \( \tau \) is given by Lemma 1.

**Lemma 1 (Probability density function for \( \tau \))** For \( s_x \) and \( v_x \) distributed according to
\[
\begin{bmatrix} s_x \\ v_x \end{bmatrix} \sim \mathcal{N}\left( \begin{bmatrix} \hat{s}_x \\ \hat{v}_x \end{bmatrix}, \begin{bmatrix} \sigma_{sx}^2 & \rho_{sx}\sigma_{sv} \sigma_{vx} \\ \rho_{sx}\sigma_{sv} & \sigma_{vx}^2 \end{bmatrix} \right),
\]
the probability density function for \( \tau \) as defined by (25) is given by
\[
p_{\tau}(t) = \int_{-\infty}^{0} -v_{sx}\cdot v_x (-vt, v) dv = \frac{g_0}{g_2^2} \left( 1 - (2\pi)^2 \frac{g_1}{g_2} \frac{v_x^2}{\sigma_{vx}^2} \Phi\left( -\frac{g_1}{g_2} \right) \right),
\]
where \( \kappa = \rho_{sx}\sigma_{sx}/\sigma_{vx} \) and
\[
\begin{align*}
g_0 &= \frac{1}{2\pi\sigma_{sx}\sigma_{vx}(1 - \rho_{sx}^2)} e^{-\frac{(s_{sx} - \hat{s}_x)^2}{2\sigma_{sx}^2(1 - \rho_{sx}^2)}}, \\
g_1 &= \frac{1}{\sigma_{sx}^2(1 - \rho_{sx}^2)} + \frac{\hat{v}_x}{\sigma_{vx}^2}, \\
g_2 &= \frac{1}{\sigma_{sx}^2(1 - \rho_{sx}^2)} + \frac{1}{\sigma_{vx}^2},
\end{align*}
\]
and \( \Phi(\cdot) \) corresponds to the standard normal distribution
\[
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\xi^2}{2}} d\xi.
\]
Proof: See [18].

We also define a distance from line-of-sight, denoted by \( s_{\perp}(t) \). This distance is a function of time-to-go \( \tau = t \), according to
\[
s_{\perp}(t) = \sqrt{(tv_y)^2 + (tv_z)^2} = t \begin{bmatrix} v_y \\ v_z \end{bmatrix}, \tag{30}
\]
and corresponds to the total displacement perpendicular to line-of-sight after \( \tau = t \) seconds. The interpretation of the event \( N_{\text{MAC}}(0,T) \) in terms of \( \tau \) and \( s_{\perp}(t) \) is that, given a time-to-go \( 0 < \tau = t < T \), if \( s_{\perp}(t) < R \) the event will occur. Since both \( \tau \) and \( s_{\perp}(t) \) are stochastic we need to compute the probability of \( N_{\text{MAC}}(0,T) \). The probability for a given ttg is provided by
\[
P(s_{\perp}(t) < R | \tau = t) = P(s_{\perp}(t) < R). \tag{31}
\]
By weighting with the probability density for \( \tau \) we have
\[
P(\text{NMAC}(0,T)) = \int_0^T P(s_\perp(t) < R)p_\tau(t)dt. \tag{32}\]

The above holds when \( s_x, v_x \) are independent of \( s_\perp(t) \). In the general case a dependency makes it difficult to express the probability of NMAC in terms of \( p_\tau(t) \). This is seen from the block diagonalization of the covariance matrix \( P \),
\[
P = \begin{bmatrix}
I & 0 & 0 \\
0 & P_{sz} & -C^TP_{sz}C \\
0 & C^TP_{sz} & I
\end{bmatrix}^T,
\]
which results in
\[
s_\perp(t) = \begin{bmatrix} tv_y \\ tv_z \end{bmatrix} - C^TP_{sz}^{-1} \begin{bmatrix} tv_x x \\ tv_x z \end{bmatrix}, \tag{33}\]
where the last step corresponds to a change of variable from \( s_x \) to \( t = -s_x/v_x \). As can be seen in (34) the dependency on \( v_x \) remains.

