Absorbed dose equations:
The general solution of the absorbed dose equation and solutions under different kinds of radiation equilibrium

Gudrun Alm Carlsson

Department of Medicine and Care
Radio Physics
Faculty of Health Sciences
Absorbed dose equations. The general solution of the absorbed dose equation and solutions under different kinds of radiation equilibrium

Gudrun Alm Carlsson

Avd för radiofysik Linköpings universitet

REPORT
LiU-RAD-R-035
Absorbed dose equations. The general solution of the absorbed dose equation and solutions under different kinds of radiation equilibrium.

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. Introduction</td>
<td>p 2</td>
</tr>
<tr>
<td>II. Absorbed dose</td>
<td>p 3</td>
</tr>
<tr>
<td>III. The general solution of the absorbed dose equation</td>
<td>p 5</td>
</tr>
<tr>
<td>IV. Absorbed dose equations at different kinds of radiation equilibrium</td>
<td>p 8</td>
</tr>
<tr>
<td>A. Radiation equilibrium</td>
<td>p 8</td>
</tr>
<tr>
<td>B. Charged particle equilibrium</td>
<td>p 13</td>
</tr>
<tr>
<td>The mass energy absorption coefficient</td>
<td>p 18</td>
</tr>
<tr>
<td>C. Delta-particle equilibrium</td>
<td>p 22</td>
</tr>
<tr>
<td>Partial delta-particle equilibrium</td>
<td>p 26</td>
</tr>
<tr>
<td>Nonequilibrium limit</td>
<td>p 29</td>
</tr>
<tr>
<td>Appendix A: Derivation of the condition $\nabla \psi = 0$ by the use of reciprocity relations</td>
<td>p 30</td>
</tr>
<tr>
<td>Appendix B: The relation between the kerma and the absorbed dose under conditions of charged particle equilibrium</td>
<td>p 34</td>
</tr>
<tr>
<td>References</td>
<td>p 37</td>
</tr>
</tbody>
</table>
Absorbed dose equations. The general solution of the absorbed dose equation and solutions under different kinds of radiation equilibrium

I. Introduction

This report is a logical continuation of two papers concerning basic concepts in dosimetry. The first paper (1) is a critical analysis of the concepts of ionizing radiation and energy imparted as defined by the ICRU (2). The second paper (3) gives a definition of the energy imparted, the fundamental quantity in radiation dosimetry, which is equivalent to that given by the ICRU but which has a different form. This alternative definition of the energy imparted is suitable in deriving a general expression, in terms of particle fluences and interaction cross sections, for the absorbed dose valid also in situations where no kind of radiation equilibrium is established. It is, however, today not possible to quantify this expression for the absorbed dose. All practical calculations of absorbed dose rely on the assumption of one or another type of radiation equilibrium. The aim of this work is to analyze different kinds of radiation equilibrium conditions and to find the corresponding exact expressions for the absorbed dose. The concept of radiation equilibrium is more carefully analyzed than has been done previously (4, 5, 6). Moreover, the definition of the mass energy absorption coefficient for indirectly (uncharged) ionizing particles is critically analyzed. A new definition is proposed relevant to calculations of the absorbed dose in cases when charged particle equilibrium exists within a homogeneous medium due to the uniform liberation of charged particles, by uncharged particles.
II. Absorbed dose

The definition of absorbed dose, \(D\), is given by the ICRU (1971)

\[
D = \lim_{m \to 0} \frac{\mathcal{E}}{dm} = \frac{d\mathcal{E}}{dm}
\]  

(1)

Here, \(\mathcal{E}\) is the expectation value of the stochastic quantity energy imparted. The energy imparted, \(\mathcal{E}\), by ionizing radiation to the matter in a volume is

\[
\mathcal{E} = \sum_{\text{in}} T_{\text{in}} - \sum_{\text{out}} T_{\text{out}} + \sum Q
\]  

(2)

where

\[
\sum_{\text{in}} T_{\text{in}} = \text{the sum of the kinetic energies of all particles, both directly and indirectly ionizing, which have entered the volume,}
\]

\[
\sum_{\text{out}} T_{\text{out}} = \text{the sum of the kinetic energies of all particles, both directly and indirectly ionizing, which have left the volume and}
\]

\[
\sum Q = \text{the sum of all changes (with sign: decreases positive, increases negative) in the rest-mass energies of nuclei and elementary particles in any transformations of such particles which have occurred in the volume.}
\]

The definition of \(\mathcal{E}\) given by the ICRU has been slightly modified according to Alm Carlsson (1). A photon’s energy is, for instance, considered as kinetic energy.

