Vibrations of partly supported concrete railway sleeper

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Abstract
Analytical and finite element solutions to the problem of a vibrating beam on an elastic foundation are presented. An application example is a concrete railway sleeper embedded in an elastic medium (the ballast). The sleeper is also elastically connected to the rails. Eigenfrequencies are calculated and vibration modes are discussed. The beam (sleeper) is divided into sections where each section may or may not be supported by the elastic foundation. The elastic connections to the rails are situated at the two joinings of the three sleeper sections. Some conclusions are that Euler-Bernoulli beam theory can be used to calculate two, or maximum three, eigenfrequencies of the sleeper. The foundation stiffness influences the lowest bending-mode eigenfrequency the most; higher eigenfrequencies are practically unaffected by the foundation stiffness. The influence of railpad (and rail) stiffness on the sleeper eigenfrequencies is negligible.
Preface

The work presented in this report was carried out during the second semester of 2009-2010 at the Department of Solid Mechanics at Linköping University. The theory part is mostly taken from the book by Tore Dahlberg with title Vibrations of mechanical systems. The special cases of MATLAB code from the reference [9] have been compiled to obtain the theoretical result. Then modelling in Finite Element software TRINITAS have been done. After that a comparison between the theoretical result and the result from FEM has been made. It should be mentioned that a summary of this report has been submitted for international publication [31].

I would like to express my gratitude towards my supervisor, Professor Tore Dahlberg, for his support and guidance through the project. I also would like to thank Dr.Bo Torstenfelt because of his support and authorized to use of his finite element software, TRINITAS. Last but not least I want to thank my beloved family and my dear friend M.Sc. Babak Sharifimajd for their supporting during the work.

Ehsan Rezaei
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پیشگفتار

وّمن یبهاری فی سبیل الله یبدّ فی الأَرْض مُزَاغًا كثيرًا وَسَعًا ... و هر که در راه خدا هجرت کند در زمین افغانی گهای فراوان و گشایشها خواهد یافت١

خداها ستاره مخصوص تو است. پس تو را شاکیم در آنجاه بر ما گذشت.

سلام بر داوودی که خداوند از غرفه و درگذشت، سلام بر آیوب که خداوند او را از بیماری سرفه بازماند، درود بر عیسی روح خدا و کلیه او، سلام بر محمد محبوب خدا و یار مخلص او، درود بر فرمانواری مؤمنان، علیه بن ابیطالب، که برادری رسول خدا به وی اختصاص یافت، سلام بر قاطعه زهراء دختر رسول الله، درود بر حسن بن علی وصی و جانشین پدرش، سلام بر حسین که جانشی نجل وی، سلام بر ابن لب های شیخِیمنه، سلام بر آن اعضا قطعه قطعه شده، سلام بر آن سرهمای بالا رفته بر نبره هزار بار، حجت بر پدری گاهنامه‌ی، سلام بر آن کشته معلوم، سلام بر آن معلوم بی یافتا، سلام بر صاحب آن بارگاه عالی رتبه، سلام بر آنیه از جامه‌های پرده‌ها جرجه تویشیت، سلام بر یو ای مولاها من و برفشانگانی به گرد بارگاه تو بی می کشند. و اطراف تربیت اجتماع کرده اند، و در استان توطاق می کنند. و برای زیارت تو وارد می شوند، سلام برتو، من به سوی تو ور او هرهم، و بستگانی در بیشگاه توامدیستیه ام، سلام برتو سلام آن کسی که به حجت ستاره، و درولات و دائما توخصی و بی را است٢.

این یک نامه در نیم‌گزار دوم سال جامعه ۲۰۰۹-۲۰۱۰ در بخش مطالب جامعه، دانشکده فنی دانشگاه لیستونیک سود انجام پذیرفته است. در این گزارش، ترفندهای دیمایکی sleeper داشته باش، ایشان در طول مدت Tore Dahlberg در انجام در پرداختشک و بهبودی از استفاده راهنماهیs توسط رشته‌های روان‌شناسی، انسان‌شناسی، حساب دادند و مرا به بهترین طریق راهنماهی که TRINITA٣ که به انجام‌های اجرای دادن تا از نرم افزار Bo Torstenfelt تمدید. ترفندهای دیمایکی از استفاده خودشان در زمینه روان‌شناسی انسان محدود نتیجه کرده است این استفاده کنم. همچنین لازم می‌باشد از دوست عهد نبرد یا راهنمایی کردن و همچنین از همسر عزیز مهسا که در این تحقیق کرد تشریح نمایم.

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Notations

$A$  Area ($m^2$)

$a$  the variable value of voiding

$E$  Young’s modulus

$EI$  bending stiffness ($Nm^2$)

$f_1, f_2$  Frequencies of rigid body modes

$G$  shear modulus

$I$  second moment of the area

$J$  mass moment of inertia

$K$  stiffness

$K_w$  the ballast bed stiffness per meter ($N/m^2$)

$M(x,t)$  Bending moment ($Nm$)

$m(kg/m)$  mass per unit length

$p (m)$  distance upwards

$q(x,t)$  distributed load ($N/m$)

$q (rad)$  angle counter clockwise

$r$  radius of gyration

$T(x,t)$  Shear force ($N$)

$w_M(x,t)$  deflection due to bending moment

$w_S(x,t)$  deflection due to bending moment ($m$)

$w(x,t)$  deflection ($m$)

$x$  coordinate

$\nu$  Poisson’s ratio

$\psi(x,t)$ (psi)  shear angle ($rad$)

$\kappa$  shear constant

$\kappa GA$  shear stiffness ($Nm^2$)

$\rho(kg/m^3)$  density

$\Delta X$  the change of length

$\omega_n$  Natural frequency

$\Delta$  the ballast deformation

$\beta$  the size of the constant unsupported part

$\theta$  angle ($rad$)
1 Introduction

1.1 Introduction

The number of passengers transported by railway has increased gradually during recent decades. The carrying capacity of the Swedish railway raised from 16372 million seat-km in 1993 to 26476 million seat-km in 2008. The seats used got its highest value in Swedish railway history in 2008. Almost 42% of the seats were used.

Totally 11017 million passenger-kilometres were transported by the Swedish railway in 2008. In addition, 23116 million ton-kilometres of goods were freighted during 2008. The amount of energy which was consumed, compared to the other types of transportation, was really low. In 2008, the electric energy consumption was 2229 GWh plus 26167 m$^3$ of Diesel [1]. See Appendix B. In Sweden rail transport use only 3.4% of the total amount of energy which is consumed in transportation [2]. The Swedish railway undertakes around 9% of total inland passenger transportation and approximately 35% of inland freight transportation in this country. Both types of transports ranked higher than the European Union (EU) average. The value of Co emission for rail transport is only 10 thousand tonnes while from road transportation it is 19396 thousand tonnes. It is clear that the emission from railway transports is negligible compared to the road transportation emission.

Figure 1.1: Sleeper in Railway; source: Trafikverket
Because of the lower energy consumption and lower emission the railway transportation is nowadays of great interest. On the other hand, the high speed required in transportation and the growing the amount of the freight have increased the demand for heavier and faster trains. Construction of many of the railway lines still in use started more than 150 years ago, and since then traffic conditions regarding speed, axel load, traffic intensity, and track construction, have developed. Today, there are urgent demands from the railway organisation and from the industry to increase axel loads and train speeds on the existing railway lines. The reasons for this are both economical and environmental [3].

Nowadays, study about the track and contain parts seems to be a crucial study regarding to the railways research. The variation of the track stiffness and optimization of it also is of interest today. The following figure which is Figure 2.1 of [4] shows a cross section of track:

![Figure 1.2: Construction of a modern railway track](image)

Railway tracks are loaded by passing trains. During the passage of a train, the track is loaded statically, due to the weight of the train, and dynamically due to imperfections of the vehicle (the wheel out-of-roundness) and the track. These loads may damage track and cause permanent settlement, and they can also cause ground vibration that can annoy inhabitants of nearby buildings. Therefore, knowledge about track dynamics and vehicle-track interaction is important for being able to deal with this type of problems.

The dynamic wheel/rail interaction force depends both on the dynamic properties of the train and on the deflection of the track due to the train load. Also, the irregular contact force then creates an increased loading and deterioration of the ballast bed below the sleeper.

The sleeper transmits vertical, lateral, and longitudinal forces from the rail down to the ballast bed. The result of unsupported sleeper will be an increase of the variations of the dynamic train/track interaction forces.

Producing a prestressed concrete railway sleeper is a complex design task that requires in-depth knowledge in several areas. The load generated from the passing trains is strongly influenced by possible irregularities of the wheels or the rail and variations of the stiffness of the underlying ballast supporting the entire track structure. The contents of dynamic loads can be substantial and must be considered or at least noticed during the design process of a sleeper. Static design system
for concrete sleeper can be used to design the sleepers for the maximum load reached during a train passage, but the effects of rapid shifts in the motion of the sleeper can only be treated by a dynamic design system [5].

The railway sleepers play an important role in:

- Uniformly transferring and distributing loads from the rail foot to underlying ballast bed;
- Sustaining and retaining the rails at the proper gauge by keeping anchorage for the rail fastening system; preserving rail inclination; and
- Providing supports for rails; restraining longitudinal, lateral and vertical rail movements by embedding itself onto substructures.

The main duty of a railway concrete sleeper is to distribute the rolling stocks’ axle loads to the supporting formation and finally the foundation. The axle load could be considered static or quasistatic when the train speeds are low to moderate [6].

When studying railway track dynamics, the behaviour of concrete railway sleepers is a central issue [7,8]. Depending on the frequency range of interest, the sleeper can be modelled in several different ways: (a) as a rigid body (at low frequencies), (b) as an Euler-Bernoulli (E-B) beam (at low to medium frequencies), (c) as a Rayleigh-Timoshenko (R-T) beam (medium to high frequencies), or (d) as a three-dimensional elastic body (at very high frequencies).

The most important factors influencing the vibration frequencies of a railway sleeper are the bending stiffness, mass and mass distribution of the sleeper itself and the influence of the surroundings. Most sleepers are embedded in ballast, but in many cases there are small gaps (voids, pockets) between the ballast and sleepers. The influence of the ballast on the sleeper vibrations is in many cases modelled by a distributed spring stiffness acting along the sleeper. The influence of the part of the track structure that is situated above the sleeper can be summarized by discrete springs acting at the rail positions. These springs take care of the stiffness of the rail pads and stiffness due to the rails and the remaining track structure. In the work presented here, the sleeper is assumed to be supported by an elastic foundation (a massless Winkler bed) acting along the full length of the sleeper, or, in case of a gap between the sleeper and ballast, along part (or parts) of the sleeper. Thus, the sleeper is in contact with the ballast only along one or several sections of its full length. There are also two discrete springs at the positions of the rails. Sleeper vibrations of the in situ sleeper will be investigated in this report. Eigenfrequencies and eigenmodes are calculated and discussed.

If the dynamic behaviour of a track is to be investigated at frequencies well below the first bending-mode eigenfrequency of a free-free sleeper, then the sleeper can, in many cases, be treated as a rigid body. The influence of the bending stiffness of the sleeper on the eigenfrequencies is then neglected. The sleeper vibrations are then influenced only by the sleeper mass, its distribution, the foundation stiffness, and the railpad and rail spring stiffness. At frequencies in the vicinity of the lowest two to three bending-mode eigenfrequencies of the sleeper, the Euler-Bernoulli beam theory of a beam on elastic foundation can be used. At the third eigenfrequency and upwards, the Rayleigh-Timoshenko beam theory should be used. Then shear deformation and inertia in rotation of a beam lamina are taken into account. Finally, at frequencies so high that the cross section of the sleeper does not remain plane during the
vibration, then a three-dimensional model and the finite element method should be used. (The Timoshenko beam theory assumes that a beam cross section remains plane during vibration, but a beam lamina will be sheared, see reference [9]. At very high frequencies the beam cross section will not remain plane and the Timoshenko theory cannot be used.)

When the author of reference [9] worked with that paper, he came across the reference [10]. Evidently there are some doubtful results reported in reference [10]. Reference [10] deals with the vibrations of a concrete sleeper either fully supported by the ballast, partly supported or not supported at all (in this last case the sleeper is hanging in the rails). Therefore the eigenfrequencies of a free-free sleeper and a partly supported in situ sleeper, connected to the rails via discrete springs, are here re-investigated, and vibration modes are discussed. Only frequencies in the lower frequency range are investigated, so the sleeper may be rigid or it may deform according to the Euler-Bernoulli beam theory. Analytical and finite element solutions are presented and compared.

To verify the dynamic models of the sleeper, calculated eigenfrequencies of a free-free sleeper will be compared with measured ones. Measured eigenfrequencies of a free-free sleeper can be found in, for example, reference [11] (measurements were performed at the Technical Research Institute of Sweden, SP). These measurements are sometimes referred to when eigenfrequencies of concrete railway sleepers are investigated [12,13,10]. In reference [13] the eigenfrequencies of a free-free concrete railway sleeper (used in Sweden) were calculated and frequencies were compared with the measured ones from [6]. Calculations were performed using both E-B (Euler-Bernoulli) and R-T (Rayleigh-Timoshenko) beam theory, and one finding was that the E-B theory gives acceptable results only for the lowest two or three eigenfrequencies of the sleeper. Eigenfrequencies of an in-situ sleeper, treated as a beam on an elastic foundation and elastically connected to the rails, were presented.

In the work reported here, an analytical solutions and finite element solutions of a vibrating beam on an elastic foundation, and also elastically connected to the rails, will be presented. The beam is divided into two or more sections that have constant properties and the different parts may or may not be supported by the elastic foundation (i.e. the ballast). The elastic connections to the rails are situated at the rail positions.

The sleeper considered in Dahlberg and Nielsen [13] is a standard gauge sleeper generally used by Swedish railways. Its natural frequencies were found by applying several theories such as Euler-Bernoulli, Timoshenko and Rayleigh-Timoshenko. Also comparisons between the calculated values and the measured ones were performed by Dahlberg and Nielsen [13].

In the study presented here, first the FEM modelling of sleepers is presented. Then, by considering five patterns (five different cases of supporting), which commonly happen in sleeper and ballast interaction, the results for these situations are discussed. In order to clearly understand and interpret the behaviour of the flexural modes, the shapes of them, for a simple theoretical case, are presented. In addition, particularly for rigid body modes, the circumstances when the mode is composed of a combination of translation and rotation, the place of the centre of oscillation will be given. (In the symmetric situation this is not so difficult to realize.) By studying the vibration of the two first modes, which are almost rigid-body modes, this can be interpreted. (Note: what will henceforth be called “rigid-body modes” are in practice just “almost
rigid-body modes”, because the bending of the sleeper will influence the mode a little, but at low frequencies (as here) this influence can be neglected. This influence of the sleeper bending stiffness on the eigenfrequencies is shown in Table 3.1.)

1.2 Literature Review

The problem of evaluating the influence of voids or pockets in the sleeper/ballast connection on the eigenfrequencies of in situ sleeper goes back to the 1980s. The dynamic response of railway track with a section of unsupported sleepers was examined experimentally by Grassie and Cox in 1985 [14] and a mathematical model of such track was presented. Through their study, it was concluded that when the support is not perfect, concrete sleepers are more prone to crack, especially if there are wheel or railhead irregularities.

At the mid of the 1980s, Ahlbeck and Hadden [15] carried out research on the load on concrete sleepers. In their paper, which was published in 1985, impact loads on concrete sleepers were measured and predicted. The sleeper model takes the first four bending modes of the sleeper into consideration.

Eight years after his first study, Grassie published another paper in this field [16]. He proposed a technique for calculating the dynamic response of railway track where the track had non-sinusoidal irregularities on the surfaces of the wheel and the rail. He concluded that adequate attention has to be paid to sleeper bending moments, caused by dynamic loads, throughout the design of a new sleeper.

Laboratory and field tests on elastic rail pads for the use on concrete sleeper were presented by Maree in 1993 [17]. In this investigation pad and ballast stiffness and damping were determined by use of receptance curves. Also, a short communication in sensitivity analysis of free vibration of an in situ concrete sleeper to variation of rail pad parameters was done by Kaewunruen and Remennikov (2006) [18]. By implementing finite element analysis, they illustrated the dependence on the rigid-body (translation and rotation) and first five flexural modes of the railpad stiffness. Sleeper is considered as a Timoshenko-beam and they have used spring element in the in situ sleeper modelling.

As mentioned above, two theories are commonly used in sleeper modelling: the Euler-Bernoulli theory and the Rayleigh-Timoshenko theory. The Rayleigh-Timoshenko theory is more accurate than the classic Euler-Bernoulli theory as it takes rotational inertia (Rayleigh) and shear deformation (Timoshenko) of the beam (sleeper) into account (Dahlberg et al., 1993 [19]). When the sleeper is modelled as a Timoshenko beam of variable thickness, several beam elements must be used. An alternative to these beam theories is to use the finite element method. By representing the sleeper as a uniform beam, an acceptable agreement is obtained in correlating the calculated frequencies of in situ sleepers in track to measured frequencies below about 700 Hz. In fact, it has been found that the dynamic response to loads at the railhead is well represented up to 1 kHz by modelling the sleeper simply as a rigid body. To be able to predict crack development in sleepers under impact loads, Ahlbeck and Hadden (1985) [15] developed existing vehicle track interaction models. They expanded and applied a validated seven degree-of-freedom nonlinear
time-domain model. The response of track to high frequency excitation (50-1500 Hz) has also been investigated by Grassie et al., (1982) [20] to study short-pitch corrugations of rail.