However, we can split \( p_\tau(t) \) into \( M \) partitions, where each partition \( m = 1, \ldots, M \), corresponds to a subset of closing speeds \( a_m < v_x < b_m \). Now, for each partition, we compute the conditional mean of \( v_x \) given \( \tau = t \) and \( a_m < v_x < b_m \). The partitioned pdf, \( \hat{p}^{(m)}_\tau(t) \), and the conditional mean, \( \hat{v}_x^{(m)}(t) \), are given by Corollary 1.

**Corollary 1 (Expressions for \( \hat{p}^{(m)}_\tau(t) \) and \( \hat{v}_x^{(m)}(t) \))** With the same assumptions as in Lemma 1 we have
\[
\hat{p}^{(m)}_\tau(t) = \frac{g_0}{g_2} e^{\frac{g_1}{g_2}} \left( e^{-\frac{1}{2}(g_2b_m - \frac{g_1}{g_2})^2} - e^{-\frac{1}{2}(g_2a_m - \frac{g_1}{g_2})^2} \right)
\cdot \sqrt{2\pi} \left( \Phi(g_2b_m - \frac{g_1}{g_2}) - \Phi(g_2a_m - \frac{g_1}{g_2}) \right),
\]
and the conditional expected value of \( v_x \) given \( \tau = t \) and \( a_m < v_x < b_m \) is
\[
\hat{v}_x^{(m)}(t) = \frac{g_0}{g_2} \left( e^{\frac{g_1}{g_2}} \left( (g_2b_m + \frac{g_1}{g_2})e^{-\frac{1}{2}(g_2b_m - \frac{g_1}{g_2})^2} - (g_2a_m + \frac{g_1}{g_2})e^{-\frac{1}{2}(g_2a_m - \frac{g_1}{g_2})^2} \right) - \sqrt{2\pi} \left( 1 + \frac{g_2}{g_1} \right) \left( \Phi(g_2b_m - \frac{g_1}{g_2}) - \Phi(g_2a_m - \frac{g_1}{g_2}) \right) \right).
\]

**Proof:** Straightforward calculations using the same technique as for \( p_\tau(t) \).

The probability \( P(\text{NMAC}(0,T)) \) is now given by applying Taylor expansion around \( \hat{v}_x^{(m)}(t) \) for each partition, see Theorem 1.

**Theorem 1 (\( \text{NMAC}(0,T) \) for a straight path)** For \( s(t) = [s_x(t), s_y(t), s_z(t)]^T \) with assumptions (2), (5) and (6) the probability of a down-crossing within \( T \) sec of a circular disc with \( x-\)axis as its normal and radius \( R \) is given by
\[
P(\text{NMAC}(0,T)) = \int_0^T \sum_{m=1}^M p^{(m)}_\tau(t)dt + P_M \tag{37}\]
where
\[
s_\perp^{(m)}(t) = \begin{bmatrix} tv_y \\ tv_z \end{bmatrix} - C^TP_{sz}^{-1} \begin{bmatrix} t^2\hat{v}_x^{(m)}(t) \\ t\hat{v}_x^{(m)}(t) \end{bmatrix}.
\]

The rest term is upper bounded by
\[
P_M \leq \int_0^T \sum_{m=1}^M \int_{-\infty}^0 \left| \frac{\partial p}{\partial v_x} \right| v_x dt \tag{38}\]

where \( a_m < \mu_\perp(t) < b_m \), \( a_1 = -\infty \) and \( b_M = 0 \).

**Proof:** See Appendix A.

**Remark 1** We can always apply Taylor expansion around a more accessible point compared to \( \hat{v}_x^{(m)} \). For example, in the case we use \( M = 1 \) a reasonable choice is \( \hat{v}_x \), the unconditional expected value of \( v_x \). The advantage using \( \hat{v}_x \) instead of \( \hat{v}_x^{(m)} \) is that the computational load is decreased. On the other hand, using \( \hat{v}_x \) in general makes the rest term \( P_M \) larger because the first order term in the Taylor expansion is no longer eliminated.