Using the vectorial energy fluence \(\Psi\) of ionizing particles, the expression in eq 1 can be written (3, 4, 5).

\[
D = \frac{d\mathcal{E}}{dm} = -\frac{1}{\rho} \text{div}\Psi + \frac{d(\sum Q)}{dm}
\]  

(3)

where by definition

\[
-\frac{1}{\rho} = \lim_{m \to 0} \frac{1}{m} \left[ \sum_{\text{in}} T_{\text{in}} - \sum_{\text{out}} T_{\text{out}} \right]
\]  

(4)

In eq 4, \(\sum_{\text{in}} T_{\text{in}}, \sum_{\text{out}} T_{\text{out}}\) and \(\sum Q\) are the expectation values of the stochastic quantities \(\sum_{\text{in}} T_{\text{in}}, \sum_{\text{out}} T_{\text{out}}\) and \(\sum Q\).
Owing to an alternative definition of the energy imparted (3), the absorbed dose can also be defined as

\[
D = \frac{1}{\rho} \frac{dN}{dV} \Delta \epsilon
\]

(5)

where \(dN/dm\) is the expectation value of the total number of basic processes (interaction by ionizing particles, nuclear decays and nuclear deexcitations) taking place per unit mass of the material and \(\Delta \epsilon\) is the expectation value of the contribution, \(\Delta \epsilon\), to the energy imparted from each basic process. The value of \(\Delta \epsilon\) of a basic process is given by

\[
\Delta \epsilon = T_b - T_a + Q
\]

(6)

where \(T_b\) is the kinetic energy of an interacting ionizing particle just before the interaction, \(\sum T_a\) is the sum of the kinetic energies of all ionizing particles appearing just after the interaction (including the kinetic energy of the interacting particle if this is still an ionizing particle) and \(Q\) is the change in the rest-mass energy of nuclei and elementary particles resulting from the interaction. For nuclear decays and nuclear deexcitations the same expression for \(\Delta \epsilon\) is valid with \(T_b\) equal to zero.

Deexcitations of the electron structure subsequent to an interaction with atomic electrons are counted as a part of the interaction. The kinetic energies of characteristic roentgen rays and Auger electrons are thus contained in \(\sum T_a\) of an interaction with atomic electrons. As nuclear deexcitations may occur with an appreciable time delay (isomeric transitions) after an interaction which leaves the nucleus in an excited state, these deexcitation processes are regarded as separate processes.

Interactions which only involve excitations and deexcitations of the electron structure have their Q-values equal to zero. As a consequence of the eqs 3 to 6 one has

\[
- \frac{1}{\rho} \text{div}\varphi = \frac{dN}{dm} \left[\bar{T}_b - \sum \bar{T}_a\right] \]

(7)

where \(\bar{T}_b\) and \(\sum \bar{T}_a\) are the expectation values of the \(T_b\) and \(\sum T_a\) of the contributing basic processes.
III. The general solution of the absorbed dose equation

Any expression for the absorbed dose containing interaction cross sections and particle fluences or other parameters pertinent to an actual radiation field is here called a solution to the absorbed dose equation, eq 3 or eq 5. Such a solution makes it possible, at least in principle, to quantify the absorbed dose in actual cases.

Starting from the definition of the absorbed dose in eq 5, the general solution of the absorbed dose equation is immediately found

\[
D = \sum_j \int \frac{d\Phi_j}{dT} \frac{\mu_j(T)}{\rho} \Delta e_j(T) dT + \sum_i \frac{dN_i}{dm} \Delta e_i
\]

where \( \frac{d\Phi_j}{dT} \) is the fluence of ionizing particles of type \( j \) with kinetic energies in the interval \( T, T+dT \), \( \mu_j(T) / \rho \) is the total interaction cross section per unit mass for particles of type \( j \) and kinetic energy \( T \), \( \Delta e_j(T) \) is the expectation value of the \( \Delta e \) from an interaction by a particle of type \( j \) and kinetic energy \( T \), \( dN_i/dm \) is the expectation value of the number of spontaneous nuclear transitions of type \( j \) per unit mass and \( \Delta e_i \) is the expectation value of the \( \Delta e \) from such a transformation.

The value of \( \Delta e \) from a nuclear decay (deexcitation) is given by that part of the released rest-mass energy of nuclei and elementary particles which is not converted into the kinetic energy of ionizing particles. The recoil energy of the daughter nuclide is contained in \( \Delta e \) in cases when the daughter nuclide cannot be regarded as an ionizing particle.

In general, the contribution to the absorbed dose from the interactions by indirectly ionizing particles is neglected. The reason for this is that mostly the number of interactions by indirectly ionizing particles per unit mass is much smaller than the corresponding number of interactions by directly ionizing (charged) particles. For instance, in cases when charged particle equilibrium is established and the kinetic energies of the charged particles liberated by the indirectly ionizing particles are large compared to their binding energies, this will be the case. In more extreme situations, however, as in build-up regions, neglect of the contribution to the absorbed dose from the interactions by indirectly ionizing particles may not be a good approximation.

In addition, the contribution to the absorbed dose from spontaneous nuclear transformations are generally neglected. In most cases, the number of the transformations taking place per unit mass is small compared to the number of interactions per unit mass by the ionizing particles emitted in such transformations. In the same way, the contribution to the absorbed dose from interactions involving nuclear and elementary particle reactions are neglected.