Grassie et al. [20], present two new dynamic models in their article: one continuous and the other one incorporated the discrete mass of the sleepers. Since rail pads are of essential importance in decreasing the dynamic loads in this frequency range, they were included in these models.

Track foundation modelling is another related topic which has been carried out up to now. From experimental measurements it was found that the ballast generally deflects in a highly non-lineal manner under the load. In general, there may be voids between sleepers and ballast, and the ballast itself may deflect non-linearly (Esveld, 1989 [21]). Energy dissipation in the foundation occurs due to dry friction and wave radiation through the substrate. Despite this, most analyses use a simple two-parameter model in the vertical direction (Knothe and Grassie, 1993 [8]).

A sleeper support can be modelled with the ballast being represented by linear stiffness and damping. This model is more accurate if high-frequency dynamic behaviour is of interest and when the wheel axle is close to the sleeper of interest. Loading and unloading when a bogie passes over a specific sleeper can be evaluated approximately by such a linear model.

Other sleeper support models are discussed in a 'State-of-the-Art' paper by Knothe and Grassie (1993) [8]. In principle, there are two different possible models for the track support, i.e. models with a completely continuous support of the rail and those with a discrete support. Although a discrete support is more representative for a track laid on discrete sleepers, the analogous continuous support is obtained by "smearing out" the discrete support through the length of the track to get a continuous elastic foundation (the ballast) and a continuous layer representing the sleepers. With these continuous layers the sleepers can be modelled as rigid bodies or as beams with distributed mass and stiffness.

At frequencies below around 500 Hz, particularly for vertical excitation, when calculating the dynamic response of the track, continuous support models are valid. A hierarchy of track models is also presented by Knothe and Grassie [8]. The simplest representation of a continuous elastic foundation was provided by Winkler in 1867. Winkler assumed the base to consist of closely spaced, independent linear springs. The only foundation constant is the foundation modulus (the spring stiffness). The Pasternak model is the most natural extension of the Winkler model for homogeneous foundations where a second foundation constant, the "shear modulus", is also taken into consideration (Kerr 1964 [22]; Dahlberg et al. 1993 [19]).

The track structure can also be modelled as being of either finite or infinite length. The type of structure is closely linked to the solution technique. Track structures of infinite length are commonly used for frequency-domain solutions whereas finite track structures are more appropriate for time-domain solutions.

To evaluate suspension features of a railway car, Newland and Cassidy (1975) [23] presented the performance of a simple single-degree-of-freedom analytical model of the suspension system. In this model the mass of the bogie frame was neglected. The model gave acceptable results although it did not take into account the bogie frame vibrations (the frame may transmit a
significant force to the vehicle). Using a two-degree-of-freedom (2DOF) model the authors subsequently analyzed the car or track behaviour as a function of a variety of track inputs.

The dynamic response of the vehicle/track system to non-sinusoidal irregularities was considered by Grassie (1993) [16]. Grassie demonstrated that such calculations underestimate sleeper bending moments while overestimating the contact forces due to stiff and resilient rail pads. The model can be applied as a good aid in track design. Grassie’s work also included experimental measurements of the dynamic loads on the track. Irregular wheels were also taken into account in the research. With respect to Grassie’s work the dynamic load can be considered to be 1.5 times the static load.

Girardi and Recchia (1991) [24] published a paper in which they investigated the dynamic behaviour of the track and its foundation. They studied the entire vehicle/track system as a unique mechanical system. A three-dimensional dissipation medium was used to model the track foundation. By applying a classical finite element method, track and vehicle movements are modelled and solved.

The dynamic interaction between a completely circular moving rigid wheel mass and a primarily straight and non-corrugated continuous railway track was modelled in a paper by Nielsen (1994) [25]. To optimize the dynamic response of the track a parametric study was performed. The purpose was to determine the maximum bending stress of the rail.

In a paper by Fryba (1987) [26] basic theoretical models and principle methods of their solution for the analysis of railway vehicles and tracks were given. The dynamic interaction between vehicle and track were accentuated and numerous basic equations were specified to show the behaviour of these elements. Possibilities of how to simplify the theoretical models with the intention to achieve a simple solution were described.
2 Theory

2.1 Beam Theory

Timoshenko's theory of beams constitutes an improvement of the Euler-Bernoulli theory, in that it incorporates shear deformation effects [27]. In this section a sleeper is considered to be a Rayleigh-Timoshenko beam, in that the two effects, \textit{i.e.} shear of a beam lamina due to shear force and influence of rotator inertia, are included in the Rayleigh-Timoshenko theory, but these effects are excluded in the Euler-Bernoulli beam theory. In particular, the first effect which is related to the shear force was suggested by the Ukrainian/Russian-born scientist Stephen Timoshenko in the beginning of the 20th century and the second effect, the influence of rotary inertia, was first studied by Rayleigh.

In Rayleigh-Timoshenko beam theory the total deflection of the beam is separated into two parts: one depends on the bending of the beam and the other part depends on shear deformation of the beam. A beam lamina, with shear force and bending moment plus the corresponding deformations, is illustrated in Figure 2.1 (this is Figure 1 in reference [9]).

![Figure 2.1: A beam lamina: load $q(x,t)$, deflection $w(x,t)$, shear force $T(x,t)$ and shear angle $w'_S(x,t)$, bending moment $M(x,t)$ and rotation angle $\psi(x,t)$. The elastic foundation influences the beam with a force $k_{bed} w(x,t)$ per unit length.](image)

The relevant deformation to the bending moment $M(x,t)$ is here indicated by $w_M(x,t)$ and the other part which is related to the shear force $T(x,t)$ is denoted $w_S(x,t)$, reference[9]. The following formula shows relation between these terms

$$w(x,t) = w_M(x,t) + w_S(x,t)$$ \hspace{0.5cm} (2.1)

The deflection due to the bending moment can be given by use of the rotation angle. In this case the relation between deflection and rotation angle is
\[ \psi(x, t) = \frac{\partial w_M(x, t)}{\partial x} \]  

(2.2)

The relationships between the bending moment \( M(x, t) \) and second derivative of the deflection \( w_M(x, t) \) or the first derivative of the angle \( \psi(x, t) \), and, furthermore, the relation between the shear force \( T(x, t) \) and the corresponding deformation can be expressed as follows

\[ M(x, t) = -EI \frac{\partial^2 w_M(x, t)}{\partial x^2} = -EI \frac{\partial \psi(x, t)}{\partial x} \]

and

\[ T(x, t) = \kappa GA \frac{\partial w_S(x, t)}{\partial x} = \kappa GA \left[ \frac{\partial w(x, t)}{\partial x} - \psi(x, t) \right] \]  

(2.3a,b)

Here \( EI \) is bending stiffness of the beam, \( E \) is Young’s modulus, \( I \) is second moment of the cross-sectional area \( A \), \( G \) is shear modulus which can be calculated from the modulus of elasticity \( E \) and the Poisson ratio \( \nu \), the factor \( \kappa \) is a constant, the Timoshenko shear factor, that depends on the form of the cross section and the Poisson ratio \( \nu \). The factor \( \kappa \) is given in reference [28].

By considering the forces in vertical direction, the equilibrium equation of the given lamina becomes:

\[ m \frac{\partial^2 w(x, t)}{\partial t^2} - \frac{\partial T(x, t)}{\partial x} = q(x, t) - k_{bed} x(x, t) \]  

(2.4a)

Here \( m \) (kg/m) is mass per unit length of the beam and \( q(x, t) \) is distributed load (N/m). The elastic foundation is assumed to give a counteracting force (per unit length) that is proportional to the deflection \( w(x, t) \); the proportionality constant is \( k_{bed} \).

Also, the rotational equilibrium equation of the beam lamina (per unit length) can be written:

\[ - \frac{\partial M(x, t)}{\partial x} + T(x, t) - \rho l \frac{\partial^2 \psi(x, t)}{\partial t^2} = 0 \]  

(2.4b)

where \( \rho \) (kg/m\(^3\)) is density, giving mass distribution \( m = \rho A \). Therefore, \( \rho l \) is mass inertia (per unit length) of the beam. The second moment of area \( l \) can be expressed \( l = Ar^2 \) where \( A \) (m\(^2\)) is cross-sectional area and \( r \) is the radius of gyration with respect to the y-direction, see Figure 2.1. Thus, \( \rho l \) can also be written \( \rho l = mr^2 \).

First, the bending moment \( M(x, t) \) and shear force \( T(x, t) \) are eliminated from the above equations (2.4a,b). For this purpose the \( T(x, t) \) value from (2.3b) should be inserted into (2.4a). It gives

\[ m \frac{\partial^2 w(x, t)}{\partial t^2} - \kappa GA \left[ \frac{\partial^2 w(x, t)}{\partial x^2} - \frac{\partial \psi(x, t)}{\partial x} \right] = q(x, t) - k_{bed} x(x, t) \]  

(2.5a)

By inserting the equation (2.3a) and (2.3b) into (2.4b), one obtains
Differential equations (2.5a, b) together with the boundary and initial conditions present the solution of the vibration problem of a beam described by Rayleigh-Timoshenko beam theory when the beam is loaded with a distributed load \( q(x,t) \).

Now, eliminate \( \psi \) from (2.5a, b). Solve for \( \frac{\partial \psi}{\partial x} \) from (2.5a) and insert into (2.5b). It gives (with \( w(x,t) \) written in short term as \( w \) as well as \( q(x,t) \) is written \( q \) only)

\[
\frac{EI}{\kappa GA} \frac{\partial^2 \psi(x,t)}{\partial x^2} + \frac{\partial w(x,t)}{\partial x} - \psi(x,t) - \rho \frac{\partial^2 \psi(x,t)}{\partial t^2} = 0
\]  

(2.5b)

This equation is the differential equation which, together with boundary conditions and initial conditions, determines the beam deflection \( w(x,t) \) according to the Rayleigh-Timoshenko beam theory. Here \( w(x,t) \) includes both the effects of bending deformation and shear deformation. By, instead, eliminating \( w(x,t) \) from (2.5a, b) one obtains a differential equation in terms of \( \psi(x,t) \). It gives

\[
\frac{EI}{\kappa GA} \frac{\partial^4 \psi}{\partial x^4} - \frac{EI k_{\text{bed}}}{\kappa GA} \frac{\partial^2 \psi}{\partial x^2} - \frac{m EI}{\kappa GA} \frac{\partial^4 \psi}{\partial x^2 \partial t^2} - \rho l \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial t^2} + \frac{m \rho l \frac{d^4 \psi}{\partial x^4}}{\kappa GA} + \frac{m \frac{\partial^2 \psi}{\partial t^2}}{\kappa GA} + \frac{\rho I \frac{\partial^2 \psi}{\partial t^2}}{\kappa GA} + k_{\text{bed}} \psi = q + \frac{\rho l}{\kappa GA} \frac{\partial^2 q}{\partial x^2} - \frac{EI}{\kappa GA} \frac{\partial^2 q}{\partial x^2}
\]

(2.6a)

It is worth noting that the form of the left hand side in (2.6a) and in (2.6b) are equal, only \( w \) and \( \psi \) are switched. Because of this the homogeneous solutions of \( w(x,t) \) and \( \psi(x,t) \) are equal. However, the particular solutions are different in these equations.

Equations (2.6a, b) have homogenous solutions \( w(x,t) \) and \( \psi(x,t) \) which contains four integration constants each. These eight constants are not independent from each other; consequently a total of four independent constants only are obtained. These are determined by boundary conditions. Boundary conditions are different from those one gets at the technical beam theory. Some of the boundary conditions that can be stated for a beam of length \( L \) (for the coordinate \( x \) for \( 0 \leq x \leq L \)) are as follows
2.2 Boundary Conditions

2.2.1 Clamped at both beam ends

For this type of boundary condition, deflection and the rotation are zero at both ends, giving

\[ w(0, t) = 0 \quad \text{and} \quad \psi(0, t) = 0 \]
\[ w(L, t) = 0 \quad \text{and} \quad \psi(x, t) = 0 \]  

(2.7a)

Note that \( w'(0, t) \) and \( w'(L, t) \) are not zero. This is because \( w(x, t) \) contains the shear angle and this angle needs not be zero at the beam end even if the beam is clamped.

2.2.2 Clamped-Simply supported ends

In this condition deflection and the rotation is zero at \( x = 0 \) and the deflection and moment are zero at \( x = L \). It gives, because of the clamping at one end

\[ w(0, t) = 0 \quad \text{and} \quad \psi(0, t) = 0 \]

and

\[ w(L, t) = 0 \quad \text{and} \quad M(L, t) = -EI \frac{\partial \psi(L, t)}{\partial x} = 0 \]  

(2.7b)

These conditions are because of the simply supported end of the beam.

2.2.3 Both ends simply supported

As mentioned at the preceding boundary condition, when the beam is simply supported at one end, this end has zero deflection and zero moment at that end. It gives

\[ w(0, t) = 0 \quad \text{and} \quad M(0, t) = -EI \frac{\partial \psi(x, t)}{\partial x} = 0 \]

(2.7c)

\[ w(L, t) = 0 \quad \text{and} \quad M(x, t) = -EI \frac{\partial \psi(x, t)}{\partial x} = 0 \]

2.2.4 Clamped-Free beam

From the previous boundary conditions it is clear that at the clamped end deflection and angle of rotation are zero. Thus at \( x = 0 \) one has

\[ w(0, t) = 0 \quad \text{and} \quad \psi(0, t) = 0 \]

The other end is in this case assumed to be free. When a beam is free at one end the moments and shear forces should be zero on that end. Therefore the following relation holds at \( x = L \)
\[
M(L, t) = -EI \frac{\partial \psi(L, t)}{\partial x} = 0 \quad \text{and} \quad T(L, t) = \kappa GA \left[ \frac{\partial w(L, t)}{\partial x} - \psi(L, t) \right]
\] (2.7d1)

By applying the condition in equation (2.5b) above, the following equation can be obtained:

\[
T(L, t) = -EI \frac{\partial^2 \psi(x, t)}{\partial x^2} + \rho l \frac{\partial^2 \psi(x, t)}{\partial t^2} = 0
\] (2.7d2)

### 2.2.5 Free-Free

Moments and shear forces are zero in the case that the beam ends are free. This gives

\[
M(0, t) = -EI \frac{\partial \psi(0, t)}{\partial x} = 0 \quad \text{and} \quad T(0, t) = \kappa GA \left[ \frac{\partial w(0, t)}{\partial x} - \psi(0, t) \right]
\] (2.7e1)

Same condition is true for the second end of the beam, at \(x=L\),

\[
M(L, t) = -EI \frac{\partial \psi(L, t)}{\partial x} = 0 \quad \text{and} \quad T(L, t) = \kappa GA \left[ \frac{\partial w(L, t)}{\partial x} - \psi(L, t) \right]
\] (2.7e2)

which is the same condition as used before.

In a last step of a Free-Free beam, instead of obtaining expression (2.7d2), the equations (2.3a) and (2.4b) can be combined to give

\[
T(L, t) = \frac{\partial M(x, t)}{\partial x} + \rho l \frac{\partial^2 \psi(x, t)}{\partial t^2} = -EI \frac{\partial^2 \psi(x, t)}{\partial x^2} + \rho l \frac{\partial^2 \psi(x, t)}{\partial t^2}
\] (2.7e3)

This expression can be useful in the case that E-B beam theory is desired as a limiting case of the R-T theory, i.e., in the situation \(\kappa GA \to \infty\) (2.7e3) should be considered in the calculation.

### 2.2.6 Prescribed quantities of beam length

6a. If a prescribed displacement, for instance \(\delta\), is given for the beam at any beam end, then

\[
w(*, t) = \delta
\] (2.7f)

where the * is the coordinate of the beam end, i.e. 0 or L.

6b. A prescribed rotation angle, for instance \(\theta\), is given at the beam end. Then

\[
\psi(*, t) = \theta
\] (2.7g)

where the * can be 0 or L.