**B. Piecewise constant velocity**

The extension of the result for a straight path to the case with a piecewise straight path given by (15) is as follows. An important observation is that if it is unlikely that \( v_x^{(j)} \) changes sign, i.e.
\[
P(\text{sign}(v_x^{(j)}) = \text{sign}(v_x^{(0)})) \approx 1, \tag{40}\]
then \( \text{NMAC}(T_j, T_{j+1}) \) for different \( j \)'s are mutually exclusive. With this assumption we can write
\[
P(\text{NMAC}(0,T)) = \sum_{j=0}^{J-1} P(\text{NMAC}(T_j, T_{j+1})), \tag{41}\]
and we can concentrate on each segment \( T_j < t < T_{j+1} \) with constant velocity \( v^{(j)} \). The principle for computing \( P(\text{NMAC}(T_j, T_{j+1})) \) is the same as for \( P(\text{NMAC}(0,T)) \). The difference is that we use the initial values \( v_x^{(j)} \) and \( s^{(j)} \) from (15b) and (18) respectively instead of \( v(0) \) and \( s(0) \). Define a stochastic time variable \( \tau^{(j)} \) according to
\[
\tau^{(j)} = \begin{cases}
\frac{s^{(j)}}{v^{(j)}} & \text{if } s^{(j)} > 0 \cap v_x^{(j)} < 0, \\
\infty & \text{otherwise}.
\end{cases}
\]

We also define a distance from line-of-sight, denoted by \( s_\perp^{(j)}(t) \), according to
\[
s_\perp^{(j)}(t) = \sqrt{(tv_y^{(j)} + s_y^{(j)})^2 + (tv_z^{(j)} + s_z^{(j)})^2}, \tag{43}\]
and corresponds to the total displacement perpendicular to line-of-sight after $\tau^{(j)}(t) = t$ seconds starting from $s^{(j)}$. Note that $s_{x}^{(j)}$ and $s_{z}^{(j)}$ are known and deterministic based on (2) and (18). The probability of NMAC for a piecewise straight path is given by Corollary 2.

**Corollary 2 (NMAC$_{(0,T)}$ for a piecewise straight path)**

For $s^{(j)}(t) = \left[ s_{x}^{(j)}(t) \ s_{z}^{(j)}(t) \right]^T$ with assumptions (2), (5) and (15) the probability of a down-crossing within $T$ sec of a circular disc with $x$–axis as its normal and radius $R$ is given by

$$P(\text{NMAC}_{(0,T)}) = \sum_{j=0}^{J-1} \int_{\tau^{(j)}}^{T_{j+1}} \int_{m=1}^{M} P(s_{\perp}^{(j,m)}(t) < R) p^{(j)}_{\tau^{(j)}}(t) dt + P^{(j)}_{M}$$

where

$$s_{\perp}^{(j,m)}(t) = \left[ \begin{array}{c} \left[ t v_{y}^{(j)} + s_{y}^{(j)} \\
 t v_{z}^{(j)} + s_{z}^{(j)} \end{array} \right] - C^{T} P^{-1} \left[ \begin{array}{c} t v_{\nu}^{(j,m)}(t) \\
 t v_{\xi}^{(j,m)}(t) \end{array} \right] \right].$$

The rest term is upper bounded by

$$\text{P}^{(j)}_{M} \leq \int_{0}^{T} \int_{m=1}^{M} | \frac{\partial^{2} P(s_{\perp}^{(j,m)}(t) < R)}{\partial v_{\nu}^{2}} | v_{\nu} = \mu^{(m)}(t) \right) dt$$

where $a_{m} < \mu^{(j,m)}(t) < b_{m}, \ a_{1} = -\infty$ and $b_{M} = 0$. 

Proof: See Appendix A with $s^{(j)}, v^{(j)}$ instead of $s(0), v(0)$.