As a net result, with the neglect of the contributions to the absorbed dose from interactions by indirectly ionizing particles, spontaneous nuclear transformations and nuclear and elementary particle reactions, the absorbed dose can be written
where the sum is over, all types $j$ of directly ionizing particles, \( (1/\rho)(dT/dx) \) is the mass stopping-power of the directly ionizing particles and $k$ is the fraction of the kinetic energy lost by the directly ionizing particles which does not reappear as kinetic energy of ionizing particles bremsstrahlung, characteristic roentgen rays, $\delta$ particles, Auger electrons). If furthermore the energy imparted in bremsstrahlung processes is neglected, then

\[
D = \sum_j \int \frac{d\phi_j(T)}{dT} \left( \frac{1}{\rho} \frac{dT}{dx} \right) k_j(T)dT \quad \text{----------------------------------------- (9)}
\]

where

\[
\sum \int \frac{d\phi_j(T)}{dT} \left( \frac{1}{\rho} \frac{dT}{dx} \right) k_{col,j}(T)dT \quad \text{----------------------------------------- (10)}
\]

where \( 1/\rho(dT/dx)_{col} \) is the mass collision stopping-power and \( k_{col} \) is the fraction of the kinetic energy lost in collisions with atomic electrons which does not reappear as kinetic energy of ionizing particles (characteristic roentgen rays, $\delta$ particles, Auger electrons).

Today values of collision stopping-powers are relatively well known. The fractions $k_{col}$ are, however, not numerically available. The expression which is today possible to quantify and which comes closest to the under the prescribed conditions general solution in eq 10 is

\[
D = \sum_j \int \frac{d\phi_j(T)}{dT} \left( \frac{1}{\rho} \frac{dT}{dx} \right) \Delta_{col,j} \quad \text{----------------------------------------- (11)}
\]

where \( 1/\rho(dT/dx)_{col,\Delta} \) is the restricted mass collision stopping-power. The expression for the absorbed dose in eq 11 corresponds to the solution under partial $\delta$ – particle equilibrium (described below).
IV. Absorbed dose equations at different kinds of radiation equilibrium

A. Radiation equilibrium

The condition of radiation equilibrium is defined by the relation

\[ \text{div} \psi = 0 \]  \hspace{1cm} (12)

This means that the expectation value of the radiation energy flowing into an infinitesimal volume element, \( \sum T_{in} \), equals the expectation value of the radiation energy, \( \sum T_{out} \), emerging from it.

When radiation equilibrium exists at a point in a medium the absorbed dose is given by, eqs 3 and 12:

\[ D = \frac{d(\sum Q)}{dm} \]  \hspace{1cm} (13)

Radioactive decay, nuclear deexcitations and nuclear and elementary particle reactions contribute (positively or negatively) to the absorbed dose. For instance, if a \( \beta^- \) - emitting radioactive nuclide is distributed in the medium both the number of nuclear decays, positron annihilation processes and nuclear reactions caused by neutrinos per unit mass should be considered in calculations of the absorbed dose at radiation equilibrium. The neutrinos are due to their weak interaction with matter often neglected as a component of the radiation field. Even if this is accepted, the above discussion demonstrates that it is not generally true, as stated by, for instance, Roesch and Attix (6), that at radiation equilibrium the absorbed dose equals the radiation energy emitted by the radioactive source per unit mass exclusive of the kinetic energies of the neutrinos.

Radiation equilibrium always exists within an infinite homogeneous medium (homogeneous in both density and atomic composition) containing a uniformly distributed radiation source. In this particular situation the vectorial energy fluence \( \psi \) is a constant throughout the medium (in an ideally infinite medium this being true also for the neutrino component of the radiation field) and the \( \text{div} \psi \), in coordinate form containing a sum of derivatives with respect to position, is, due to this constancy, zero at all points in the medium. The absorbed dose equation valid for this particular state of radiation equilibrium will now be investigated.

Ionizing particles emitted from the radiation source are called primary particles. As these primary particles slow down secondary ionizing particles may be created in interactions with atomic electrons or in nuclear and elementary particle reactions. If radioactive or excited nuclei are thereby formed, these are regarded as radiation sources emitting primary particles on their subsequent decay or deexcitation.

In case the medium is infinite the primary particles are completely stopped together with all generations of their secondary particles.
A primary ionizing particle emitted with initial kinetic energy $T_p$ gives, together with all the generations of its secondary ionizing particles, a contribution $\bar{\varepsilon}_p T_p$ to the mean energy imparted (the expectation value of the energy imparted) to the infinite medium given by ($\sum T_{in} = T_p$ and $\sum T_{out} = 0$ in eq 2).

$$\bar{\varepsilon}_p T_p = T_p + \sum Q$$ ................................................................. (14)

Here $\sum Q$ is the expectation value of the sum of all those changes in the rest-mass energies of nuclei and elementary particles occurring in the interactions by the primary particle and all its secondary particles. In principle, $\sum Q$ may depend on the atomic composition of the medium. For instance, in treating ideal cases with radiation equilibrium also for neutrinos, due regard must be taken to the fact that neutrinos initiate different kinds of nuclear reactions in media of differing atomic composition.

Within an infinite homogeneous medium containing a uniformly distributed source a reciprocity relation is valid such that

$$\frac{d\bar{\varepsilon}(P \rightarrow P')}{dV} = \frac{d\bar{\varepsilon}(P' \rightarrow P)}{dV}$$ ......................... (15)

where $d\bar{\varepsilon}(P_i \rightarrow P_{i+1})/dV$ is the mean energy imparted (the expectation value of the energy imparted) per unit volume at point $P_{i+1}$ due to one primary particle being emitted from another point $P_i$ and $P''$ is a point at the same distance from $P$ as $P'$ but in the opposite direction. This formulation of the reciprocity relation meets the demands of a general case with nonisotropic emission of the primary particles and nonisotropic absorption properties of the medium.