6c. In this case there is a prescribed moment such as \(M_0\) (positive) which can be applied at the beam end (‘*’ represents 0 or L)
\begin{equation}
M(\ast, t) = -EI \frac{\partial \psi(\ast, t)}{\partial x} = M_0 \tag{2.7h}
\end{equation}

6d. A prescribed shear force such as \( P_0 \) (positive) can be applied at a beam end (\( \ast \) is 0 or \( L \)).

\begin{equation}
T(\ast, t) = \kappa GA \left[ \frac{\partial w(\ast, t)}{\partial x} - \psi(\ast, t) \right] = P_0 \tag{2.7i}
\end{equation}

In the case studied here the sleeper can be considered as a Free-Free beam and the boundary conditions are thus as given in case No. 2.2.5 above.
The sleeper mentioned above is modelled with the finite element software TRINITAS, which is suitable for linear problems. To obtain a FEM-model with the same behaviour as the real sleeper, a ballast model is connected to the sleeper. Only the ballast behaviour in the vertical direction is of interest so that from the material properties point of view the ballast material shall be orthotropic without any stiffness and lateral strain in the directions perpendicular to the vertical one.

Figure 3.1: Real sleeper

Figure 3.2: Sketch of Swedish sleeper; source: Trafikverket
First some comparisons are made between theoretic results by use of beam theory and results obtained from 3D-FEM modelling and 2D-FEM modelling. From the comparisons it is concluded that when the 3D model was fixed in the lateral direction (free to move only in the vertical direction) it gave reasonable results as compared to the 2D model. Furthermore the 2D model showed accurate results compared to the theory. Also some investigations of the theoretic model finally lead to the selection of the 2D model. Another issue that is paid attention to is the mesh properties. It will be investigated how the selection of the mesh properties affects the accuracy of the results. Another important step regarding the FEM modelling is the material selection, namely material properties of sleeper, ballast, railpad and rail. Finally, the boundary conditions need to be defined so that similarity to the real case is obtained.

3.1 Model

The FEM model is a model of a simplified concrete sleeper. It is assumed that the sleeper is embedded in a foundation with a given foundation stiffness $k_{bed} = 13$ MN/m$^2$. In addition, the sleeper is assumed to be connected to the surroundings (to the rails) by two springs of equal stiffness $k_{rail} = 17$ MN/m at the position of the rails (this stiffness thus symbolizes the combined stiffness of the railpads and the rail).

The sleeper is modelled as an equivalent uniform beam. It has a quadratic cross-sectional area with 0.2m width and 0.2m height and the area is constant along the length of the sleeper. The length of the sleeper is 2.5m. In the FEM modelling, this volume is created as an area that has constant thickness.

Also the ballast is modelled as a volume with constant thickness. The ballast geometry has the same length and width as the sleeper, but different depth, which is 1m.

The distance between the centre lines of the rails is assumed to be 1500mm (giving a rail gauge of about 1430mm). The two springs symbolizing the railpads and the rails are then placed (to agree with the theoretical model) at 0.5m and 2m from one end of the sleeper. Each rail spring is, in the FEM model, modelled as a cylinder with constant cross section; the cylinder is one meter high and it has a circular cross-sectional area of $0.8293 \times 10^{-4}$ m$^2$.

Figure 3.3 depicts the final shape of the model used in the analyses.

![Figure 3.3: A schematic view of the FEM model of sleeper](image-url)
3.2 Material properties

In this study the concrete sleeper has a simplified geometry (as described above) and the material properties are the same throughout the length of the sleeper so it is considered to be a structure with homogenous properties. The material (concrete) in the sleeper has Young’s modulus 36.6 GPa [29]. The total mass of the sleeper is 251 kg [9] (thus, with a sleeper volume of 0.1m$^3$ the overall density of the concrete and reinforcement is 2510 kg/m$^3$). The second moment of inertia of the sleeper cross section is $I=1.33\times10^{-4}$ m$^4$ giving the flexural rigidity $EI=4.79$ MN/m$^2$.

The shear modulus of the concrete can be obtained as follows:

$$G = \frac{E}{2(1 + \nu)} \quad (3.1)$$

Where the ‘$\nu$’ is Poisson’s ratio and for concrete $\nu = 0.2$ is used in this study. It gives $G = 15$ GPa. The value of shear rigidity can be calculated by applying the following relation:

$$\kappa GA = 498 \text{ MN}$$

where ‘$A$’ is the cross-sectional area of the sleeper and ‘$\kappa$’ is the shear constant, which depends on the geometry of the cross section. The value $\kappa = 0.83$ is used here for the simplified sleeper [28]. (The original trapezoidal cross section with only a gentle taper is simplified to the quadratic area with side 0.2m).

As mentioned in the preceding section the bed and rail have stiffness $k_{bed} = 13$ MN/m$^2$ and $k_{rail} = 17$ MN/m, respectively. Because of the lack of the spring tools in TRINITAS it is not possible to create springs with desired properties. Instead, springs are modelled as bars having the same stiffness as the springs.

Some quick calculation in solid mechanics should be performed to find the proper material for replacing a spring with a bar. Hook’s low can, in this case, be stated as follows:

$$F = K \Delta X \quad (3.2)$$

Where $K$ is the stiffness and $\Delta X$ is the change of length when applying the force $F$. Here, for replacing the rail stiffness, a solid cylinder with the length of one meter is considered. The material of this cylinder is steel with Young’s modulus $E=205$GPa and Poisson’s ratio 0.3, respectively. It is worth emphasizing that, because of the requirements in this case, it is only the stiffness property of the material that is important. The density should be set to zero. This will remove the effect of the spring mass. A tensile load of 1000 N applied to the cylinder, and using the formula,

$$\delta = \frac{FL}{AE} \quad (3.3)$$

will give, for a cross-sectional area $0.8293\times10^{-4}$ m$^2$, the elongation $\delta$ around $0.5882\times10^{-4}$m. Inserting the assumed values to the above formula will give the stiffness $K = 17$ MN/m = $k_{rail}$. 


Next step is to find a suitable material for the bed. Also in this case the same approach is suitable. The only difference is the dimension of the bed which is 3D in this model. Here it extends in the lateral and length directions and the ballast is loaded in vertical direction. Therefore, deformation only in the load direction is permitted. Because of this the material used is made orthotropic rather than isotropic.

In this case the length and width of the ballast should be the same as the sleeper, and for simplicity, the depth of the ballast bed is assumed to be one metre. Thus, one has to find a proper model by varying the material properties. The following formulas illustrate the way of finding the Young’s modulus of the ballast model.

\[
F = K_w \Delta L \tag{3.4}
\]

where \( F \) is force, \( K_w \) is the ballast bed stiffness per meter (thus \( \text{N/m/m} = \text{N/m}^2 \)), \( L \) is the length of the sleeper, and \( \Delta \) is the ballast deformation.
Load the sleeper by total force:

\[ P = q L t \]  \hspace{1cm} (3.5)

From Hook’s low:

\[ \varepsilon = \frac{\sigma}{E} \]  \hspace{1cm} (3.6)

one obtains

\[ \delta = \varepsilon b = \frac{\sigma b}{E} = \frac{Pb}{AE} \]  \hspace{1cm} (3.7)

Finally one obtains:

\[ P = \frac{E L t}{b} \delta \]  \hspace{1cm} (3.8)

The comparison between above formula and (3.4) gives that:

\[ K_v L = \frac{E L t}{b} \]  \hspace{1cm} (3.9)

Then Young’s modulus of the ballast model can be found as

\[ E = \frac{b K_v}{t} = \frac{1 \text{m} \times 13 \text{ MN/m}}{0.2 \text{ m}} = 65 \text{ MPa} \]  \hspace{1cm} (3.10)

The obtained value is related to the \( E \) in vertical direction and all other Young’s and shear modulii should be zero. Furthermore, the assumed value of Poisson’s ratio is 0.3.

### 3.3 Mesh and Elements

The two rails and railpad springs are meshed by 2D bar elements and each one consists of only one element. To increase the number of elements in this case would not improve the results. Thus, this element is suitable for the rail springs.

When calculating the eigenfrequencies of the sleeper not fully supported, \( i.e. \) when there are voids under the sleeper, then 5% of the bed is removed in each step. Therefore the meshing of the sleeper and the ballast is made so that each section of 5% consists of one finite element in the length direction and one element in the vertical direction.

The eigenfrequencies of the model are evaluated when the lower support of it is lacking. In order to obtain an accurate enough result the step of 5% of the voided length is considered (in most of the cases). In addition, to facilitate the assessment, the model essentially is created by the volume elements which each holds 5% of the length of the sleeper as well as ballast. Each volume has the same thickness and width as the sleeper. Since the ballast is connected to the neutral line of the
sleeper, which is parallel to the length, the sleeper is broken into the two volumes in the vertical direction. To keep the fine meshing, each volume in sleeper has two elements in vertical direction and one element in length direction

Thus, totally the sleeper contains 80 elements (40 volumes and each one contain 2 elements). Each part of the ballast has only one element in each direction, which will give totally 20 elements of the ballast. Apart from the rail springs, the model contains 100 elements. Because of the required accuracy when assessing the eigenfrequencies, the model should be meshed with high-order elements, and Lagrange type elements are selected for this study, implying that the model contains one hundred 2D 9-node quadrilateral Lagrangian membrane elements.

When the calculations are performed the voided parts of the sleeper can easily be created in the model by deleting the appropriate volume elements of the ballast.

3.4 Boundary Conditions

Boundary conditions for the model are prescribed so that they reflect the real case as close as possible. Also, the model results depict a small difference with the ones obtained from [13].

The mentioned description leads to a model with boundary conditions that have fixed points in all direction for the upper point of the rail springs (the bars). Also, the lower surface of the ballast should be fixed in all directions. The other connections between rails and sleeper, and also between sleeper and ballast, are already considered since they have some common points.

3.5 Contact patterns

The arrangements of the contact between sleeper and ballast bed can take various shapes. In this study it is attempted to evaluate some of the most common situations. From previous assessments, five situations of contact boundary conditions are of practical concern in real railway track problems. In particular, these defects (voids or hanging sleepers) can be detected through vibration tests. The stated patterns are depicted in Figure 3.7. The first part of Figure 3.7 shows the type of imperfection that is called ‘centre void’. The void starts from the centre of the sleeper and after that the void grows out to the sides symmetrically. In the first case the sleeper is
fully supported and the void grows until the sleeper is completely hanging in the rail. To evaluate the alteration of the dynamic behaviour of the sleeper during the increase of the voided part, the ratio of the central void length to the sleeper length is, in this case, used as a non-dimensional variable. The relation below presents this value:

\[ \alpha_c = \frac{L_c}{L} \]

The next case considers another problem which occurs in the sleeper/ballast contact. This case is referred to as a ‘single hanging’ sleeper. In this type, the void is assumed to form from one of the ends of the sleeper and it expands incrementally to the other end. Also in this situation the initial case is the fully supported sleeper and the last situation is the totally hanging sleeper. As mentioned in the preceding situation, for assessing the variation of the dynamic behaviour of the sleeper during these imperfections a non-dimensional parameter is used which is the ratio of the single-side void length to the length of the model. This relation can be written as follows:

\[ \alpha_s = \frac{L_s}{L} \]

Another contact situation that can occur is double-side hanging sleeper. An unsupported part starts to grow from each end of the sleeper, and the contact remains only in the middle segment of the sleeper. This situation is known as the ‘double hanging’ sleeper. The third part of Figure 3.7 displays this situation. Because of the double voiding two parameters are needed in this case to cover all the possible situations. By assuming any value less than 50% for the voided part of one sleeper end, and having the void on the other side grow from zero to highest possible value, it can be guaranteed that no case is missed. Parameters describing in this case are

\[ \alpha_d = \frac{L_{dL}}{L} \quad \beta_d = \frac{L_{dR}}{L} \]

The next pattern has a more complicated shape. The sleeper is assumed unsupported along three parts; one part in the centre of the sleeper plus two more imperfections (voids) at both ends of the sleeper. It is worth emphasizing again that, in this study, only symmetrical voids in the centre and also at the ends are considered. Thus, two variables can define this situation. One of the non-dimensional parameters is related to the central void and the other to the unsupported parts at the ends of the sleeper. One obtains

\[ \alpha_t = \frac{L_t}{L} \quad \beta_t = \frac{L_{tc}}{L} \]

The last case contains two imperfections (voids); one of them is a centre void and the other is an end void. This situation is mentioned ‘side-central voids’ and this situation is displayed in the last part of Figure 3.7. The asymmetric contact configurations are considered in this case, and two corresponding non-dimensional parameters can be stated as follows:

\[ \alpha_{s-c} = \frac{L_s}{L} \quad \beta_{s-c} = \frac{L_c}{L} \]
Dynamic response of an *in situ* concrete railway sleeper is evaluated when each one of the mentioned situations occurs. The normalized (eigen-) frequency $\omega_N$ is defined as the ratio of the deteriorated (voided) sleeper’s frequency $\omega_{\text{voided}}$ and the frequency of the fully supported *in situ* sleeper $\omega_{\text{ideal}}$ with the ideal contact condition. To make the evaluation faster and easier the frequency $\omega_N$ is displayed versus the variation of the contact non-dimensional length (in percent), where

$$\omega_N = \frac{\omega_{\text{voided}}}{\omega_{\text{ideal}}}$$

---

**Figure 3.7:** Sleeper/ballast contact patterns, (a) central void, (b) single hanging, (c) double hanging, (d) triple hanging, and (e) side-central voids
3.6 Comparison and Validation

Before going forward to the results presented in this section, a comparison between the FEM model, which is used in this study, and two analytical models have been made. Two tables, Table 3.1 and Table 3.2, have been prepared to present the results and differences between them. In the first table, Table 3.1, the analytical solutions and hand calculations were used to obtain the results. The details of the task and related formulas will be presented when the result of the single hanging sleeper is discussed. The relevant MATLAB code is presented in Appendix B. Here it should be noted that the analytical solution is based on rigid-body motion. If Young’s modulus $E$ in the FEM model goes to infinity, then the solution is equivalent to the rigid-body condition, and then the values obtained will be more close to the analytical values. Without considering this matter, still the obtained values show good agreement with the analytical results. There is at most just over 3% difference between these two situations.

The following figure, Figure 3.8, shows the variables used and the manner of connection between the sleeper and the ballast.

![Figure 3.8: Rigid-body modelling of partly supported railway sleeper.](image_url)
<table>
<thead>
<tr>
<th>Voided portion</th>
<th>Analytical</th>
<th>FEM modelling</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a/L$</td>
<td>$f_1$</td>
<td>$f_1$</td>
</tr>
<tr>
<td>0,00</td>
<td>81,92</td>
<td>83,58</td>
</tr>
<tr>
<td>0,05</td>
<td>79,21</td>
<td>82,55</td>
</tr>
<tr>
<td>0,10</td>
<td>75,92</td>
<td>82,49</td>
</tr>
<tr>
<td>0,15</td>
<td>72,96</td>
<td>82,49</td>
</tr>
<tr>
<td>0,20</td>
<td>70,37</td>
<td>82,48</td>
</tr>
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<td>0,25</td>
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<td>82,40</td>
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<td>62,32</td>
<td>81,84</td>
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<td>0,50</td>
<td>61,49</td>
<td>81,46</td>
</tr>
<tr>
<td>0,55</td>
<td>60,85</td>
<td>80,92</td>
</tr>
<tr>
<td>0,60</td>
<td>60,36</td>
<td>80,20</td>
</tr>
<tr>
<td>0,65</td>
<td>59,99</td>
<td>79,26</td>
</tr>
<tr>
<td>0,70</td>
<td>59,71</td>
<td>78,04</td>
</tr>
<tr>
<td>0,75</td>
<td>59,51</td>
<td>76,49</td>
</tr>
<tr>
<td>0,80</td>
<td>59,36</td>
<td>74,53</td>
</tr>
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<td>59,25</td>
<td>72,08</td>
</tr>
<tr>
<td>0,90</td>
<td>59,16</td>
<td>69,03</td>
</tr>
<tr>
<td>0,95</td>
<td>59,04</td>
<td>65,27</td>
</tr>
<tr>
<td>1,00</td>
<td>58,58</td>
<td>60,87</td>
</tr>
</tbody>
</table>

Table 3.1: Eigenfrequencies of partly supported sleeper. Analytical values for rigid sleeper ($EI$ is infinite) and FEM values for sleeper when bending stiffness $EI$ is finite.

In a real situation the concrete sleeper is not a rigid body. Because of this, in its free vibration, there would be some bending in what is here calculated as rigid-body modes, and at higher frequencies bending modes should be considered.

At higher frequencies, *i.e.* at bending-mode frequencies of the sleeper, the MATLAB code used in Dahlberg [9] was used. For the evaluating of theoretical results, six special cases of ballast and sleeper contact were investigated. The results for these cases and for the corresponding cases calculated with the FEM model, plus a comparison between these cases, are given in Table 3.2. In the table it can be seen that there is no noticeable difference between these two cases which implies that the FEM model almost completely match with the theory of the Rayleigh-Timoshenko beam.
Table 3.2: Eigenfrequencies of partly supported beam calculated with FEM and with different beam theories. R-T: rotation and translation with rotation dominating; T-R translation and rotation with translation dominating, then bending mode No.