**VI. IMPLEMENTATION OF THE DISC APPROXIMATION**

**A. Computing $P(s_{\perp}^{(j,m)}(t) < R)$**

For notational convenience we ignore index $j$ and study the probability $P(s_{\perp}^{(j,m)}(t) < R)$, keeping in mind that $s_{\perp}^{(j,m)}(t)$ is actually $s_{\perp}^{(j,m)}(t)$ as given in (45). Let us define a new random variable representing orthogonal displacement per time unit according to

$$v_{\perp}(t) = \frac{s_{\perp}(t)}{t} = \left[ \begin{array}{c} v_{y}(t) \\
 v_{z}(t) \end{array} \right],$$

where we have using (45)

$$\left[ \begin{array}{c} v_{y}(t) \\
 v_{z}(t) \end{array} \right] = \left[ \begin{array}{c} v_{y} + \frac{\tau}{t} \\
 v_{z} + \frac{\tau}{t} \end{array} \right] - C^{T} P_{xx}^{-1} \left[ \begin{array}{c} tv_{\nu}(t) \\
 tv_{\xi}(t) \end{array} \right].$$

The covariance matrix for $v_{y}$ and $v_{z}$ is from (33) given by $P_{yz} = C^{T} P_{xx}^{-1} C$. We assume that $v_{y}$ and $v_{z}$ are uncorrelated, i.e. the matrix $P_{yz} = C^{T} P_{xx}^{-1} C$ is diagonal. This is no restriction since correlation is handled by applying a change of variables

$$\left[ \begin{array}{c} v_{y}' \\
 v_{z}' \end{array} \right] = U \left[ \begin{array}{c} v_{y} \\
 v_{z} \end{array} \right],$$

where $U$ is a unitary matrix given by

$$P_{yz} = C^{T} P_{xx}^{-1} C = U D U^{T} = U \left[ \begin{array}{cc} d_{y}^{2} & 0 \\
 0 & d_{z}^{2} \end{array} \right] U^{T}.$$
3) Compute $\hat{P}_L(v_1^2 < \frac{R^2}{l^2})$:

$$\hat{P}_L(v_1^2 < \frac{R^2}{l^2}) = \frac{\Delta R^2}{2\pi} + \frac{2}{\pi} \sum_{\ell=1}^{L} \text{Re} \left( \phi v_2 (\Delta \ell) \right) \sin \frac{\Delta l R^2}{l^2}.$$

**Example 1** ($P(v_1^2 < \frac{R^2}{l^2})$)

Consider the case with $C = 0$, $\delta_y = \delta_z = 0$ and 

$$\begin{bmatrix} \hat{v}_y \\ \hat{v}_z \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}, \quad P_{yz} = \begin{bmatrix} 2^2 & 0 \\ 0 & 1^2 \end{bmatrix}. \quad (57)$$

This is a typical result of angle-only tracking where the velocity orthogonal to line-of-sight is estimated accurately. The result of computing $P(v_1^2 < \frac{R^2}{l^2})$ with $R = 150$ using Algorithm 1 is given by Figure 5. We see that using no more than $15 - 20$ terms ($= L$) gives equivalent accuracy compared to using the Monte Carlo solution,

$$\hat{P}_{mc}(v_1^2 < \frac{R^2}{l^2}) = \frac{1}{M} \sum_{i=1}^{M} I \left( (v_{1y}^{(i)})^2 + (v_{1z}^{(i)})^2 < \frac{R^2}{l^2} \right), \quad (58)$$

with $M = 1000000$.