In case the radiation source emits monoenergetic primary particles with initial kinetic energy $T_p$, the mean energy imparted to an infinitesimal volume element $dV_p$ at $P$ due to the interactions by primary and secondary ionizing particles in that volume element is given by

$$\int \frac{dN(T_p)}{dV} dV_p \frac{d\bar{\varepsilon}(P \rightarrow P)}{dV} dV_p = \int \frac{dN(T_p)}{dV} dV_p \frac{d\bar{\varepsilon}(P \rightarrow P')}{dV} dV_p.$$ ......................... (16)

where $dN(T_p)/dV$ is the number of emitted primary particles per unit volume.

Ultimately, when the primary particles are of different kinds and have a spectrum of initial kinetic energies the absorbed dose at point $P$ can be written

$$D = \frac{dS}{dm} \left[ \bar{\varepsilon}_p + \sum Q \right] + \frac{1}{n} \frac{dS}{dn} \Delta \varepsilon_p$$ ............................................ (17)
where \( \frac{dS}{dm} \) is the expectation value of the number of emitted primary particles per unit mass,

\[
\overline{T}_p + \sum Q
\]

is a mean value of \( \varepsilon_r \) averaged over particle types and energies, \( \overline{n} \) is the expectation value of the number of primary particles emitted in each source process and \( \overline{\Delta \varepsilon}_p \) is the expectation value of the \( \Delta \varepsilon \) of the source processes (nuclear decays and nuclear deexcitations).

While \( (dS/dm)_{\varepsilon_r} \) gives the contribution to the absorbed dose from the P interactions by (primary and secondary) ionizing particles, the last term on the right hand side of eq 17 gives the contribution to the absorbed dose from the source processes themselves.

The reciprocity relation in eq 16 is of fundamental importance in evaluating the absorbed dose equation 17. The generally valid condition that at radiation equilibrium the \( \text{div} \Psi = 0 \) is not sufficient for deriving eq 17. This will now be demonstrated.

The condition \( \text{div} \Psi = 0 \) can be written, cf eq 7.

\[
\frac{1}{\rho} \text{div} \Psi = \frac{dN_i}{dm} \overline{T}_{b,i} - \frac{dN_i}{dm} (\sum T_a) - \frac{dS}{dm} \overline{T}_p = 0 \quad \text{.................................................. (18)}
\]

where \( dN_i/dm \) is the expectation value of the number of interactions by ionizing particles per unit mass and \( \overline{T}_{b,i} \) and \( (\sum T_a) \) are the expectation values of the \( T_b \) and \( \sum T_a \) of the \( \Delta \varepsilon \) (cf eq 6) of these interaction processes.

According to eq 5 the absorbed dose at any point can be written

\[
D = \frac{1}{n} \frac{dS}{dm} \overline{\Delta \varepsilon}_p + \frac{dN_i}{dm} \overline{\Delta \varepsilon}_i \quad \text{.......................................................... (19)}
\]

Substituting the expression in eq 6 for the \( \Delta \varepsilon \) of the processes one has

\[
D = \frac{dN_i}{dm} \overline{T}_{b,i} - \frac{dN_i}{dm} (\sum T_a) - \frac{1}{n} \frac{dS}{dm} (\sum T_a) + \frac{dN_i}{dm} \overline{Q}_i \quad \text{.................................................. (20)}
\]

Here, \( \frac{1}{n}(dS/dm)(\sum T_a) \) equals \( (dS/dm)\overline{T}_p \). Eq 17 can be derived by substituting the condition \( \text{div} \Psi = 0 \), eq 18, into eq 20, by reintroducing the expression \( \overline{\Delta \varepsilon}_p \) through elimination of \( Q_i \), and by utilizing the fact that, due to the reciprocity relation

\[
\frac{dS}{dm} \overline{Q} = \frac{dN_i}{dm} \overline{Q}_i \quad \text{.................................................. (21)}
\]
Inversely, starting from eq 19 and utilizing the relations in eqs 17 and 21, resulting from the reciprocity properties of the case in hand, the condition \( \text{div} \vec{\psi} = 0 \), eq 18, can be derived. This implies that radiation equilibrium can also be obtained at a point P within a finite homogeneous medium containing a uniformly distributed source provided that the primary particles, together with all their secondary particles, have a finite range, i.e., provided that \( \frac{d\vec{E}(P' \rightarrow P)}{dV} \) is zero for all points P' sufficiently far away from P in the case with an infinite medium.

If the neutrino component of the radiation field is disregarded, a state of approximate radiation equilibrium is possible to obtain within a finite homogeneous medium if the thickness of the material surrounding the point considered is equal to a few times the maximum mean free paths of the indirectly ionizing particles and if within these distances the primary particles are emitted uniformly.

The absorbed dose equation 17 is valid also in situations where Fanos theorem is applicable although the reciprocity relation in eq 15 may not be satisfied in the case with density variations within the medium. However, Fanos theorem implies that the absorbed dose in the medium is not dependent on density variations (all fluences of ionizing particles remain constant). This is consistent with the fact that \( \tilde{\epsilon}_r \) in eq 17 is independent of the density state of P the medium.