<table>
<thead>
<tr>
<th></th>
<th>fully supported</th>
<th>middle and one side supported</th>
<th>two sides supported</th>
<th>Only middle supported</th>
<th>only one side supported</th>
<th>free</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FEM</td>
<td>Theory</td>
<td>Differ.%</td>
<td>FEM</td>
<td>Theory</td>
<td>Differ.%</td>
</tr>
<tr>
<td>R-T</td>
<td>81,28</td>
<td>81,33</td>
<td>0,06</td>
<td>68,92</td>
<td>68,95</td>
<td>0,04</td>
</tr>
<tr>
<td>T-R</td>
<td>82,66</td>
<td>82,66</td>
<td>0,00</td>
<td>81,66</td>
<td>81,69</td>
<td>0,04</td>
</tr>
<tr>
<td>1</td>
<td>135,00</td>
<td>134,86</td>
<td>0,10</td>
<td>131,80</td>
<td>131,60</td>
<td>0,15</td>
</tr>
<tr>
<td>2</td>
<td>331,90</td>
<td>331,44</td>
<td>0,14</td>
<td>331,10</td>
<td>330,64</td>
<td>0,14</td>
</tr>
<tr>
<td>3</td>
<td>612,10</td>
<td>610,12</td>
<td>0,32</td>
<td>611,60</td>
<td>609,64</td>
<td>0,32</td>
</tr>
<tr>
<td>4</td>
<td>950,10</td>
<td>944,38</td>
<td>0,61</td>
<td>949,70</td>
<td>944,03</td>
<td>0,60</td>
</tr>
<tr>
<td>5</td>
<td>1332,00</td>
<td>1321,87</td>
<td>0,77</td>
<td>1332,00</td>
<td>1321,64</td>
<td>0,78</td>
</tr>
</tbody>
</table>
4 Results and Discussion

In this study eigenfrequencies of a fully or partly supported concrete railway sleeper, and the corresponding rigid body and bending modes, are studied. The first and the second mode are associated with almost rigid-body motion of the sleeper; a motion that is a combination of translation and rotation. Higher modes are associated with bending. From the basic theory, it can be concluded that the lower value the eigenfrequencies have, the more sensitive the modes are to the support boundary condition. Thus, evaluation is done to determine how many of the bending modes are influenced by the boundary conditions (i.e. the ballast support). It proved that up to the fifth bending mode is sensitive to the variation of the length of the contact zone between the sleeper and the ballast; higher modes are not influenced by change of length of the contact zone between ballast and sleeper.

Regarding the mechanical properties of the sleeper, numerical values applied in the calculations presented here are stated in the preceding chapter. Other parameters, such as geometry, boundary conditions and so on, are given in the modelling explanations above. The author has strived to keep the finite element model similar to the analytical model. Analyses are carried out for an in situ Rayleigh-Timoshenko beam in free vibration. Results obtained for concrete sleepers in the track system for all contact patterns discussed earlier are presented separately in the following sections.

First of all a rough estimation of the natural frequencies is considered. The ideal case is regarded to have full contact between the sleeper and the ballast. The natural frequency (for a rigid sleeper) can be calculated as follows

\[ \omega_n = \sqrt{\frac{2K_r + LK_w}{m}} \]  

When applying the given values to the above formula \( \omega_n \) will be around 514 rad/sec. For the totally free case the value of \( K_w \) is zero, and then the sleeper is hanging in the rails. Then only the value of rail stiffness should be considered, and the above formula gives \( \omega_n \) equal to 368 rad/sec. From these two cases, which are the two extreme cases, it can be concluded that the final value of natural frequency is about 70% of its initial value.

4.1 Central void

Figure 4.1 depicts the normalized frequency obtained for the symmetrical central void in the in situ railway concrete sleeper versus length of the void. From the figure one can obviously find that an increasing unsupported section diminishes all frequencies either it is the rigid-body modes or bending modes. Additionally, as the number of the bending mode increases the effect of the voided part decreases. Thus, the eigenfrequencies of the three highest modes studied have decreased less than one percent. In particular, the first two flexural modes, as well as the translational and rotational modes of the rigid-body dynamics, are more sensitive to the support boundary condition. Thus, the change is close to 30 percent in the rigid-body modes when the
completely unsupported sleeper is compared with the ideal (fully supported) case. The largest decrease of the eigenfrequency in the flexural modes belongs to the first mode and is 9 percent. Table 4.1 expresses better the exact value corresponding to each voiding steps.

<table>
<thead>
<tr>
<th>Modes</th>
<th>Voiding(%) 0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rigid Body(1)</td>
<td>81.28</td>
<td>78.81</td>
<td>76.13</td>
<td>73.38</td>
<td>70.67</td>
<td>68.14</td>
<td>65.83</td>
<td>63.73</td>
<td>61.76</td>
<td>59.82</td>
<td>57.68</td>
</tr>
<tr>
<td>Rigid Body(2)</td>
<td>82.66</td>
<td>82.64</td>
<td>82.52</td>
<td>82.18</td>
<td>81.50</td>
<td>80.35</td>
<td>78.57</td>
<td>75.96</td>
<td>72.25</td>
<td>67.08</td>
<td>59.80</td>
</tr>
<tr>
<td>Bending (1)</td>
<td>135.0</td>
<td>133.5</td>
<td>132.3</td>
<td>131.5</td>
<td>131.2</td>
<td>131.1</td>
<td>130.7</td>
<td>129.5</td>
<td>127.0</td>
<td>127.0</td>
<td>122.6</td>
</tr>
<tr>
<td>Bending (2)</td>
<td>331.9</td>
<td>331.8</td>
<td>331.6</td>
<td>330.9</td>
<td>330.1</td>
<td>329.3</td>
<td>328.8</td>
<td>328.6</td>
<td>328.6</td>
<td>328.3</td>
<td>327.1</td>
</tr>
<tr>
<td>Bending (3)</td>
<td>612.1</td>
<td>611.6</td>
<td>611.3</td>
<td>611.2</td>
<td>611.0</td>
<td>610.5</td>
<td>610.2</td>
<td>610.1</td>
<td>610.1</td>
<td>609.6</td>
<td></td>
</tr>
<tr>
<td>Bending (4)</td>
<td>950.1</td>
<td>950.0</td>
<td>949.8</td>
<td>949.5</td>
<td>949.4</td>
<td>949.3</td>
<td>949.2</td>
<td>948.9</td>
<td>948.8</td>
<td>948.8</td>
<td>948.5</td>
</tr>
<tr>
<td>Bending (5)</td>
<td>1332</td>
<td>1332</td>
<td>1332</td>
<td>1332</td>
<td>1332</td>
<td>1332</td>
<td>1332</td>
<td>1331</td>
<td>1331</td>
<td>1331</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.1: Central Void

It is seen in the table that when the void ratio goes from nil to about 30 percents, the rotational frequency is almost unaffected. This is because the central void grows from centre of the sleeper which is the point of the node of the rotational mode. When the voided part is below the fixed point (the oscillation centre), i.e. close to the node, the ballast stiffness (or lack of stiffness) has no effect on the eigenfrequency. On the other hand, when the stiffness changes at the part of the sleeper where it has the highest amplitude of vibration, the stiffness affects the value of the eigenfrequency a lot. A simple manifestation of this is that when the size of the un-supported part increases, growing from the centre of the sleeper to the sleeper ends, the importance of the removed stiffness will increase. In this process the slope of the eigenfrequency curve, as function of the size of the voided part, will increase accordingly.

For the translation mode, because there is no node in the vibration mode, there is a more or less linear relationship between the voided part and the normalized eigenfrequency.

The same discussion as for the trend of the rotational mode curve is true for the flexural modes. The only difference is the number of nodes which is increased by one for each mode. When bed stiffness is removed at (or close to) the nodes, the eigenfrequency is not affected at all (or affected very little when the stiffness close to the node is removed).
When a free-free beam is considered the first flexural mode has two nodes that are placed around 22% and 78% of the length. The points of the largest amplitude are placed at the ends of the beam and at the centre [30]. The free-free beam can be a good approximation of the vibration of the sleeper (the rail and bed stiffness will affect the vibration mode, but this influence is assumed to be small). Regarding the vibration amplitude of the free-free beam, one should keep in mind that in the central void case the first bending mode curve should demonstrate two large-slope regions corresponding to the creation of voids at the middle and the ends of the beam and zero slope is expected when the void is created at the places of the two symmetric nodes. It is seen in the figure that when the void grows from, say, 30% to 70% of the sleeper length, the eigenfrequency of the sleeper is hardly affected at all.

Thus, the fundamental conclusion is that when the void starts to grow from the centre of the sleeper, the eigenfrequency curve shows a pronounced slope. After that, by increasing the voided section further so that it grow in the vicinity of the nodes, then the slope decreases very little, and finally, when the void grows beyond the region of the nodes, again the slope goes to the higher value.

In order to get more clarity on this, the next two bending modes are shown in a separated figure. For the second bending mode the same interpretation works. As can be seen in Figure A.2 the second flexural mode has three nodes: one node exactly at the middle point of the beam. Because of this the curve related to 0% void has no gradient, see Figure 4.2. Then, because of the void symmetry, when the void grows from 20% to 60% of the beam, there is a larger slope. Here stiffness is removed where the vibration amplitude is large. At 60% to 80% the void grows at the other nodes giving a small slope of the curve, and finally, the bed stiffness at the ends influences more the eigenfrequency.
4.2 Single Hanging

In this section, vibration frequencies of a sleeper under conditions called ‘single hanging’ are evaluated. By ‘single hanging’ is here meant that one part at the end of the sleeper is hanging. To start with the sleeper is fully supported by the ballast along its full length. After that the contact length between sleeper and ballast diminishes by steps of five percent of the sleeper length until all support from the ballast is removed and the sleeper is free (then only hanging in the rails). All results related to the two almost rigid-body modes plus five bending modes are given in the following table:

![Figure 4.2: Eigenfrequencies of second and third flexural modes.](image)
Table 4.2: Eigenfrequencies of 'single hanging’ sleeper

<table>
<thead>
<tr>
<th>Voiding(%)</th>
<th>Modes</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>95</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rigid Body(1)</td>
<td>81,3</td>
<td>78,9</td>
<td>74,6</td>
<td>71,5</td>
<td>68,9</td>
<td>66,8</td>
<td>65,1</td>
<td>63,7</td>
<td>62,5</td>
<td>61,6</td>
<td>60,8</td>
<td>60,1</td>
<td>59,6</td>
<td>59,1</td>
<td>58,8</td>
<td>58,6</td>
<td>58,5</td>
</tr>
<tr>
<td>Rigid Body(2)</td>
<td>82,7</td>
<td>79,4</td>
<td>75,1</td>
<td>71,8</td>
<td>68,4</td>
<td>66,2</td>
<td>64,5</td>
<td>63,1</td>
<td>61,8</td>
<td>60,9</td>
<td>60,2</td>
<td>59,5</td>
<td>58,9</td>
<td>58,4</td>
<td>58,0</td>
<td>57,6</td>
<td>57,3</td>
</tr>
<tr>
<td>Bending (1)</td>
<td>135,0</td>
<td>133,8</td>
<td>132,6</td>
<td>131,6</td>
<td>130,6</td>
<td>129,6</td>
<td>128,8</td>
<td>128,0</td>
<td>127,4</td>
<td>126,8</td>
<td>126,2</td>
<td>125,7</td>
<td>125,2</td>
<td>124,8</td>
<td>124,4</td>
<td>124,0</td>
<td>123,7</td>
</tr>
<tr>
<td>Bending (2)</td>
<td>331,9</td>
<td>331,7</td>
<td>331,4</td>
<td>331,3</td>
<td>331,2</td>
<td>331,1</td>
<td>330,9</td>
<td>330,8</td>
<td>330,6</td>
<td>330,4</td>
<td>330,2</td>
<td>330,0</td>
<td>329,8</td>
<td>329,6</td>
<td>329,4</td>
<td>329,2</td>
<td>329,0</td>
</tr>
<tr>
<td>Bending (3)</td>
<td>612,1</td>
<td>611,9</td>
<td>611,6</td>
<td>611,4</td>
<td>611,2</td>
<td>611,1</td>
<td>610,9</td>
<td>610,7</td>
<td>610,5</td>
<td>610,3</td>
<td>610,0</td>
<td>609,7</td>
<td>609,4</td>
<td>609,1</td>
<td>608,8</td>
<td>608,5</td>
<td>608,2</td>
</tr>
<tr>
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<td>950,1</td>
<td>949,9</td>
<td>949,6</td>
<td>949,4</td>
<td>949,2</td>
<td>949,0</td>
<td>948,8</td>
<td>948,6</td>
<td>948,4</td>
<td>948,1</td>
<td>947,9</td>
<td>947,7</td>
<td>947,5</td>
<td>947,3</td>
<td>947,0</td>
<td>946,8</td>
<td>946,6</td>
</tr>
<tr>
<td>Bending (5)</td>
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<td>1332</td>
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<td>1332</td>
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<td>1332</td>
</tr>
</tbody>
</table>

Table 4.3: Comparison between in situ sleeper with and without rail stiffness

To investigate the influence of rail and rail pad stiffness a comparison between the ballast supported sleeper, with and without rail springs, have been made. The results are tabulated in the following table. As can be seen, the lack of rail stiffness only affects the rigid-body modes and it does not affect a lot the bending modes.
In the pattern of the results, because of deficiency of symmetry, the rigid-body modes except for the first and last points are neither pure translation nor pure rotation. Only in the two extreme cases fully supported and not supported sleeper the rigid-body modes are purely translation (in one mode) and rotation (in the other mode). In a real situation for the intermediate cases (partly supported) the modes are combinations of rotation and translation (and also some bending, but this can be neglected for these low frequencies). For the evaluation in this case the sleeper is assumed to be a rigid body (i.e. the influence of the bending stiffness of the beam is neglected here). By calculating the eigenfrequencies and eigenmodes for some different lengths of the unsupported part, more details about the nature of the modes can be obtained. The following figure illustrates the sleeper considered.

![Figure 4.3: Rigid-body modelling of partly supported railway sleeper.](image)

The mass centre of the sleeper moves a distance $p$ (m) upwards and rotates an angle $q$ (rad) counter clockwise. The equations of motion become (coordinate $x = 0$ at sleeper mass centre)

$$M\ddot{p} + \int_{-\frac{L}{2}+a}^{\frac{L}{2}} k_w(p + xq)dx + 2k_r p = 0$$

$$J\ddot{q} + k_w(p + xq)dx + 2k_r b^2q = 0$$

(4.2a,b)

Here ‘$b$’ is the distance (in the $x$ direction) between the centre of the sleeper and the rail position (thus $2b$ between the rails). In this study the centre to centre distance between the rails is assumed to be 1500mm and the length of the sleeper is 2500mm so that $b$ is 750mm. Also ‘$J$’ is mass moment of inertia (in rotation)

$$J = J_q = \frac{ML^2}{12}$$

(4.3)

Other notations are the same as before.

After integration and some simplification one obtains
\[
M\ddot{p} + k_w (L - a)p + k_w \frac{1}{2} a(L - a)q + 2k_r p = 0 \tag{4.4a,b}
\]
\[
J\ddot{q} + k_w \frac{1}{2} a(L - a)p + k_w \frac{1}{12} \{L^3 - 3L^2a + 6La^2 - 4a^3\} q + 2k_r b^2 q = 0
\]

For further simplification the following definitions are introduced:

\[
L_1 = L - a \quad L_2 = \frac{1}{2} a(L - a) \quad L_3 = \frac{1}{12} \{L^3 - 3L^2a + 6La^2 - 4a^3\} \tag{4.5}
\]

By applying the above definitions to equations (4.4a,b), one obtains:

\[
M\ddot{p} + k_w L_1 p + k_w L_2 q + 2k_r p = 0 \tag{4.6a,b}
\]
\[
J\ddot{q} + k_w L_2 p + k_w L_3 q + 2k_r b^2 q = 0
\]

For solving the coupled system of differential equations the following assumptions are made:

\[
p = A \sin \omega t \quad \text{and} \quad q = B \sin \omega t \tag{4.7a,b}
\]

In order to obtain the characteristic equation the determinant of the following matrix should be equal to zero.

\[
\begin{bmatrix}
-M\omega^2 + k_w L_1 + 2k_r & k_w L_2 \\
k_w L_2 & -J\omega^2 + k_w L_3 + 2k_r b^2
\end{bmatrix}
\begin{bmatrix}
A \\
B
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix} \tag{4.8}
\]

This gives the frequency equation as follows:

\[
MJ\omega^4 - (Mk_w L_3 + Jk_w L_1 + 2Mk_r b^2 + 2Jk_r)\omega^2 + k_w^2 (L_1 L_3 - L_2^2) + 2k_r k_w (L_3 + L_1 b^2) + ak_r^2 b^2 = 0 \tag{4.9}
\]

Solving for \( \omega \) gives the eigenfrequencies

\[
\omega_{1,2}^2 = s \pm \sqrt{(s)^2 - \frac{k_w^2 (L_1 L_3 - L_2^2) + 2k_r k_w (L_3 + L_1 b^2) + ak_r^2 b^2}{MJ}} \tag{4.10}
\]

Where

\[
S = \frac{(Mk_w L_3 + Jk_w L_1 + 2Mk_r b^2 + 2Jk_r)}{2MJ} \tag{4.11}
\]

The following equation gives the final relation for the eigenfrequencies:
\[
\omega_{12}^2 = s \left\{ 1 \mp \frac{(k_w^2(L_1 L_2 - L_3^2) + 2k_r k_w(L_3 + L_1 b^2) + ak_r^2 b^2)4MJ}{(Mk_w L_3 + Jk_w L_1 + 2Mk_r b^2 + 2Jk_r)^2} \right\}
\] (4.12)

Study the cases \(a=0, a=0.1L, a=0.2L\) and so on.