---

**B. Computing $P(\text{NMAC}_{C(0,T)})$**

A sound and simple numerical approximation for computing a one-dimensional integral is given by Simpson’s rule [23]

$$P(\text{NMAC}_{(T_j,T_{j+1})}) = \int_{T_j}^{T_{j+1}} f(t) dt$$

$$= \frac{\Delta t}{3} \left( f(t_0) + 2 \sum_{k=1}^{K/2-1} f(t_{2k}) + 4 \sum_{k=1}^{K/2} f(t_{2k-1}) \right) + R_K, \quad (59)$$

where $\Delta t = (T_{j+1} - T_j)/K$, $t_k = k\Delta t + T_j$ and 

$$f(t) = \sum_{i=1}^{M} f(s_{i,m}^{(j)}(t) < R)p_{x,(i,m)}(t). \quad (60)$$

From [23] we know that the approximation error is bounded by

$$R_K < \frac{\Delta t^5}{90} \sum_{k=0}^{\max_{t_{2k+1} < t < t_{2k+2}}} \left| \frac{\partial^5 f}{\partial t^5} \right|. \quad (61)$$

The implementation of Corollary 2 is given by Algorithm 2.

**Algorithm 2 (Implementation of Corollary 2)**

1) Set $j = 0$.
2) For each $t_k = k\Delta t_j + T_j$ and $m = 1, \ldots, M_j$ compute 

$$p_{x,(i,m)}(t_k), \quad \hat{P}_{x,(i,m)}(t_k), \quad (62)$$

given by Corollary 1.
3) For each $t_k = k\Delta t_j + T_j$ and $m = 1, \ldots, M_j$ compute 

$$P(s_{j,m}^{(j)}(t_k) < R) \quad (63)$$

using Algorithm 1.
4) Compute the probability for segment $j$

$$\hat{P}_{\text{simp}}(C(T_j,T_{j+1})) = \frac{\Delta t_j}{3} \left( f(T_j) + f(T_{j+1}) + 2 \sum_{k=1}^{K_j-1} f(t_{2k}) + 4 \sum_{k=1}^{K_j} f(t_{2k-1}) \right), \quad (64)$$

where 

$$f(t) = \sum_{m=1}^{M} P(s_{j,m}^{(j)}(t) < R)p_{x,(i,m)}(t). \quad (65)$$

5) Set $j = j + 1$ and iterate from step 2 until $j = J$.
6) Compute the total probability of conflict

$$\hat{P}_{\text{simp}}(C_{(0,T)}) = \sum_{j=0}^{J-1} \hat{P}_{\text{simp}}(C(T_j,T_{j+1})). \quad (66)$$

---

Fig. 5. The upper plot shows estimated $P(v_1^2 < \frac{R^2}{l^2})$, the lower left shows the difference compared to Monte Carlo solution (solid line) including the $3 - \sigma$ levels for the Monte Carlo solution (dotted lines), and the lower right shows the number of terms used in Algorithm 1.
VII. SIMULATION RESULTS

Using the notation from Figure 6 we let $\alpha$ and $\psi$ determine the direction of the intruders and own speeds, $v_{\text{int}}$ and $v_{\text{own}}$ respectively, relative to line-of-sight. The corresponding relative velocity is then given by

$$v_x = -v_{\text{own}} \cos \psi - v_{\text{int}} \cos \alpha,$$

$$v_y = v_{\text{own}} \sin \psi - v_{\text{int}} \sin \alpha. \quad (67)$$

Inserting (67) into (68) yields an expression for $\psi$ according to

$$\psi = \arcsin \left( \frac{v_{\text{int}}}{v_{\text{own}}} \sin (\alpha + \arcsin \left( \frac{s_{\min}}{s_x(0)} \right)) \right) + \arcsin \frac{s_{\min}}{s_x(0)}. \quad (69)$$

Here we place the circular disc with $R = 150 \text{ m}$ at $x = 22.5$ instead of $x = 0$ to approximate the crossing of the sphere better when performing an avoidance maneuver, see Figure 7. The value $x = 22.5$ corresponds to the $x-$coordinate where a straight path starting from $s_x(0) = 1000$ is tangent to the circle.

Fig. 6. Collision geometry in absolute coordinates.