**B. Charged particle equilibrium**

The energy fluence of ionizing particles can be divided into the sum of the energy fluences of indirectly (uncharged) and directly (charged) ionizing particles. The \( \text{div} \vec{\psi} \) can be written

\[
\text{div} \vec{\psi} = \text{div} \vec{\psi}_u + \text{div} \vec{\psi}_c
\]

Charged particle equilibrium is defined by the condition that \( \text{div} \vec{\psi}_c = 0 \).

The absorbed dose equation valid in the particular case when charged particle equilibrium exists within an infinite homogeneous medium containing a uniformly distributed radiation source will now be investigated.

Consider a charged particle being emitted with initial kinetic energy \( T_c \). When this particle is slowed down in the infinite homogenous medium new ionizing particles (both uncharged and charged) are created, fig 1.
Charged ionizing particles emitted in a nuclear decay or nuclear deexcitation are called primary charged particles. Charged ionizing particles generated in primary charged particle interactions are called secondary charged particles together with all the charged ionizing particles these subsequently generate. Charged ionizing particles liberated by uncharged particles are counted as primary charged particles.

The contribution to mean energy imparted, $\bar{\varepsilon}_{\mathcal{T}_c}$, to the infinite medium through the interactions by a primary charged particle, with initial kinetic energy $T_c$, and all generations of its secondary charged particles is given by

$$
\bar{\varepsilon}_{\mathcal{T}_c} = T_c - \left( \sum T_u \right)_{\mathcal{T}_c} + \left( \sum Q \right)_{\mathcal{T}_c} 
$$

where $\left( \sum T_u \right)_{\mathcal{T}_c}$ is the expectation value of the sum of the kinetic energies of all those uncharged ionizing particles which have been created in the interactions by the primary charged particle and all its secondary charged particles, and $\left( \sum Q \right)_{\mathcal{T}_c}$ is the expectation value of the total change in the rest-mass energies of nuclei and elementary particles resulting from the interactions by the primary charged particle and its secondary charged particles.

$\bar{\varepsilon}_{\mathcal{T}_c}$ is obtained through the interactions along the lines in fig 1. $\bar{\varepsilon}_{\mathcal{T}_c}$ depends on the atomic composition of the medium.

Often $\left( \sum Q \right)_{\mathcal{T}_c}$ is zero as in the cases when the charged particles do not initiate any nuclear or elementary particle reactions. In these cases, $\left( \sum T_u \right)_{\mathcal{T}_c}$ contains the energies of bremsstrahlung photons and characteristic roentgen rays, the latter being created through ionization and excitation processes.

However, when positrons are involved, annihilation processes contribute to $\left( \sum Q \right)_{\mathcal{T}_c}$ and the energies of the annihilation photons are contained in $\left( \sum T_u \right)_{\mathcal{T}_c}$. 

Fig 1: When a charged particle, emitted at point P, decelerates in a medium new ionizing particles (both uncharged and charged) are created. The paths of charged particles are illustrated as lines. Uncharged particles are illustrated as waves.
Within an infinite homogeneous medium containing a uniformly distributed radioactive nuclide, the emission of primary charged particles is uniform since the fluence of uncharged ionizing particles, liberating part of the primary charged particles, remains uniform throughout this infinite medium.

When primary charged particles are emitted uniformly in an infinite homogeneous medium, a reciprocity relation, valid for the emitted primary charged particles and their secondary charged particles, can be utilized in a similar way as in the case with radiation equilibrium, such that the absorbed dose at any point can be written

$$D = \frac{dn_c}{dm} \tilde{\epsilon}_c + \frac{dN_{i,a}}{dm} \Delta \epsilon_{i,a} + \frac{dN_s}{dm} \Delta \epsilon_s, \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (24)$$

where $dn_c/dm$ is the expectation value of the number of emitted primary charged particles per unit mass, $\tilde{\epsilon}_c$ is a mean value of $\epsilon_c$, eq 23, averaged over the initial kinetic energies of the primary charged particles, $dN_{i,u}/dm$ is the expectation value of the number of interaction processes by uncharged particles per unit mass, $dN_s/dm$ is the number of source processes (nuclear decays and deexcitations) per unit mass and $\Delta \epsilon_{i,u}$ and $\Delta \epsilon_s$ are the expectation values of the $\epsilon$ of the respective basic processes.

The first term on the right hand side of eq 24 gives the contribution to the absorbed dose from the interactions by primary charged particles and secondary charged particles. The last two terms give the contribution to the absorbed dose from all those processes in which the primary charged particles are emitted.

Within an infinite homogenous medium containing a uniformly distributed source, the fluence of charged particles is, as well as the fluence of uncharged particles, uniform. It follows that $\text{div} \bar{\psi}_c$ is zero and that, according to the definition, charged particle equilibrium exists at all points in the medium.

The relation $\text{div} \bar{\psi}_c$ equal to zero can also be demonstrated, cf Appendix A, in a direct way by utilizing the characteristic reciprocity properties of the case in hand. As in the case with radiation equilibrium, this implies that charged particle equilibrium is a possible condition at single points within a finite homogeneous medium, as in cases when the ranges of the charged particles are finite and the primary charged particles are emitted uniformly over distances equal to the maximum range of the charged particles.