In the case that \(a=0\) one obtains

\[
L_1 = L \quad L_2 = 0 \quad L_3 = \frac{1}{12} [L^3]
\] (4.13)

Because of \(L_2 = 0\) the equations (4.8) are uncoupled; one equation is obtained for the translation \(p\) and one for the rotation \(q\). It can be seen in the table that the corresponding value of \(A_1\) is minus infinity (when \(B_1=1\)), which implies pure translation. On the other hand the value of \(A_2\) is in this case zero; this represents pure rotation.

Table 4.4 illustrates the nature of the modes. The frequencies are in Hertz and the value of \(B_i\) is assumed to be one. From the table it can be seen that the eigenmodes are composed of both translation and rotation with different ratios. The fact which must always be borne in mind is that, once the absolute value of the \(A_i\)'s are small, then the dominant rigid-body modes are rotation. However, for values outside the range \(-1\) to \(+1\) (say), translation dominates the modes. It is concluded that the eigenmodes related to \(f_1\) has an oscillation centre \(O_1\) to the right of the mass centre (\(A_1\) and \(B_1\) have different signs). The oscillation centre \(O_1\) is further away the lower the ratio \(a/L\) is. For \(a/L = 0\) the oscillation centre \(O_1\) is at infinity to the right. For eigenfrequency \(f_2\) the oscillation centre \(O_2\) starts at the mass centre and moves to the left (\(A_2\) and \(B_2\) have the same sign).

When \(a/L = 0.41\) the two oscillation centres are situated at the same distance from the mass centre, the ratios being, respectively, \(A_1 = -0.722B_1\) and \(A_2 = 0.722B_2\).

When \(a/L > 0.41\) and increasing, then the oscillation centre \(O_1\) continues to move away from the mass centre to the right whereas the oscillation centre \(O_2\) continues to move toward the mass centre (coming from the left). When \(a/L = 1\) one has again one oscillation centre, now \(O_2\), at the mass centre, giving rigid body rotation \(q\) of the sleeper (at frequency 60.87), and the other oscillation centre \(O_1\) has moved to infinity (to the right), giving rigid body translation \(p\) of the sleeper (at frequency 58.58).
Figure 4.4 depicts the behaviour of the modes during various amount of the voided part for the ‘Single hanging’ condition. It is seen that the rigid-body frequencies decreased by increasing the unsupported part. The final values of these frequencies are around 70 percent of the initial values. Therefore, these changes are important. The situations for the first and last points are explained above (the modes are uncoupled).

A brief explanation of the behaviour of the modes when there is a combination of rotation and translation is now given. In both modes, because of the existence of both rotation and translation, there must be a node in the modes. The place of the node can clarify the trend of the rigid-body mode. For the mode which has pure translation at the beginning (fully supported sleeper), the place of the node is at infinity. The pure rotational mode has its node in the middle of the sleeper. When the voided part is small, for instance 10 percent, the node from the ‘rotational’ mode moves from the centre towards the side with the void.

There is a long distance, around 45 percent of the beam length, between the node and the voided part at the first step. The more the void extends the closer will the node come to the void end. For instance, when the voided part increases by 10 percent the node moves away from the voided part by 5 percent only. Consequently the edge of the void part will be closer to the node by 5 percent of the beam length. That is to say that the slope of the first rigid body mode frequency curve diminishes by increasing the voided part. The following figure shows this matter for the first rigid-body mode.

<table>
<thead>
<tr>
<th>$a/L$</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$A_1$</th>
<th>$A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>81.92</td>
<td>83.58</td>
<td>- inf</td>
<td>0</td>
</tr>
<tr>
<td>0.001</td>
<td>81.90</td>
<td>83.52</td>
<td>-34.090</td>
<td>0.015</td>
</tr>
<tr>
<td>0.01</td>
<td>81.63</td>
<td>83.09</td>
<td>-2.886</td>
<td>0.180</td>
</tr>
<tr>
<td>0.10</td>
<td>75.92</td>
<td>82.49</td>
<td>-0.595</td>
<td>0.875</td>
</tr>
<tr>
<td>0.15</td>
<td>72.96</td>
<td>82.49</td>
<td>-0.583</td>
<td>0.893</td>
</tr>
<tr>
<td>0.20</td>
<td>70.37</td>
<td>82.48</td>
<td>-0.594</td>
<td>0.876</td>
</tr>
<tr>
<td>0.40</td>
<td>63.37</td>
<td>82.11</td>
<td>-0.711</td>
<td>0.732</td>
</tr>
<tr>
<td>0.41</td>
<td>63.06</td>
<td>82.05</td>
<td>-0.722</td>
<td>0.722</td>
</tr>
<tr>
<td>0.60</td>
<td>60.36</td>
<td>80.20</td>
<td>-0.894</td>
<td>0.582</td>
</tr>
<tr>
<td>0.70</td>
<td>59.71</td>
<td>78.04</td>
<td>-1.013</td>
<td>0.514</td>
</tr>
<tr>
<td>0.80</td>
<td>59.36</td>
<td>74.53</td>
<td>-1.165</td>
<td>0.447</td>
</tr>
<tr>
<td>0.99</td>
<td>58.78</td>
<td>61.74</td>
<td>-4.456</td>
<td>0.117</td>
</tr>
<tr>
<td>0.999</td>
<td>58.60</td>
<td>60.96</td>
<td>-35.753</td>
<td>0.015</td>
</tr>
<tr>
<td>1</td>
<td>58.58</td>
<td>60.87</td>
<td>- inf</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4.4: Rigid-body frequencies and centre of oscillation; Single hanging
The method given in the preceding paragraph is true also for the second rigid-body mode. But here, at the first step, the node is placed in the middle of beam, and just by starting the voiding the node moves towards the middle of the voided part. For example, once the unsupported part increases by 10 percent, the node place shifts by half of that. Hence, the slope of the corresponding curve enhances progressively up to achieving the totally free beam.

The voiding starts from one end of the sleeper and increases (bed stiffness is removed by start at one end). The first bending mode of the sleeper has large amplitudes at the sleeper ends, and at the central part of the sleeper, see Appendix A. Consequently this eigenfrequency changes the most when support stiffness is removed at the ends of the sleeper and at the centre. Also, in this particular case, when the removal of stiffness comes closer to a node, the eigenfrequency changes very little and there is a low-slope section of the curve when stiffness is removed close to the node. After that, when removing stiffness around the beam centre, there is another high-amplitude region and because of this, when the voided part advances to the centre, the slope of the eigenfrequency curve again increases. By continuing the removal of stiffness (removal of ballast support) the other node will be passed which again implies another section of the frequency curve.

**First and higher bending modes**

The frequency of the first bending mode decreases by 10 percent of its initial value, through the voiding variation. This is less than the decrease of the rigid-body modes. Further, the second and third bending modes show less than 1.5 and 0.5 percent decrease, respectively, when the ballast bed is removed. Because the change is so small, this is not further considered in this part of the report. Furthermore, the two next bending modes stay approximately constant during the extension of the unsupported section.

**Figure 4.4: Normalized eigenfrequencies of single hovering sleeper.**
with a low slope. Finally, upon reaching the other end of the sleeper, because of the large amplitude, the slope of the curve is to some extent enhanced.

4.3 Double Hanging

Double hanging is the case when the voided parts start to form from the ends of the sleeper, where after they increase towards the centre. Finally, one gets a completely free beam; only the rail springs remain. This case can be divided into sub-cases, where only one of the cases is symmetric. In all other cases there is a constant unsupported part at one end and then the voided part at the other end starts to grow and extends through the sleeper.

4.3.1 Equally double hovering (symmetric case)

As mentioned before this pattern is symmetric. As a result of the symmetry the modes corresponding to the rigid-body frequencies are either pure rotation or pure translation. The Table 4.5 illustrates the results obtained regarding to the rigid-body and bending modes for this situation.

<table>
<thead>
<tr>
<th>Voiding(%)</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rigid Body(1)</td>
<td>81.28</td>
<td>76.77</td>
<td>71.81</td>
<td>67.88</td>
<td>64.90</td>
<td>62.77</td>
<td>61.33</td>
<td>60.45</td>
<td>59.99</td>
<td>59.82</td>
<td>57.68</td>
</tr>
<tr>
<td>Rigid Body(2)</td>
<td>82.66</td>
<td>79.40</td>
<td>77.22</td>
<td>74.93</td>
<td>72.67</td>
<td>70.46</td>
<td>68.25</td>
<td>65.97</td>
<td>63.51</td>
<td>60.79</td>
<td>59.80</td>
</tr>
<tr>
<td>Bending (1)</td>
<td>135.0</td>
<td>131.1</td>
<td>129.0</td>
<td>128.2</td>
<td>128.0</td>
<td>128.0</td>
<td>127.8</td>
<td>127.1</td>
<td>125.9</td>
<td>124.3</td>
<td>122.6</td>
</tr>
<tr>
<td>Bending (2)</td>
<td>331.9</td>
<td>330.8</td>
<td>330.5</td>
<td>330.5</td>
<td>330.3</td>
<td>329.8</td>
<td>329.0</td>
<td>328.1</td>
<td>327.5</td>
<td>327.2</td>
<td>327.1</td>
</tr>
<tr>
<td>Bending (3)</td>
<td>612.1</td>
<td>611.6</td>
<td>611.5</td>
<td>611.4</td>
<td>611.1</td>
<td>610.7</td>
<td>610.4</td>
<td>610.3</td>
<td>610.3</td>
<td>610.0</td>
<td>609.6</td>
</tr>
<tr>
<td>Bending (4)</td>
<td>950.1</td>
<td>949.8</td>
<td>949.8</td>
<td>949.6</td>
<td>949.4</td>
<td>949.2</td>
<td>949.2</td>
<td>949.1</td>
<td>948.8</td>
<td>948.5</td>
<td>948.5</td>
</tr>
<tr>
<td>Bending (5)</td>
<td>1332</td>
<td>1332</td>
<td>1332</td>
<td>1332</td>
<td>1332</td>
<td>1332</td>
<td>1332</td>
<td>1332</td>
<td>1332</td>
<td>1331</td>
<td>1331</td>
</tr>
</tbody>
</table>

Table 4.5: Eigenfrequencies of equally double hovering sleeper.

As can be seen from the table the rigid-body frequencies have decreased by about 30 percent when the sleeper is totally free as compared to the fully supported case. The first bending mode shows a decrease of the eigenfrequency of about 10 percent. The decrease will be smaller at the higher bending-mode frequencies; it is just under 1.5, 0.5, 0.15 and 0.07 percent for the bending-mode number two to mode number five, respectively.
Because of the small influence of the support, the author prefers to omit plotting of the higher bending frequencies. Figure 4.6 depicts the first and second rigid-body frequencies versus the length of the unsupported part of the sleeper.

As mentioned, the rigid-body modes are, due to the symmetry, either pure rotation or pure translation. The rotational mode has a node at the centre of the beam whereas the translation mode has no node (or, the node is at infinity). When stiffness is removed symmetrically, starting from the two ends of the beam, the rotational frequency will be influenced the most (the rotational mode has its largest amplitudes at the beam ends). The frequency curve will then have a higher slope. When the removal of stiffness approaches the centre of the beam (i.e. the node) the influence of this stiffness is less (on the rotational mode) and the slope of the curve is less. Therefore, when removing stiffness close to the node, the eigenfrequency will hardly change at all and the curve becomes more or less horizontal. On the other hand, the translational mode will be influenced the same amount independently of where the stiffness is removed. Therefore the slope of the curve giving the translational eigenfrequency will be approximately the same for all values of the stiffness removed; it does not matter from where the stiffness is removed. (It is noted that the curves intersect, but as the normalization frequencies are different, this does not imply that the two eigenfrequencies are the same.)
The mode’s shape of the first and the second rigid body modes are illustrated in the following table, note that this shapes are obtained from the FEM model.

<table>
<thead>
<tr>
<th>Void (%)</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>First rigid body mode</td>
<td>T</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td></td>
</tr>
<tr>
<td>Second rigid body mode</td>
<td>R</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>R</td>
</tr>
</tbody>
</table>

*R for Rotation and T for Translation

Table 4.6: Mode shapes for Rigid-Body modes

Both rigid body modes for the first and last step show switching from translation to rotation or vice versa. From this matter it can be induced that there should be a certain value of the voiding where the switching from one mode shape to the other happens. At these points the values of frequencies should be same. By applying the MATLAB code of appendix B, this issue for the beam as a rigid body can be investigated. The obtained results from the codes are tabulated as follows:

<table>
<thead>
<tr>
<th>a/l</th>
<th>f₁</th>
<th>f₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,00</td>
<td>81,92</td>
<td>83,58</td>
</tr>
<tr>
<td>0,0224</td>
<td>81,02</td>
<td>81,02</td>
</tr>
<tr>
<td>0,10</td>
<td>73,38</td>
<td>77,81</td>
</tr>
<tr>
<td>0,20</td>
<td>66,44</td>
<td>73,48</td>
</tr>
<tr>
<td>0,30</td>
<td>62,57</td>
<td>68,87</td>
</tr>
<tr>
<td>0,40</td>
<td>61,09</td>
<td>63,93</td>
</tr>
<tr>
<td>0,4578</td>
<td>60,89</td>
<td>60,89</td>
</tr>
<tr>
<td>0,50</td>
<td>58,58</td>
<td>60,87</td>
</tr>
</tbody>
</table>

Table 4.7: Rigid body frequencies, Equally double hanging

The first and second modes have the same value when a/L is 2.24% and when a/L is 45.78%. Since this case is symmetric it states that the switching between mode shapes happens when the sleeper is voided totally by 4.48% and by 91.56%.

From the above table, Table 4.6, it can be seen that, although the higher frequency, when the sleeper is fully supported, has a rotation mode, it shows pure translation behaviour for a small voided part. In consequence of pure rotation at the first step, and also because of that the first voiding step is under the side part which holds the highest amplitude in the rotation mode, the curve has high slope and the slope diminish when the void moves towards the centre. From the second step, 10 percent, to the one of the final step, 90 percent in all the steps this frequency (the higher) possesses pure translation. As explained in the preceding section, in the case of pure translation there is not any node in this mode. It does not matter from where one removes the support, and the frequency decreases with approximately the same rate. Consequently, the relevant curve is (more or less) a straight line in this range. At the very last step again the mode
would be pure rotation. Therefore, the curve from the 90 to 100 percent is ‘quadratic’ (bent) as it was in the early step. The only difference here is that the voided part is located under the middle of the sleeper, the place of the node in the rotation mode, and because of the small amplitude here the slope of the curve is taking a decreasing value and goes to zero.

For the lowest frequency exactly the reverse of the first one happens. It starts and ends with translation and because of this it behaves linearly in the first and the last steps of voiding. Except from the first and last stages this mode contains rotation; the further the voiding goes, the lower the slope of curve is. When the support of 20 percent, which is closed to the ends, is removed the curve has a higher gradient, because of the higher amplitude at this part. On the other hand just before the 90 percent, because of arriving at the end of the un-supported part, close to the centre of the sleeper, it describing by the lower ratio. The last part of the lowest frequency curve possesses linear behaviour because of the pure translation mode at the last step.

Figure 4.6: Two Rigid-Body modes

The next graph shows the eigenfrequencies of the first two flexural modes. The behaviour of the modes can be interpreted by applying the discussion in Appendix A.

The initially voided part in this case is at the ends where the modes have the highest amplitude in both first and second bending modes. As a result of this both curves in Figure 4.7start with a large gradient. When the voided parts extend towards the centre, stiffness will be removed at the nodes, and the eigenfrequencies are not influenced very much. The first bending mode, as pointed out in Appendix A, has the nodes when length fraction is equal 22 and 78 percent. In this situation, because of the symmetry, the curve has zero gradient at these points (this is for the first mode). Then, by continue to increase the size of the voided parts, stiffness is removed where the
mode has a high amplitude, see the section which is placed at 50 % in Figure A.2. Then the curve again shows a high gradient.

The frequency curve related to the second bending mode behaves in the same way at the node places, see Figure A.2. Now there are three nodes, at 13, 50 and 87 percent, and the curve depicts zero gradient at these three points. These point locations correspond to 26 and 100 percent of the voided amount in the Figure 4.7. Furthermore, in view of the fact that there is two inflection points at 30 and 70 percent, the curve shows high slope there, corresponding to points with large amplitude in the mode.