There are two simulated scenarios with avoidance maneuvers performed with a $60\text{ deg}$ turn in the horizontal plane. The first is given by Table I and Figure 7 and corresponds to the case with $\alpha = 0 \text{ deg}$. The second is given by Table I and Figure 8 and corresponds to $\alpha = 25 \text{ deg}$. The estimated state vector and corresponding covariance matrix given in Table I are the result from a tracking filter using simulated angle-only sensor measurements. Both simulations assume a reaction time set to 3 sec, and reflects the time from initiation of the avoidance maneuver until the turn actually begins. All turns are performed with 10 deg/sec, meaning that the turning time is 6 seconds. The trajectories numbered 2 and 3 correspond to the situation two and four seconds later respectively during which the constant velocity according to the initial conditions applies.

Table II

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Initial Distance</th>
<th>$v_y(0)$</th>
<th>$v_x(0)$</th>
<th>$v_{\text{own}}(0)$</th>
<th>$v_{\text{int}}(0)$</th>
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<tr>
<td>I.1</td>
<td>100</td>
<td>0.91</td>
<td>0.0008</td>
<td>-9%</td>
<td>-6%</td>
</tr>
<tr>
<td>I.2</td>
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<td>0.93</td>
<td>0.0060</td>
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<td>-2%</td>
</tr>
<tr>
<td>I.3</td>
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<td>0.95</td>
<td>0.0050</td>
<td>1%</td>
<td>-2%</td>
</tr>
<tr>
<td>II.1</td>
<td>600</td>
<td>0.045</td>
<td>0.045</td>
<td>-2%</td>
<td>9%</td>
</tr>
</tbody>
</table>

Table III

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Initial Distance</th>
<th>$v_y(0)$</th>
<th>$v_x(0)$</th>
<th>$v_{\text{own}}(0)$</th>
<th>$v_{\text{int}}(0)$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>-1%</td>
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<tr>
<td>II.2</td>
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<td>0.92</td>
<td>0.0008</td>
<td>-2%</td>
<td>-2%</td>
</tr>
<tr>
<td>II.3</td>
<td>100</td>
<td>0.92</td>
<td>0.0008</td>
<td>-2%</td>
<td>-2%</td>
</tr>
</tbody>
</table>

VIII. CONCLUSIONS

This paper presents a novel solution to the problem of NMAC for a three-dimensional predicted relative trajectory. A
NMAC occurs if the distance between two aircraft becomes less than a threshold, which is determined by the radius of a safety sphere. The method does not rely on sampling techniques, but uses theory for level-crossings. By appropriate approximations, crossing of a circular disc instead of a sphere and Taylor expansion to deal with correlation, we derive an expression for the probability of NMAC. By using the proposed expression it is possible to decrease the computational load by at least three orders of magnitude compared to the Monte Carlo solution.

**APPENDIX A**

**PROOF OF THEOREM 1**

Using the joint probability density for \( s_x, v_x, v_y, v_z \),

\[
p_{s_x,v_x,v_y,v_z}(s,v,y,z) = p(s,v,y,z) = \frac{1}{(2\pi)^2 \det P^{1/2}} e^{\frac{1}{2} (x-\hat{x})^T P^{-1}(x-\hat{x})}
\]

(70)

we have

\[
P(NMAC(0,T)) = \int_0^T \int_{-\infty}^0 \int_{-\infty}^0 p(s,v,y,z) dy dz ds dv.
\]

(71)

where \( t = -s/v \). Block diagonalize \( P \) according to

\[
P = \begin{bmatrix} I & 0 \\ K & I \end{bmatrix} \begin{bmatrix} P_{sv} & 0 \\ 0 & P_{y_z} \end{bmatrix} \begin{bmatrix} I & 0 \\ K & I \end{bmatrix}^T,
\]

(72)

\[
P^{-1} = \begin{bmatrix} I & 0 \\ -K & I \end{bmatrix}^T \begin{bmatrix} P_{sv}^{-1} & 0 \\ 0 & (P_{y_z})^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -K & I \end{bmatrix},
\]

(73)

where \( P_{y_z} = P_{y_z} - C^T P_{sv}^{-1} C \) and \( K = C^T P_{sv}^{-1} \). The inner double integral over \( y, z \) is, using the block diagonalized covariance matrix, given by

\[
F(s,v) = \int_{(y)^2+(z)^2 < R^2} \int_{(y)^2+(z)^2 < R^2} p(y,z|s,v) dy dz
\]