As soon as radiation equilibrium exists charged particle equilibrium also exists. Within an infinite homogeneous medium containing a uniformly distributed source both radiation equilibrium and charged particle equilibrium exist and the absorbed dose equations 17 and 24 are both exactly valid in this case. However, charged particle equilibrium may exist also in cases where radiation equilibrium is not established. Approximate charged particle equilibrium is, for instance, attainable at points within a finite homogeneous medium irradiated with uncharged particles having their mean free paths much larger than the maximum range of the charged particles they liberate, provided that these points are surrounded by
thicknesses of the medium which are at least equal to the maximum range of the charged particles. This case is of great significance in practical dosimetry.

The condition of charged particle equilibrium has been discussed by Rossi and Roesch (4) and Roesch (5). These authors seem not to be aware of the fact that the charged particle equilibrium discussed here defines a particular instance where charged particle equilibrium exists. Roesch states that "\( \nabla \cdot \vec{G}_2 \) is also frequently zero or negligible; the condition in which this is true is called charged particle equilibrium". (\( \nabla \cdot \vec{G}_2 \) is another expression for \( \text{div}\vec{\psi}_c \)). However, later on he states that "\( \nabla \cdot \vec{G}_2 \) can be zero for other reasons at a point or on a line or plane. \( \Phi \) (T) may be much different in these cases from the distribution in charged particle equilibrium. In charged particle equilibrium \( \nabla \cdot \vec{G}_2 \) vanishes throughout some nonzero volume". These two statements are obviously contradictory. Generally, charged particle equilibrium is defined by the condition that \( \text{div}\vec{\psi}_c \) is equal to zero. Then, charged particle equilibrium exists as soon as this condition is fulfilled and can therefore also exist at a "point or on a line or plane". The \( \text{div}\vec{\psi}_c \) contains a sum of derivatives with respect to position. It can be zero either due to an extreme situation (max or min values for the coordinate functions \( \psi_x, \psi_y, \psi_z \)) or due to a constancy of the vectorial energy fluence \( \vec{\psi}_c \). Possibly, a constancy of \( \vec{\psi}_c \) can only be achieved within a homogeneous medium containing a uniformly, distributed source. On the other hand, within a finite homogeneous medium with a uniform emission of primary charged particles, charged particle equilibrium may theoretically exist, and the \( \text{div}\vec{\psi}_c \) be zero, at a single point only. The particular charged particle equilibrium situation discussed here is better described by its reciprocity properties than by the condition that "\( \text{div}\vec{\psi}_c \) vanishes throughout some nonzero volume".

The absorbed dose equation, eq 24, is also valid within a homogeneous medium with density variations under the conditions prescribed for the validity of the Fano theorem.

The mass energy absorption coefficient

The ICRU (2) defines a quantity the kerma, \( K \), for indirectly (uncharged) ionizing particles. With the notations used here, the kerma is defined by

\[
K = \frac{dN_{i,u}}{dm} \left( \sum T_{a,e} \right)_{i,u} \tag{25}
\]

where \( dN_{i,u}/dm \) is the expectation value of the number of interactions by uncharged particles per unit mass and \( \left( \sum T_{a,e} \right)_{i,u} \) is the expectation value of the sum of the kinetic energies of all those charged particles appearing just after the interactions by the uncharged particles. \( \sum T_{a,e} \) is a part of \( \sum T_a \) in eq 6 for the \( \Delta E \) of interaction processes.
Alternatively, the kerma can be written

\[ K = \sum_j \frac{\mu_{u,j}(T)}{\rho} \frac{d\psi_{u,j}(T)}{dT} dT \] .......................... (26)

where \( \mu_u/\rho \) is the mass energy transfer coefficient and the summation is over all types \( j \) of the uncharged ionizing particles.

In cases when no radioactive or excited nuclei are present in the medium and no such nuclei are formed through the interactions by the ionizing particles, \( dn/dm \) in eq 24 is equal to the expectation value of the number of primary charged particles liberated by uncharged ionizing particles per unit mass. Neglecting the contribution to the absorbed dose from the interactions by the uncharged particles one has

\[ D = \frac{dn_c}{dm} \left[ \sum_{T_c} - (\sum T_u)_{T_c} + \sum Q_{T_c} \right] \]

\[ = K - \frac{dn_c}{dm} \left[ (\sum T_u)_{T_c} + \sum Q_{T_c} \right] \] ............................... (27)

for the case when charged particle equilibrium exists within a homogeneous medium due to the uniform liberation of primary charged particles by uncharged particles.

In cases when the liberated charged particles do neither generate any uncharged ionizing particles as they slow down, i.e., \( \sum T_u \) equals zero, nor do initiate any nuclear or elementary particle reactions, i.e., \( \sum Q \) equals zero, the absorbed dose equals the kerma.

The mass energy absorption coefficient for indirectly ionizing particles, \( \mu_{en}/\rho \), is by the ICRU (2) defined as

\[ \frac{\mu_{en}}{\rho} = \frac{\mu_u}{\rho} (1 - g) \] .......................... (28)

where \( g \) is the fraction of the initial kinetic energy of the liberated charged particles that is lost to bremsstrahlung when the charged particles and their secondary charged particles slow down in the medium.