![Equally Double Hovering](image)

**Figure 4.7: First and second flexural modes**

Figure A.2 shows the eigenfrequencies of the third bending mode. This mode encompasses four nodes, at 10, 35, 65 and 90 percent, giving zero slopes in the curve in addition to inflection points at 0, 22, 78 and 100 percent. The behaviour of the curve is affected by the place of these critical points.

Figure 4.8 portrays the third bending mode. Because of the voiding starts from the ends of the sleeper the curve initially shows high slope, but by continuing to increase the size of the voids the slope decreases accordingly. After that the other high gradient area will be obtained.

Since there are four nodes in the third bending mode, and because of the symmetry in the node places as well as in the voiding manner, there are two zero gradient point in the relevant curve. Between and outside the node places there are high slope parts.
4.3.2 One end hovering is 10 percent (non-symmetric case)

In this case there is a constant voided part at one end of the sleeper and the evaluation will be carried out when a second voided part starts to grow from the other end.

In this case, $\beta$ is the value giving the constant voided amount at one end; here $\beta$ is 10 percent. The voiding from the other end starts and continues step by steps with 5 percent in each step and finally at 90 percent the sleeper will be totally unsupported by the ballast.

The equations of motion with respect to the mass centre are investigated to determine the nature of the rigid-body modes. Now the modes are a combination of rotation and translation: it is worth emphasizing that the same idea as before, with the single hanging, is applied here. However, the values of the $L_1$, $L_2$ and $L_3$ are modified in the MATLAB code (see Appendix B) because of the change of the ballast support. The modified $L_i$’s are as follow

$$L_1 = L(1 - \beta) - a \quad \text{and} \quad L_2 = \frac{1}{2} a(L - a) + ((0.5 - \beta)^2 - 0.5^2)L^2$$

(4.14)

$$L_3 = \frac{1}{12} \{4 \ast ((0.5 - \beta)^3 + 0.5^3)L^3 - 3L^2a + 6La^2 - 4a^3\}$$

Here ‘$\beta$’ gives the size of the constant unsupported part at one end and ‘$a$’ is a value which is varying and gives the size of the void starting from the other end.

The variation of the first rigid-body frequency can be seen in Table 4.8. In this table the values of the frequencies are roughly the same as those obtained with the FEM-model; the difference can be deduced from the differences of stiffness of the FEM-model and the rigid body, which was pointed out for a sleeper in the preceding section (the stiffness of a rigid body is infinite).
Anyhow, even with the small difference we can interpret the behaviour of the finite element model by applying the values obtained from the MATLAB code for a rigid body.

<table>
<thead>
<tr>
<th>$a/L$</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$A_1$</th>
<th>$A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>75.92</td>
<td>82.49</td>
<td>0.60</td>
<td>-0.87</td>
</tr>
<tr>
<td>0.01</td>
<td>75.75</td>
<td>81.54</td>
<td>0.52</td>
<td>-1.00</td>
</tr>
<tr>
<td>0.10</td>
<td>73.38</td>
<td>77.81</td>
<td>0.00</td>
<td>-inf</td>
</tr>
<tr>
<td>0.20</td>
<td>68.84</td>
<td>76.73</td>
<td>-0.29</td>
<td>1.79</td>
</tr>
<tr>
<td>0.30</td>
<td>65.17</td>
<td>76.17</td>
<td>-0.43</td>
<td>1.22</td>
</tr>
<tr>
<td>0.39</td>
<td>62.95</td>
<td>75.64</td>
<td>-0.52</td>
<td>1.00</td>
</tr>
<tr>
<td>0.40</td>
<td>62.65</td>
<td>75.53</td>
<td>-0.54</td>
<td>0.97</td>
</tr>
<tr>
<td>0.50</td>
<td>61.07</td>
<td>74.56</td>
<td>-0.65</td>
<td>0.80</td>
</tr>
<tr>
<td>0.60</td>
<td>60.16</td>
<td>72.99</td>
<td>-0.77</td>
<td>0.67</td>
</tr>
<tr>
<td>0.70</td>
<td>59.66</td>
<td>70.49</td>
<td>-0.93</td>
<td>0.56</td>
</tr>
<tr>
<td>0.73</td>
<td>59.55</td>
<td>69.39</td>
<td>-1.00</td>
<td>0.52</td>
</tr>
<tr>
<td>0.80</td>
<td>59.35</td>
<td>66.54</td>
<td>-1.21</td>
<td>0.43</td>
</tr>
<tr>
<td>0.90</td>
<td>58.58</td>
<td>60.88</td>
<td>-inf</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 4.8: Rigid-body frequencies and centre of oscillation; Double hovering when One end hovering is 10 percent

Here the value of $B_i$ like in the case of the single hovering sleeper is assumed to be one. Then, when $A_i$ is small the nature of mode is mainly rotation, and when $A_i$ tends to zero the mode is pure rotation. Then, for other values of $A_i$ except infinity, the mode is a combination of translation and rotation. Finally, when $A_i$ tends to infinity, the motion is translation only.
Figure 4.9 shows the variation of $A_i$’s versus the $\alpha_d$, which helps us to understand the behaviour of the rigid-body modes. For the lower frequency, to start with, the centre of oscillation is somewhere to the left of the sleeper centre ($A_1/B_1$ is positive), but by starting the voiding it goes to the centre of the sleeper very fast. Because of the voided part starts far away from the centre of oscillation where the amplitude is high, the slope of the curve will be large. Figure 4.10 depicts the first and second rigid body frequencies.

The mode of the lower frequency will be pure rotation when the void is exactly 10 percent, because this makes the model symmetric. The sign of $A_1/B_1$ is changed from plus to minus at 10 percent. This change is presented in the graph by switching the sign of curvature; in this case the 10-percent point is an inflection point of the curve.

After 10 percent the centre of oscillation moves to the right of the mass centre ($A_1/B_1$ is negative) and goes away from the centre. On the other hand, the edge of the void still moves to the right by the same rate but here the movement of the void edge is much faster than the movement of centre of oscillation. Consequently they will come more close to each other step by step. When the void grows under the low amplitude part of the mode, then the slope of the curve diminishes to become more or less horizontal (i.e. zero).

Above mentioned idea is true up to 73 percent. At this point the $A_1/B_1$ value is one which means that the nature of mode is more translation instead of rotation (giving pure translation when $A_1/B_1$ is minus infinity). Due to this change there is another inflection point in the curve and after that, again, the curvature of the graph will be switched.
From 73 percent the centre of oscillation goes to infinity very fast. Thus, the distance between the voided part edge and centre of oscillation is going to increase. For this reason the gradient of the relevant curve increases accordingly as the voided part grows under the high amplitude sections.

![Graph showing double hovering with one end void 10 percent](image)

Figure 4.10: Double hovering with one end void 10 percent

When the ratio between the $A_i$'s and the $B_i$'s are outside the range $[-1, 1]$ there is translation in mode shape as well as rotation and the centre of oscillation is somewhere away from the beam centre. When the ratio is in the range $(-1, 1)$ there is less translation in the mode shape and the centre of oscillation is somewhere along the beam. For the special case \textit{i.e.}, ration is zero the centre of oscillation is located exact at the mass centre which implies pure rotation. Also for the other special case, \textit{i.e.} the value for this ratio is infinity; the centre of oscillation is located at infinity, implying that the mode shape is pure translation.

Apart from the first one percent of voiding, which is negligible, the second frequency has a mode which is similar to translation since then $A_2/B_2$ is -1 or less. This is due to the fact that by starting the removal of ballast (removal of stiffness) from the intact side the centre of oscillation goes rapidly to infinity as shown in the Figure 4.9, which has a high ratio compared to the movement of the voided part expansion. Anyhow, the voided part grows below the lower amplitude section and for this reason the slope of the curve diminished.

At the 10 percent void, the mode will be pure translation as the $A_2/B_2$ value is infinite. At pure translation it does not matter of where the growth of the void is placed, and the frequency decreases with the size of the void (and not depending on where the void is created).
At around 40 percent the dominant mode changes to rotation and the centre of oscillation goes towards the centre of mass. On the other hand the voided end grows to the right giving a larger ratio $A_2/B_2$. As a result of this the distance between the centre of oscillation and edge of the voided part goes far from each other. Consequently, when the voided part grows below the high amplitude section, then an increasing gradient can be seen in Figure 4.10.

The following figure depicts the movement of centre of oscillation regarding to the ratio of $A_i/B_i$.

![Figure 4.11: Movement of centre of oscillation](image)

All results obtained from FEM-modelling are tabulated in Table 4.9(a). From this table, it can be seen that the rigid-body frequencies show around 23 and 27 percent decreasing, respectively. The flexural modes do not show this high amount of decrease. Although the frequencies of the two first bending modes decrease by just over seven and just over one percent, the higher frequencies stay, approximately, the same throughout the elimination of the ballast support.

The manner of how the frequencies vary in the flexural modes depends strongly on the location of nodes and the positions of the large amplitudes of the mode. The shapes of the bending modes are studied in Appendix A, and they can be applied here for explaining the trend of the bending-mode frequencies. The first steps of voiding are at the end of the sleeper where the first bending mode has a high amplitude. Consequently, as Figure 4.12 depicts, the frequency curve starts by a large gradient. By continuing the voiding the curve shows a region with zero gradient. This region is related to the place of the node, which is at 22.4 percent of the sleeper length. As
pointed out before, removing stiffness at the parts of the sleeper which are located at the node does not change the eigenfrequency noticeably. Next stage of the curve trend is another high slop region which is related to the middle part of the first bending mode. In this area the frequency variation is more sensitive to the ballast stiffness removal. After that the gradient of the curve changes slightly to approach a horizontal line. This horizontal part occurs when the void grows below the next node, which is at 77.6 percent of the sleeper length. Then the unsupported part grows to the other end of the beam, with a large vibration amplitude of the mode, and then, as a result of this, the curve again increase in gradient.

The second bending-mode frequency follows the illustrated curve shape in Figure A.2. Due to large amplitude of the mode at the end, the frequency curve starts with a large gradient, which is going to zero very fast because of the node at 13.2% of the beam length. The corresponding curve to the second bending mode shows two places of high slope; the first one occurs because of the large amplitude of the mode between the first node and the second node, where the second node is located at 50 percent.

The last region of frequency decrease is related to the area of large amplitude between 50 and 86.8 percent and, as mentioned, the slope is then going to zero at the place of the third node. Due to the initial void at the second sleeper end, the curve ends at 90 percent. If this were not the case, another part of a large gradient would be expected.

Figure 4.12: Eigenfrequencies for first and second flexural modes.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{double_hovering.png}
\caption{Double Hovering ($\beta=10\%$)}
\end{figure}
4.3.3 Other values of $\beta$

Increasing the size $\beta$ of the initial void does not have any special effect on the behaviour of the bending modes. The discussion presented in the preceding section is applicable here for other cases. The same study as in Appendix A will be considered in these cases. The only difference is that in some cases, since the initial value of $\beta$ is increased; the evaluation cannot be performed at some nodes or areas of high amplitude because these areas might be voided already in the initial conditions. As a result of this the corresponding curves of the bending-mode eigenfrequencies look like they are cut off. This is because of the initial voiding. Eigenfrequencies for different cases are tabulated in the Error! Reference source not found. and the corresponding figures can be found in Figure 4.13.
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Table 4.9: Double hanging, unsymmetric cases. (a) $\beta = 10\%$, (b) $\beta = 20\%$, (c) $\beta = 30\%$, (d) $\beta = 40\%$, (e) $\beta = 50\%$. 
The figures portray the normalized graphs of double hovering cases for various amount of initial voiding $\beta$. 

Figure 4.13: Double hanging, unsymmetric cases
The first rigid-body frequency for a higher value of $\beta$, shows the same behaviour as the 10 percent initial voiding that was studied in the preceding section. The differences are only in the place of the events (for example zero slope) but not on the trends of the curves. For instance, all modes start with a combined mode which contains rotation and translation. Then, all modes are going to hold pure rotation. This point also is found like an inflection point and the curvature sign is supposed to change at this point.

After that for all cases even for the last one the mode property will be translation. The precise fact is that all modes after holding the translation have tendency to maintain the pure translation since $A_1$ jumps to infinity, see Table 4.10.

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<td>60,87</td>
<td>inf</td>
<td>0,00</td>
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Table 4.10: Double hovering, Rigid-body frequencies
In Table 4.10, it is worth to note that the lowest frequency for all value of $\beta$, holds pure translation. But before that, all cases have a mode which is similar to rotation somewhat. Because of this switching from rotation to translation there is an inflection point at the last part of each curve. In addition, the end of the curves shows the behaviour of pure translation which is linear.

For the 50 percent initial voiding, at the last step the centre of oscillation of lower frequency is located at the left side of the centre of mass and in infinity ($A$ and $B$ hold the same sign). Thus around the final step its mode looks translation, consequently the end of the first rigid body frequency of the last situation ($\beta = 50\%$), shows increasing in gradient sharply.

The results obtained by the MATLAB program are tabulated in Table 4.17. The table shows the amplitude $A_2$ (when $B_2 = 1$) giving the centre of oscillation corresponding to the second rigid-body mode. At first $A_2$ takes some value between zero and minus one. This implies that in this mode the dominant motion is rotation. Then, except for the last case, the node goes to the minus infinity, which is analogous to pure translation.

Next stage for all the cases is switching from the mode with translation nature to one that hold pure rotation once the sleeper is completely unsupported from the bed.

Figure 4.13 depicts the behaviour of the normalized second rigid body frequency versus the amount of ballast voiding. At least for the first three cases it is clear that there is an inflection point at the position where the two unsupported sides are equal and the model is symmetric. In addition, around this point an almost linear behaviour is expected because of the pure translation mode.

### 4.4 Triple Hanging

This case contains an initial void at the sleeper centre and two symmetric voids which start from the sleeper ends and grow towards the centre of the beam. The size of the initial central void is increased by steps of 10 percent and the end voids are increased by steps of 5 percent. For instance, in the first case the central unsupported part is 10 percent and symmetric with respect to the centre of the sleeper. This value is fixed and do not change during the evaluation. The end voids are at first zero and then increase by steps of 5 percent. This continues until the beam is free (i.e. not in contact with the ballast bed).

It is noticed that at all steps the symmetry is maintained. Consequently the (almost) rigid-body modes are pure translation and pure rotation, respectively.
The mode shape for each situation have obtained from the FEM model and stated in the following table. Here ‘T’ represents the translation mode and ‘R’ the rotation.

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<th>α value</th>
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<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>45</th>
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<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mode 2</td>
<td>R</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>R</td>
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<tr>
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<td>T</td>
<td>T</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>T</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>Mode 2</td>
<td>R</td>
<td>R</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>R</td>
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</tr>
<tr>
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<td>T</td>
<td>T</td>
<td>T</td>
<td>R</td>
<td>R</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>Mode 2</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>T</td>
<td>T</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
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<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>Mode 2</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>-</td>
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<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
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<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
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<td>-</td>
</tr>
</tbody>
</table>

Table 4.11: Mode shapes for rigid-body modes; Triple hovering

### 4.4.1 Parameter β is 10 percent

In this case the void in the middle is considered to be 10 percent (β=0.1) of the total sleeper length. The unsupported parts at the sleeper ends grow towards the centre by steps of 5 percent. The values obtained for this pattern are tabulated in Table 4.12.

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<th>45</th>
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<td>67.86</td>
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<td>62.74</td>
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<td>59.97</td>
<td>57.68</td>
</tr>
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<td>77.11</td>
<td>75.07</td>
<td>72.85</td>
<td>70.59</td>
<td>68.31</td>
<td>65.96</td>
<td>63.48</td>
<td>60.76</td>
<td>59.80</td>
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<td>330.4</td>
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<td>1332</td>
<td>1331</td>
<td>1331</td>
<td>1331</td>
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Table 4.12: Central void is 10 percent

The rigid body modes behaviour is depicted in Figure 4.14. In this figure the natural frequency values are not normalized.
The first rigid-body mode, except for the first and last steps, is of rotational shape. Because of this, when the voiding starts at the sleeper ends (far from the middle point) the frequency curve has a large slope (the rotational mode has large amplitudes at the sleeper ends) and then when the voided regions come closer to the centre of the sleeper the slope of the curve diminishes accordingly (stiffness is removed close to the node).

The second rigid-body mode is opposite, i.e., at the beginning and at the end the mode is pure rotational but then it changes to be translation. For this reason (translation dominates) the frequency curve is more or less linear. It is noted in the figure that the rotational mode and the translational mode have the same frequency at about 5% and 41%. Thus, following the frequency of the rotational mode, this frequency is the largest for \( \alpha_t < 5\% \). Then, for \( 5\% < \alpha_t < 41\% \), the rotational mode has the lowest eigenfrequency, and finally, when \( \alpha_t > 41\% \) again the rotational eigenfrequency will be the highest.
The trend of the bending modes can be interpreted by the idea put forward in the preceding section. The places of the nodes and the high amplitude regions can be seen form of the bending-mode frequency curves. For the first bending mode, the initial step of voiding is at the high amplitude area and then, as a result of this, the frequency curve shows a high slope. When the voided region extends to the nodes, consequently the curve shows a low slope. After that, when the unsupported part moves close to the midpoint of the sleeper, the curve again depicts a large slope.