(74)

where

\[
p(y,z|s,v) = \frac{1}{2\pi \det(P_{y_z})^{1/2}} e^{-\frac{1}{2} \left[ y - \hat{y}_z \right] \left[ z - \hat{v}_y \right] \left[ s - \hat{s}_x \right] \left[ v - \hat{v}_x \right] (P_{y_z}^{-1})^{-1}}.
\]

The probability of \( \overline{NMAC}(0,T) \) is given by

\[
P(\overline{NMAC}(0,T)) = \int_{-\infty}^0 \int_0^{-s/v} F(s,v) p_{s_x,v_x}(s,v) ds dv.
\]

(75)
Consider partition the inner integral over \( v \) in \( M \) intervals, i.e.

\[
\int_{-\infty}^{0} -vF(-v, t) p_{s, v_x} (-v, t) dv = \sum_{m=1}^{M} \int_{a_{m}}^{b_{m}} -vF(-v, t) p_{s, v_x} (-v, t) dv dt,
\]

where \( a_1 = -\infty \) and \( b_M = 0 \). For each partition, Taylor expansion of \( F(-v, t) \) around \( \tilde{\nu}_x^{(m)}(t) \) yields

\[
F(-v, t) = \sum_{r=0}^{1} \frac{1}{r!} \left. \frac{\partial^r F(-v, t)}{\partial v^r} \right|_{v=\tilde{\nu}_x^{(m)}(t)} \left( v - \tilde{\nu}_x^{(m)}(t) \right)^r.
\]

Inserting into (77) yields

\[
\int_{-\infty}^{0} -vF(-v, t) p_{s, v_x} (-v, t) dv = \sum_{m=1}^{M} \int_{a_{m}}^{b_{m}} \left( \tilde{\nu}_x^{(m)}(t) \right)^r \left. \frac{\partial F(-v, t)}{\partial v} \right|_{v=\tilde{\nu}_x^{(m)}(t)} \cdot \int_{a_{m}}^{b_{m}} -v\left( v - \tilde{\nu}_x^{(m)}(t) \right)^r p_{s, v_x} (-v, t) dv dt,
\]

Choose \( \tilde{\nu}_x^{(m)}(t) \) such that

\[
\int_{a_{m}}^{b_{m}} -v\left( v - \tilde{\nu}_x^{(m)}(t) \right)^r p_{s, v_x} (-v, t) dv = 0,
\]

which means that the first order terms corresponding to \( r = 1 \) in (79) are eliminated. The result is

\[
P(\text{NMAC}(0, T)) = \int_{0}^{T} \sum_{m=1}^{M} F\left( -\tilde{\nu}_x^{(m)}(t) t, \tilde{\nu}_x^{(m)}(t) t \right) \cdot \int_{a_{m}}^{b_{m}} -v \left( v - \tilde{\nu}_x^{(m)}(t) \right)^2 p_{s, v_x} (-v, t) dv dt + P_M
\]

\[
= \int_{0}^{T} \sum_{m=1}^{M} F\left( -\tilde{\nu}_x^{(m)}(t) t, \tilde{\nu}_x^{(m)}(t) t \right) p_{s, v_x} (-v, t) dv dt + P_M,
\]

where

\[
P_M \leq \int_{0}^{T} \sum_{m=1}^{M} \int_{a_{m}}^{b_{m}} \frac{1}{2} \left. \frac{\partial^2 F(-v, t)}{\partial v^2} \right|_{v=\mu^{(m)}(t)} \left( v - \mu^{(m)}(t) \right)^2 p_{s, v_x} (-v, t) dv dt
\]

\[
\cdot \left( v - \mu^{(m)}(t) \right)^2 p_{s, v_x} (-v, t) dv dt
\]

for \( a_m < \mu^{(m)}(t) < b_m \).