In cases when bremsstrahlung photons are the only uncharged particles contributing to \( \sum T_u \) and \( \sum Q \) zero, the absorbed dose can for the case in hand be written

\[ D = \sum_j \frac{\mu_{en,j}(T)}{\rho} \frac{d\psi_{en,j}(T)}{dT} dT \] ............................... (29)

However, this is a rather specific situation. For instance, when charged particles slow down in a high atomic medium not only bremsstrahlung photons but also characteristic roentgen rays may be produced.

When positrons slow down, even annihilation processes contribute to both \( \sum T_u \) and \( \sum Q \) .
Every annihilation process contributes an amount equal to $2m_e c^2$ to $(\sum Q)_{T_e}$. If the annihilation takes place at rest, the two annihilation photons contribute the same amount to the term $(\sum T_u)_{T_e}$ such that the net effect of annihilations at rest is zero. However, annihilations taking place in flight give a net contribution to the expression in the parenthesis on the right hand side of eq 27.

From a fundamental point of view a more satisfactory definition of the mass energy absorption coefficient is given by

$$\frac{\mu_{em}}{\rho} = \frac{\mu_u}{\rho} \left[ 1 - \frac{(\sum T_u)_{T_e} - (\sum Q)_{T_e}}{T_e} \right] \equiv \frac{e_{T_e}}{T_e}$$

(30)

If the mass energy absorption coefficient is defined as in eq 30, it is achieved that the absorbed dose equation 29 is generally valid for the case that charged particle equilibrium exists within a homogeneous medium due to the uniform liberation of charged particles, provided that the contribution to the absorbed dose from the interactions by the uncharged particles is neglected and no radioactive or excited nuclei are formed due to the interactions by the ionizing particles.

It is of interest to note that in some tabulations of mass energy absorption coefficients for photons the annihilations of positrons in flight have been taken into consideration, cf Hubbell (7). Thus, there exists a discrepancy between the mass energy absorption coefficient as defined by the ICRU and the quantity represented as the mass energy absorption coefficient in tabulations. For heavy elements the quantitative discrepancy amounts to 2-3% at 10 MeV photon energy.

The quantity kerma may be used for describing the field of uncharged ionizing particles. Its usefulness in dosimetry is, however, limited. In spite of this, the ICRU gives much emphasis to this quantity and states that, in cases when charged particle equilibrium exists and bremsstrahlung losses are negligible, the absorbed dose equals the kerma. This statement is only approximately valid since even the generation of characteristic roentgen rays in charged particle interactions as well as annihilations by positrons in flight must be negligible (cf Appendix B). The product of the mass energy absorption coefficient as defined in eq 30 and the energy fluence of the uncharged ionizing particles is a quantity of greater significance in dosimetry. It seems that the ICRU (2) underestimates the fundamental importance of the mass energy absorption coefficient. It is not only a quantity which can be related to the exposure but it is applicable in a large number of instances through the absorbed dose equation 29. This equation is commonly used in practical dosimetry but has not as yet been established by the ICRU as a relation of fundamental importance in dosimetry.

Some of the bremsstrahlung photons and characteristic roentgen rays have mean free paths which are much less than the ranges of the primary charged particles. When charged particle equilibrium prevails under the conditions treated here, this low energy photon radiation will also be a condition of equilibrium.

The energies of low energy photons can be excluded from the sum $(\sum T_u)_{T_e}$ contained in the
Then, in calculating the absorbed dose according to eq 29 the fluences of these photons must, however, as well be excluded from the fluences of the uncharged ionizing particles.

C. Delta-particle equilibrium

Here, primary charged particles are all charged ionising particles emitted from an accelerat, emitted in a nuclear decay or deexcitation, generated in interactions by uncharged particles or created in nuclear and elementary particle reactions.

The first generation of secondary charged particles are generated in interactions by primary charged particles. As charged particles created in nuclear and elementary particle reactions, the first generation of secondary charged particles consists exclusively of $\delta$ -- particles and Auger electrons, are created. It is assumed that none of the $\delta$ -- particles or Auger electrons are energetic enough to initiate any nuclear reactions. This corresponds to a large number of practically important situations.

The radiation field can be split into three components with respect to uncharged ionizing particles, primary charged particles and secondary charged particles ($\delta$ -- particles, Auger electrons) such that

$$\text{div}\,\mathbf{\psi} = \text{div}\,\mathbf{\psi}_u + \text{div}\,\mathbf{\psi}_{c,p} + \text{div}\,\mathbf{\psi}_{\delta,A} \quad \text{............... (31)}$$

Delta-particle equilibrium exists when the $\text{div}\,\mathbf{\psi}_{\delta,A}$ is equal to zero.

Within an infinite homogeneous medium containing a uniformly distributed radiation source, $\delta$ -- particle equilibrium exists as well as both radiation equilibrium and charged particle equilibrium. An expression for the absorbed dose equivalent to both that in eq 17 and that in eq 24 is

$$D = \frac{dn_{\delta,A}}{dm} \left[ \bar{T}_{\delta,A} - \left( \sum T_u \right)_{\delta,A} \right] + \frac{dn_{i,c,p}}{dm} \bar{\Delta\epsilon}_{i,c,p} + \frac{dn_{i,u}}{dm} \bar{\Delta\epsilon}_{i,u} + \frac{dn_s}{dm} \bar{\Delta\epsilon_s} \quad \text{............... (32)}$$

where $dn_{\delta,A}/dm$ the expectation value of the number of $\delta$ -- particles and Auger electrons of the first generation emitted with mean kinetic energy $\bar{T}_{\delta,A}$, $\left( \sum T_u \right)_{\delta,A}$ is the expectation value of the sum of the energies of all those uncharged ionizing particles which are created in the interactions by a secondary charged particle of the first generation and all its associated secondary charged particles, and the last three terms give the contributions to the absorbed dose from the interactions by primary charged (i,c,p) and uncharged (i,u) ionizing particles and from nuclear decays and deexcitations (s).