### 4.4.2 Other values of $\beta$

When the size of the end voids approaches the central void the shape of the rigid-body modes switch from translation to rotation or vice versa. Also, at the final step the mode shapes will switch back to the initial shapes. By increasing the size of the central void the switching back to the initial shape will come earlier. When the size of the central void increases, particularly to the value of $e.g.$ 40 to 50 percent, then the switching and switching back come closer, and finally the switching disappears. The frequencies, connected to each value of $\beta$, obtained are tabulated in Table 4.13.
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<th>15</th>
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<td>61.14</td>
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<td>72.68</td>
<td>70.52</td>
<td>68.25</td>
<td>65.89</td>
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<td>59.80</td>
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<td>124.4</td>
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<td>330.1</td>
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<td>328.6</td>
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<td>610.8</td>
<td>610.7</td>
<td>610.4</td>
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<td>948.9</td>
<td>948.8</td>
<td>948.5</td>
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<td>1332</td>
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(a)

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</thead>
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<td>67.33</td>
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<td>68.01</td>
<td>65.71</td>
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<td>60.70</td>
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<td><strong>Bending (1)</strong></td>
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<td>127.1</td>
<td>124.7</td>
<td>123.6</td>
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<td>123.3</td>
<td>123.1</td>
</tr>
<tr>
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<td>329.5</td>
<td>329.3</td>
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<td>610.8</td>
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<td>609.9</td>
<td>609.6</td>
</tr>
<tr>
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<td>949.2</td>
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<td>948.6</td>
</tr>
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(b)

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<th>20</th>
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<td>70.67</td>
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<td>65.40</td>
<td>63.05</td>
<td>60.50</td>
</tr>
<tr>
<td><strong>Rigid Body(2)</strong></td>
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<td>70.55</td>
<td>66.54</td>
<td>63.50</td>
<td>61.29</td>
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<tr>
<td><strong>Bending (1)</strong></td>
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<td>124.2</td>
<td>123.1</td>
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<td>122.7</td>
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<td>328.6</td>
<td>328.5</td>
<td>327.9</td>
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<td>610.7</td>
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<td>1332</td>
<td>1332</td>
<td>1332</td>
<td>1332</td>
<td>1332</td>
</tr>
</tbody>
</table>

(c)
Table 4.13: Triple hanging, other values of β

As explained and stated before in the Table 4.11, for the 20 and 30 percent of central void, the mode shape is pure translation or pure rotation depending on the size of the end voids. To show the situation of rigid body modes for these two cases a graph is drawn in which is kept the mode shape. It is seen that the curves intersect giving equal eigenfrequency for the two modes at some points. Also, sometimes the translational mode has the lowest eigenfrequency and sometimes the rotational mode.
As before the following graphs shows normalized natural frequencies for triple hovering.
For the bending modes, the same structure as before is obtained. It should be noted that by increasing the centrally voided part, the graphs will be cut off earlier.

4.5 Side-central voids

This case sometimes occurs in the support of the sleeper by the ballast. To guarantee that all situations in this case are considered, the following method is applied:

First the initial value for the central void is fixed. Then the value of the side void grows from zero up to it reaches the central unsupported part. This case is evaluated in this part of the study. In the following cases the size of the central void is increased up to half the sleeper length. Furthermore, for each case the side void increases from zero to when it reaches the central void.

Apart from the symmetry situations, the rigid body modes are neither pure translation nor pure rotation. Because of this, to determining the nature of the mode shapes a similar study as performed before is required.

4.5.1 Parameter $\beta$ is 10 percent

Because of the shape of the rigid-body modes, which are now combinations of rotation and translation, a task is now to determine the nature of the modes. From the results obtained, shown in Table 4.14, it can be seen that the first mode at the starting point, because of the symmetry, is pure translation. The smallest imperfection at the end will destroy the symmetry and as a result of that the first mode will have a shape which is a combination of translation and rotation. But, as can be seen, the relation between the coefficients $A_i$ and $B_i$ is greater than one which indicates that the dominant shape of the mode is translation. This situation will continue up to 7 percent of end voiding, after which the dominant part will switch to be rotation.
The obtained results from the FEM model are tabulated as follows:

Table 4.14: Central void is 10 percent; rigid-body frequencies

<table>
<thead>
<tr>
<th>a/L</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$A_1$</th>
<th>$A_2$</th>
</tr>
</thead>
<tbody>
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<td>79,89</td>
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<td>-inf</td>
<td>0,00</td>
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<td>76,81</td>
<td>81,61</td>
<td>-1,00</td>
<td>0,52</td>
</tr>
<tr>
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<td>74,86</td>
<td>81,45</td>
<td>-0,82</td>
<td>0,64</td>
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<tr>
<td>0,20</td>
<td>69,30</td>
<td>81,37</td>
<td>-0,71</td>
<td>0,73</td>
</tr>
<tr>
<td>0,30</td>
<td>65,04</td>
<td>81,34</td>
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<td>0,71</td>
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<tr>
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<td>61,99</td>
<td>81,14</td>
<td>-0,80</td>
<td>0,65</td>
</tr>
<tr>
<td>0,45</td>
<td>60,85</td>
<td>80,92</td>
<td>-0,84</td>
<td>0,62</td>
</tr>
</tbody>
</table>

Table 4.15: Central void is 10 percent

<table>
<thead>
<tr>
<th>Voiding(%)</th>
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<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>45</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rigid Body(1)</td>
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<td>77,02</td>
<td>73,83</td>
<td>70,76</td>
<td>68,13</td>
<td>65,96</td>
<td>64,14</td>
<td>62,61</td>
<td>61,30</td>
<td>60,16</td>
</tr>
<tr>
<td>Rigid Body(2)</td>
<td>82,64</td>
<td>80,56</td>
<td>80,16</td>
<td>80,10</td>
<td>80,08</td>
<td>80,01</td>
<td>79,86</td>
<td>79,60</td>
<td>79,20</td>
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</tr>
<tr>
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<td>131,5</td>
<td>130,5</td>
<td>130,1</td>
<td>130,1</td>
<td>130,1</td>
<td>130,0</td>
<td>129,6</td>
<td>129,1</td>
<td>128,4</td>
</tr>
<tr>
<td>Bending (2)</td>
<td>331,8</td>
<td>331,3</td>
<td>331,1</td>
<td>331,1</td>
<td>331,0</td>
<td>330,8</td>
<td>330,4</td>
<td>330,0</td>
<td>329,7</td>
<td>329,5</td>
</tr>
<tr>
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<td>611,6</td>
<td>611,3</td>
<td>611,3</td>
<td>611,3</td>
<td>611,1</td>
<td>610,9</td>
<td>610,7</td>
<td>610,7</td>
<td>610,7</td>
<td>610,6</td>
</tr>
<tr>
<td>Bending (4)</td>
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<td>949,9</td>
<td>949,9</td>
<td>949,8</td>
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<td>1332</td>
<td>1332</td>
</tr>
</tbody>
</table>
All the variations stated for the first rigid-body mode can be followed in the related graph Figure 4.18. Because the nature of mode is translation, the relevant curve start with an increasing slope area and the oscillation centre comes closer to the sleeper centre. The voided region goes towards the centre faster than the centre of oscillation. Therefore, the slope of the curve is increasing to some extent. At around seven percent there is an inflection point in the curve. This implies that there is a switching in the nature of the mode’s shape. The first rigid-body mode frequency decreases by the lower rate when the unsupported part increases.

![Figure 4.18: Central void 10 percent, side central voids](image)

For the second rigid body mode, apart from the first step which is pure rotation, the other steps are combination of two shapes with the rotation tendency. This issue is obtainable from Table 4.14, where the value of the amplitude ratio (giving the centre of oscillation) is less than one. It is worth to note that the value of the $A_2$ column is increased from zero to 0.73 at 20 percent and then diminished to 0.62 at the final step. This means that the rotational property of the mode decreases (the closer to zero $A_2$ is, the more rotational contents of the mode, and in the reverse way: the closer $A_2$ is to infinity, the more translational content). Thus, the diminishing in the curve slope for the first part, and increasing in the curve gradient for the second part of the curve is expected.

In the graph the curve that give the behaviour of the second rigid-body mode starts with a high slop, but step by step the voided area goes closer to the midpoint. The centre of rotation is located at the middle of the unsupported part. The initial void is close to the end. On the other hand, in these steps the centre of oscillation is close to the midpoint of sleeper. At the following steps the void will grow somewhere much closer to the centre of oscillation. Because of this the slope of the curve is almost zero . By continuing the voiding through to the midpoint of sleeper the slope of the curve increase because the voiding boundary have passed the centre of oscillation and again goes from this point.
4.5.2 The other value of $\beta$

In the following section the behaviour of the eigenfrequencies and the modes due to the extended central voiding will be considered. By increasing the central unsupported part area the trend of the related curve to the rigid-body and flexural mode will behave in a same manner as before, but the location of the critical points will shift somewhat.

From the Table 4.16, which are for the 20, 30, 40 and 50 percent of central voiding respectively, it can be seen that for the 20 and 30 percent of central void the inflection point in the first rigid body mode curve is moved by 5 percent to the right. This means that switching the dominant shape from translation to rotation is postponed. Furthermore, when the $\beta$ value take such a high value, e.g. 40 and 50 percent, the mentioned switching would be cancelled and the manner of the first rigid body modes from start to end will be translation. This issue is indicated by the $A_1$ value which holds a greater that one absolute value.

<table>
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<th>$A_1$</th>
<th>$A_2$</th>
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<td>0,00</td>
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<td>80,61</td>
<td>-1,09</td>
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</tr>
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<td>80,51</td>
<td>-1,00</td>
<td>0,52</td>
</tr>
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<td>67,95</td>
<td>80,35</td>
<td>-0,84</td>
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<td>60,36</td>
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<td>76,49</td>
<td>-1,08</td>
<td>0,48</td>
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(d)

Table 4.16: Rigid-body frequencies, for other value of central void
The results for the entire situation from FEM model are tabulated in Table 4.17

<table>
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<tr>
<th>Modes</th>
<th>Voiding(%)</th>
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<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
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<td>69,55</td>
<td>66,96</td>
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<td>62,79</td>
<td>61,11</td>
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</tr>
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<td>78,63</td>
<td>78,58</td>
<td>78,58</td>
<td>78,55</td>
<td>78,44</td>
<td>78,23</td>
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<td>129,2</td>
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<td>128,7</td>
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<td>330,8</td>
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<td>330,1</td>
<td>329,7</td>
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<td>611,1</td>
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<td>949,4</td>
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(a)

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<th>Voiding(%)</th>
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<td>77,12</td>
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<td>127,8</td>
<td>127,7</td>
<td>127,4</td>
<td></td>
</tr>
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<td>329,9</td>
<td>329,5</td>
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</tr>
<tr>
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<td>611,1</td>
<td>611,0</td>
<td>611,0</td>
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<td>610,6</td>
<td>610,5</td>
<td>610,4</td>
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(b)

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<th>25</th>
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<tr>
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<td>78,65</td>
<td>76,83</td>
<td>76,01</td>
<td>75,74</td>
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<td>75,68</td>
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<tr>
<td>Bending (1)</td>
<td>131,2</td>
<td>129,0</td>
<td>127,9</td>
<td>127,5</td>
<td>127,4</td>
<td>127,4</td>
<td>127,3</td>
<td></td>
</tr>
<tr>
<td>Bending (2)</td>
<td>330,1</td>
<td>329,5</td>
<td>329,4</td>
<td>329,4</td>
<td>329,3</td>
<td>329,0</td>
<td>328,6</td>
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</tr>
<tr>
<td>Bending (3)</td>
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<td>611,0</td>
<td>611,0</td>
<td>610,9</td>
<td>610,8</td>
<td>610,5</td>
<td>610,4</td>
<td></td>
</tr>
<tr>
<td>Bending (4)</td>
<td>949,4</td>
<td>949,2</td>
<td>949,2</td>
<td>949,1</td>
<td>949,0</td>
<td>948,9</td>
<td>948,9</td>
<td></td>
</tr>
<tr>
<td>Bending (5)</td>
<td>1332</td>
<td>1332</td>
<td>1332</td>
<td>1332</td>
<td>1332</td>
<td>1332</td>
<td>1332</td>
<td></td>
</tr>
</tbody>
</table>

(c)
From the corresponding graph it is achieved that the first rigid-body mode has an inflection point around 12 and 18 percent for $\beta$ values of 20 and 30 percent, respectively. The other graphs which are related to the higher values of $\beta$ do not hold such a point.

By having a look at the $A_2$ value in the tables, it can be seemed that in many situations the value of this parameter is between zero and one. Hence, the second rigid-body mode contains more rotation and less transition. For 20 percent as well as 30 percent of central void, the $A_2$ value increases then decreases ($A_2=0$ means pure rotation) this increasing and decreasing slope are illustrated in the corresponding curves.

Since the curve of higher value for central void is cut off for lower value of $\beta$, the part which holds the higher slope in the second rigid-body mode will not exist anymore for the two last graphs. It is noticed that the starting point of the void is located at the end of the sleeper and is far from the centre of oscillation. By extending the unsupported part towards the midpoint, the voided region and the centre of oscillation will come closer to each other. As a result of this the gradient of the second rigid-body mode decreases gradually. The final steps of voiding are more close to the centre of oscillation which is located around the midpoint of sleeper. Therefore the relevant part of the curve is a straight line with a zero gradient.

By increasing the $\beta$ value nothing will be changed in the behaviour of the flexural modes. The first step of evaluation in side-central case is focused in the voiding which starts from one of the sleeper ends. Thus, it is located at the place which has high amplitude for any bending mode. Then by continuing the voiding, the removal of stiffness will move to the place which is under the node (it happens for some bending modes sooner or later than for the others). Consequently a low-gradient area in the corresponding curves will appear.

Particularly, for the first flexural mode, when the central void is not so much extended, the final steps of the voiding are placed under a high-amplitude area. Hence, the second high slop area in the relevant curve is predictable. This last variation of the curve gradient cannot happen for the high value of central void because the high amplitude part of the sleeper is already unsupported.
by the central void and there is no stiffness to be removed when the voiding starts from the end and goes towards the centre of the sleeper.

Figure 4.19: Side central voids, other value of central void
5 Summary and Conclusions

By use of FEM, vibration, eigenfrequencies and eigenmodes of an *in situ* concrete railway sleeper have been investigated. The sleeper is modelled either as a rigid body or as an elastic body. The sleeper is connected to the rails by two discrete springs and it is supported by a continuously distributed spring modelling the ballast. Voids (pockets) can appear between the sleeper and the ballast. Five patterns of imperfections have been considered and evaluated. Also a comparison between *in situ* sleeper with rail stiffness and *in situ* sleeper without rail stiffness has been made.

In the symmetric cases the rigid-body modes are pure translation or rotation. Once the model is not symmetric the first and second rigid-body modes are combination of rotation and translation. When the mode is pure translation the centre of oscillation is at infinity and for pure rotation the centre of oscillation coincides with the mass centre of the beam.

The following conclusions from this report can be drawn:

The foundation stiffness influences the two rigid-body eigenfrequencies the most (decreasing up to 70 percent of its initial value when the foundation stiffness is removed). The higher eigenfrequencies are more or less unaffected by the foundation stiffness.

The rate of eigenfrequency reducing for bending modes is directly related to the place of the pocketing (the voiding). If the voiding part is placed under the high amplitude section of the bending mode, it decreases the eigenfrequency a lot, and vice versa. The fifth bending mode stays the same during the voiding of ballast.

The influence of railpad (and rail) stiffness on the sleeper rigid-body eigenfrequencies is considerable, but for the bending-mode eigenfrequencies this influence is negligible.
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Appendices

Appendix A

Bending modes of free-free beam

This appendix provides the case of a free-free beam as an element for evaluating bending modes of it. Engineering beam theory (E-B, Euler-Bernoulli beam theory) was used to calculate the related eigenfrequencies and eigenmodes. The beam considered in this section has length \( L \) (m). A constant mass distribution \( m \) (kg/m) is assumed for the beam. In addition, constant bending stiffness \( EI \) (Nm\(^2\)) is considered. Boundary conditions: as the beam is assumed to be totally free in this case the boundary conditions are “free-free”, meaning that these are no bending moments and no shear forces at the beam ends. Generally, boundary condition on the deflection \( w(x, t) \) and its first, second and third derivatives can be given. Since the differential equation is expressed in the deflection \( w(x, t) \), the physical quantities that lead to these derivatives are deflection \( w(x, t) \), the beam slope \( w'(x, t) \), bending moment \( M(x, t) = -E I w''(x, t) \) and shear force \( T(x,t) = -E I w'''(x,t) \). From these four possibilities two relevant boundary conditions at each beam end should be chosen. (For example, for a beam that is clamped at one end, the displacement and angle (slope) is zero.) After that, the boundary conditions lead to a frequency equation in terms of \( \sin, \cos \) and so on. In some cases, the minimum eigenfrequency can be zero (0). This means that the beam has one or more rigid-body movements. It implies that the beam can translate and/or rotate as a rigid body. Eigenfrequencies corresponding to rigid-body motion are zero.

In this appendix the lowest bending modes (self-oscillation forms) at bending vibration in the plane have been sketched and the coordinates of the nodes (points that do not move when the beam bends) are marked. The two rigid-body modes (in the plane) are not drawn.

\[
\begin{align*}
M(0,t) &= 0, T(0,t) = 0, M(L,t) = 0 \text{ and } T(L,t) = 0
\end{align*}
\]

Two first boundary conditions are related to the left end of the beam which is free and the same boundary condition is true for the right end of the beam as it is free too.

By using the differential equation of the beam motion and applying the boundary conditions given, the frequency equation can be obtained as follows:

\[
1 - \cosh \mu L \cos \mu L = 0 \text{ Where } \mu^4 = \frac{m \omega^2}{EI}
\]

Solutions for \( \omega_c \neq 0 \) are:
\( \mu_1 L = 4.730; \quad \mu_2 L = 7.853; \quad \mu_3 L = 10.996; \quad \mu_4 L = 14.137; \)

From this series it can be included that, for \( n \geq 2 \), one has, approximately, \( \mu_n L = (n+1/2)\pi \)

Then eigenfrequencies are as follows:

\[
\omega_{en} = \beta_n \pi^2 \sqrt{\frac{EI}{mL^4}} \quad \text{Where}
\]

\( \beta_1 = 2.267; \quad \beta_2 = 6.249; \quad \beta_3 = 12.250; \) and \( \beta_n = (n+1/2)^2 \); approximately, for \( n \geq 2 \).

Eigenmodes can be calculate from the next formula (two rigid body modes when \( \omega_e = 0 \) is not included)

\[
X_n(x) = A_n \left\{ \cosh \mu_n x + \cos \mu_n x - \frac{\cosh \mu_n L - \cos \mu_n L}{\sinh \mu_n L - \sin \mu_n L} \{\sinh \mu_n x + \sin \mu_n x\} \right\}
\]

This equation gives the place of the nodes when the left hand side is equal to zero. One obtains for the first five modes:
Figure A.2: Bending modes of free-free beam
Appendix B

MATLAB code for calculating the rigid-body modes as well as centre of oscillation when the sleeper is considered as a rigid-bogy centre of oscillation.

```matlab
format short eng
L=2.5;
beta=0.5;
i=1;

for h=0:0.05:0.5 %(depend on the case)
a=h*L;
M=251;
J=M*L^2/12; k=13000000; kr=17000000; b2=0.5625;

% Rigid body (rigid sleeper)

For Single hanging
L1=L-a; L2=a*(L-a)/2; L3=(L^3-3*L^2*a+6*L*a^2-4*a^3)/12;

For double hanging
L1=L-beta*L-a
L2=(beta^2*L^2-L^2*beta-a^2+L*a)/2
L3=((L/2-beta*L)^3-(-L/2+a)^3)/3

For side-central
L1=L-beta*L-a; L2=(L-a)*a/2; L3=((L/2-beta*L)^3-(-L/2+a)^3+(L/2)^3-
(b*L/2)^3)/3;

For Central Void
L1=L-a; L2=0; L3=1/12*(L^3-a^3);

o1=(M*k*L3+J*k*L1+2*M*kr*b2+2*J*kr)/(2*M*J);
x=(M*k*L3+J*k*L1+2*M*kr*b2+2*J*kr)^2;
o2=(4*M*J*(k^2*L1*L3-k^2*L2^2+2*kr*k*L3+2*kr*k*L1*b2+4*kr^2*b2))/x;
om1sq=o1*(1-sqrt(1-o2)); om1sq=o1*(1-sqrt(1-o2));
om1=sqrt(om1sq); om2=sqrt(om2sq);
f1=om1/2/pi;
f2=om2/2/pi;

% Modes: calculate A when B = 1;
Amplo1=-k*L2/(k*L1-M*om1sq+2*kr);
Amp11=-(-J*om1sq+k*L3+2*kr*b2)/(k*L2);
Amplo2=-k*L2/(k*L1-M*om2sq+2*kr);
Amp12=-(-J*om2sq+k*L3+2*kr*b2)/(k*L2);
z(i,1)=f1;z(i,2)=f2;z(i,3)=Amplo1;z(i,4)=Amp11;z(i,5)=Amplo2;z(i,6)=Amp12;
i=i+1;
end
z
```
## Appendix C

### Figure A.3: Historical overview

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<th>Penomenalt</th>
<th>Godstrafik</th>
<th>Linjedelning av</th>
<th>Årliga treubel</th>
<th>Dicke</th>
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### Table A1: Historisk översikt / Historical overview

- **Transport performance:**
  - **Passenger traffic:**
    - Regional traffic
    - Long distance traffic
    - Total traffic
  - **Freight traffic:**
    - Domestic transportation
    - Full wagon traffic
  - **Energy consumption by rail transport:**
    - Million passenger-kilometers
    - Million tonne-kilometers

### Footnotes:

    - M. "tonkilometer" av tåg, av källa, i likhet med 1988-
    - "tonkilometer" av tåg, av källa, i likhet med 1988-
    - "tonkilometer" av tåg, av källa, i likhet med 1988-
    - "tonkilometer" av tåg, av källa, i likhet med 1988-

    - M. "tonkilometer" av tåg, av källa, i likhet med 1968-
    - "tonkilometer" av tåg, av källa, i likhet med 1968-
    - "tonkilometer" av tåg, av källa, i likhet med 1968-
    - "tonkilometer" av tåg, av källa, i likhet med 1968-

SIKASatsik 2008:22
Appendix D

%Three-part sleeper on elastic foundation and with rail stiffness
spring
clear
clc
L=2.5;
a=0.2*L;
b=0.8*L;
M=251;
J=M*L^2/12; k=17000000;
%Böjsvängande en- två- tre-delad sliper
%EI1=8400000; %bending stiffness
%EI1=6410000;
%EI2=4600000;
EI1=4800000;
EI2=EI1;
EI3=EI1;
%I1=0.2*0.2^3/12; %second moment of area
%I2=I1;
%I3=I1;
%A1=0.2*0.2; %area
%A2=A1;
%A3=A1;
%r1=I1/A1; %radius of gyration
%r2=I2/A2;
%r3=I3/A3;
r1=0.0600^2;
r2=0.0568^2;
r3=r1;
%m1=114; % mass distribution (kg/m)
m1=100.4;
m2=m1;
%m2=91.2;
m3=m1;
mass=m1*a+m2*(b-a)+m3*(L-b);
% kGA1=628000000; %skjuvstiffness, shear stiffness
%kGA2=502000000;
KGA1=498000000;
KGA2=kGA1;
kGA3=kGA1;
s1=EI1/KGA1;
s2=EI2/KGA2;
s3=EI3/KGA3;
% %
%s1=0;
%s2=s1;
%s3=s1;

k1=13000000; %Bäddstiffness, bed stiffness
k2=13000000;
k3=13000000;
k1=0;
% k2=0;
k3=0;
% kral=17000000; %rail spring stiffness
% kral=0;
kap1=k1/EI1;
kap2=k2/EI2;
kap3=k3/EI3;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%om=0*2*pi:1:150*2*pi;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
for j=1:length(om)
b1=m1*om(j).^2./EI1;
b2=m2*om(j).^2./EI2;
b3=m3*om(j).^2./EI3;
mu1=sqrt(sqrt(b1-kap1));
mu2=sqrt(sqrt(b2-kap2));
mu3=sqrt(sqrt(b3-kap3));

CA1=b1*(r1+s1)-kap1*s1;
CB1=b1*(1-b1*r1*s1)+kap1*b1*r1*s1-
kap1;
alf1=sqrt(-CA1/2+sqrt(CA1^2/4+CB1));
bet1=sqrt(CA1/2+sqrt(CA1^2/4+CB1));

CA2=b2*(r2+s2)-kap2*s2;
CB2=b2*(1-b2*r2*s2)+kap2*b2*r2*s2-
kap2;
alf2=sqrt(-CA2/2+sqrt(CA2^2/4+CB2));
bet2=sqrt(CA2/2+sqrt(CA2^2/4+CB2));

CA3=b3*(r3+s3)-kap3*s3;
CB3=b3*(1-b3*r3*s3)+kap3*b3*r3*s3-
kap3;
alf3=sqrt(-CA3/2+sqrt(CA3^2/4+CB3));
bet3=sqrt(CA3/2+sqrt(CA3^2/4+CB3));

C1p=1/alf1*(alf1^2+(b1-kap1)*s1);
C2p=C1p;
C3p=1/bet1*(-bet1^2+(b1-kap1)*s1);
C4p=-C3p;
D1p=1/alf2*(alf2^2+(b2-kap2)*s2);
D2p=D1p;
D3p=1/bet2*(-bet2^2+(b2-kap2)*s2);
D4p=-D3p;
Elp=1/alf3^2+(b3-kap3)*s3;
E2p=Elp;
E3p=1/bet3^2+(b3-kap3)*s3;
E4p=-E3p;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%
%Tredelad balk 0 < x < a, a < x < b och b < x < L R-T teori
%Boundary condition
%RV x = 0
RT3(1,1)=alf1*C1p;
RT3(1,2)=0;
RT3(1,3)=bet1*C3p;
RT3(1,4)=0;
RT3(1,5)=0;
RT3(1,6)=0;
RT3(1,7)=0;
RT3(1,8)=0;
RT3(1,9)=0;
RT3(1,10)=0;
RT3(1,11)=0;
RT3(1,12)=0;

RT3(2,1)=0;
RT3(2,2)=-bet1;
RT3(2,3)=0;
RT3(2,4)=alf1;
RT3(2,5)=0;
RT3(2,6)=0;
RT3(2,7)=0;
RT3(2,8)=0;
RT3(2,9)=0;
RT3(2,10)=0;
RT3(2,11)=0;
RT3(2,12)=0;

%Continuity conditions at x = a
%Deflection Sätt w lika
RT3(3,1)=cosh(alf1*a);
RT3(3,2)=sinh(alf1*a);
RT3(3,3)=cos(bet1*a);
RT3(3,4)=sin(bet1*a);
RT3(3,5)=-cosh(alf2*a);
RT3(3,6)=-sinh(alf2*a);
RT3(3,7)=-cos(bet2*a);
RT3(3,8)=-sin(bet2*a);
RT3(3,9)=0;
RT3(3,10)=0;
RT3(3,11)=0;
RT3(3,12)=0;

%Slope Sätt psi lika
RT3(4,1)=C1p*sinh(alf1*a);
RT3(4,2)=C2p*cosh(alf1*a);
RT3(4,3)=C3p*sin(bet1*a);
RT3(4,4)=C4p*cos(bet1*a);
RT3(4,5)=-D1p*sinh(alf2*a);
RT3(4,6)=-D2p*cos(alf2*a);
RT3(4,7)=-D3p*sin(bet2*a);

RT3(4,8)=-D4p*cos(bet2*a);
RT3(4,9)=0;
RT3(4,10)=0;
RT3(4,11)=0;
RT3(4,12)=0;

%Bending moment Sätt momenten lika
RT3(5,1)=alf1*C1p*cosh(alf1*a);
RT3(5,2)=alf1*C2p*sinh(alf1*a);
RT3(5,3)=bet1*C3p*cos(bet1*a);
RT3(5,4)=-bet1*C4p*sin(bet1*a);
RT3(5,5)=-EI2/EI1*alf2*D1p*cosh(alf2*a);
RT3(5,6)=-EI2/EI1*alf2*D2p*sinh(alf2*a);
RT3(5,7)=-EI2/EI1*bet2*D3p*cos(bet2*a);
RT3(5,8)=-EI2/EI1*bet2*D4p*sin(bet2*a);
RT3(5,9)=0;
RT3(5,10)=0;
RT3(5,11)=0;
RT3(5,12)=0;

%Shear force, including rail spring stiffness
%Sätt tvärfraft lika (enligt alternativt uttryck på tvärfraften)
RT3(6,1)=EI1*C1p*(alf1^2-b1*r1)*sinh(alf1*a)-krail*cosh(alf1*a);
RT3(6,2)=EI1*C2p*(alf1^2-b1*r1)*cosh(alf1*a)-krail*sinh(alf1*a);
RT3(6,3)=EI1*C3p*(-bet1^2-b1*r1)*sin(bet1*a)-krail*cos(bet1*a);
RT3(6,4)=EI1*C4p*(-bet1^2-b1*r1)*cos(bet1*a)-krail*sin(bet1*a);
RT3(6,5)=-EI2/D1p*(alf2^2-b2*r2)*sinh(alf2*a);
RT3(6,6)=-EI2/D2p*(alf2^2-b2*r2)*cosh(alf2*a);
RT3(6,7)=-EI2/D3p*(-bet2^2-b2*r2)*cos(bet2*a);
RT3(6,8)=-EI2/D4p*(-bet2^2-b2*r2)*sin(bet2*a);
RT3(6,9)=0;
RT3(6,10)=0;
RT3(6,11)=0;
RT3(6,12)=0;

%Continuity conditions at joint two
%Deflection
RT3(7,1)=0;
RT3(7,2)=0;
RT3(7,3)=0;
RT3(7,4)=0;
RT3(7,5)=cosh(alf2*b);
RT3(7,6)=sinh(alf2*b);
RT3(7,7)=cos(bet2*b);
RT3(7,8)=sin(bet2*b);
RT3(7,9)=-cosh(alf3*b);
RT3(7,10)=-sinh(alf3*b);
RT3(7,11)=-cos(bet3*b);
RT3(7,12)=-sin(bet3*b);

%Slope Sätt psi lika
RT3(8,1)=0;
RT3(8,2)=0;
RT3(8,3)=0;
RT3(8,4)=0;
RT3(8,5)=D1p*sinh(alf2*b);
RT3(8,6)=D2p*cosh(alf2*b);
RT3(8,7)=D3p*sin(bet2*b);
RT3(8,8)=D4p*cos(bet2*b);
RT3(8,9)=-E1p*sinh(alf3*b);
RT3(8,10)=-E2p*cosh(alf3*b);
RT3(8,11)=-E3p*sin(bet3*b);
RT3(8,12)=-E4p*cos(bet3*b);

%Bending moment Sätt momenten lika
RT3(9,1)=0;
RT3(9,2)=0;
RT3(9,3)=0;
RT3(9,4)=0;
RT3(9,5)=alf2*D1p*cosh(alf2*b);
RT3(9,6)=alf2*D2p*sinh(alf2*b);
RT3(9,7)=bet2*D3p*cos(bet2*b);
RT3(9,8)=-bet2*D4p*sin(bet2*b);
RT3(9,9)=-EI3/EI2*alf3*E1p*cosh(alf3*b);
RT3(9,10)=-EI3/EI2*alf3*E2p*sinh(alf3*b);
RT3(9,11)=-EI3/EI2*bet3*E3p*cos(bet3*b);
RT3(9,12)=-EI3/EI2*bet3*E4p*sin(bet3*b);

%Shear force including rail spring stiffness
%Enligt alternativ (tvärkraft)
RT3(10,1)=0;
RT3(10,2)=0;
RT3(10,3)=0;
RT3(10,4)=0;
RT3(10,5)=E12*D1p*(alf2^2-b2*r2)*sinh(alf2*b)-
krail*cosh(alf2*b);
RT3(10,6)=E12*D2p*(alf2^2-b2*r2)*cosh(alf2*b)-
krail*sinh(alf2*b);
RT3(10,7)=E12*D3p*(-bet2^2-b2*r2)*sin(bet2*b)-
krail*cos(bet2*b);

%Boundary at x = L
%Moment
RT3(11,1)=0;
RT3(11,2)=0;
RT3(11,3)=0;
RT3(11,4)=0;
RT3(11,5)=0;
RT3(11,6)=0;
RT3(11,7)=0;
RT3(11,8)=0;
RT3(11,9)=alf3*E1p*cosh(alf3*L);  
RT3(11,10)=alf3*E2p*sinh(alf3*L);
RT3(11,11)=bet3*E3p*cos(bet3*L);  
RT3(11,12)=-bet3*E4p*sin(bet3*L);

%Shear force
RT3(12,1)=0;
RT3(12,2)=0;
RT3(12,3)=0;
RT3(12,4)=0;
RT3(12,5)=0;
RT3(12,6)=0;
RT3(12,7)=0;
RT3(12,8)=0;
RT3(12,9)=bet3*sinh(alf3*L);    
RT3(12,10)=bet3*cosh(alf3*L);
RT3(12,11)=alf3*sin(bet3*L);    
RT3(12,12)=-alf3*cos(bet3*L);

RT3r(j)=det(RT3);

end

RT3max=max(abs(RT3r(j)));