Delta-particle equilibrium may exist also in cases when charged particle equilibrium is not established. Approximately, $\delta$ -- particle equilibrium can, for instance, be obtained at points within a finite homogeneous medium irradiated by primary charged particles as soon as the ranges of these particles are much larger than the ranges of the $\delta$ -- particles and Auger electrons they liberate, the requirement being that the points are surrounded by thicknesses of the medium at least equal to the maximum range of the $\delta$ -- particles and Auger electrons. At these points, the absorbed dose equation 32 is applicable.
An electron may lose as much as half of its kinetic energy to a $\delta$ - particle. In cases when the primary charged particles are electrons, $\delta$ - particle equilibrium only exists when charged particle (electronic) equilibrium also exists. A state of $\delta$ - particle equilibrium without charged particle equilibrium is, however, possible with heavy charged particles.

The expression for the absorbed dose in eq 32 can be reduced to more familiar ones by introducing a number of simplifying assumptions:

1. the $\delta$ - particles and Auger electrons do not generate any uncharged ionizing particles as they slow down, i.e., $(\sum T_{u})_{T_{\delta,A}}$ is zero
2. the contribution to the absorbed dose from all processes in which primary charged particles are emitted is negligible
3. the contribution to the absorbed dose from bremsstrahlung collisions by primary charged particles is negligible compared to the contribution from ionization and excitation processes.

Then, eq 32 reduces to

$$D = \frac{dN_{i,c,p}}{dm} + \left( \frac{dN_{i,c,p}}{dm} \right)_{col} (\Delta \varepsilon)_{i,c,p}$$

$$= \left( \frac{dN_{i,c,p}}{dm} \right)_{col} \left( \sum T_{\delta,A} \right)_{i,c,p} - (\sum T_{a,u})_{i,c,p}$$

$$= \sum_{j} \left[ \frac{d\phi_{p,j}(T)}{dT} \left( \frac{1}{\rho} \frac{dT}{dx} \right)_{col} \right] \left[ 1 - m_{col}(T) \right] dT$$

Here, $(dN_{i,c,p}/dm)_{col}$ is the expectation value of the number of electronic collisions by primary charged particles per unit mass, $\sum T_{u}$ contained in the $\Delta \varepsilon$ has been split into the sum of three terms:

- $\sum T_{u,\delta,A}$, $T_{a,p}$ and $\sum T_{a,u}$ corresponding to the secondary charged particles, the primary charged particle and the uncharged ionizing particles appearing just after an electronic collision by a primary charged particle, $\phi_{p,j}$ is the fluence of primary charged particles of the type $j$, $(1/\rho)(dT/dx)_{col}$ is the mass collision stopping power and $m_{col}$ is that fraction of the kinetic energy lost by the primary charged particle in an electronic collision which reappears as energy of characteristic roentgen rays.

In low Z-materials the probability of emission of characteristic roentgen rays is low and the energies of the roentgen rays are small such that $m_{col}$ can be neglected. Then, eq 33 reduces to the well known expression

$$D = \sum_{j} \left[ \frac{d\phi_{p,j}(T)}{dT} \left( \frac{1}{\rho} \frac{dT}{dx} \right)_{col} \right] dT$$

In cases when the mean free paths of the characteristic roentgen rays are much smaller than the maximum range of the $\delta$ - particles, the roentgen radiation will be in a state of equilibrium as soon as $\delta$ - particle equilibrium exists. Eq 34 is applicable also in these cases.

In high atomic media it is questionable if $m_{col}$ can be neglected.
In principle, the absorbed dose equations 33 and 34 are valid when a particle equilibrium exists for all the types \( j \) of primary charged particles. When characteristic roentgen rays and bremsstrahlung photons are generated to a great extent, electrons must also be considered among the primary charged particles. If a particle equilibrium is not established for the primary electrons, the absorbed dose from the primary electrons and all generations of their a-particles and Auger electrons must be calculated according to the general absorbed dose equation. If, on the other hand, electronic equilibrium prevails, the absorbed dose from the primary electrons liberated by the photons and their secondary electrons can be calculated from eq 29.

In high atomic media it may furthermore be unrealistic to assume that the a-particles do not cause the creation of characteristic roentgen rays and bremsstrahlung photons as they decelerate. If this is the case, \( \sum T_{\alpha} \alpha \) in eq 32 has to be added to \( \sum T_{\alpha, \alpha} \) in calculating the fraction \( m \) in eq 33.

**Partial \( a \) – particle equilibrium**

When a homogeneous medium is irradiated externally with electrons, a particle equilibrium cannot be established. One can, however, for this case think of a condition of approximate equilibrium for a-particles andAuger electrons with kinetic energies less than a certain value \( \Delta \), ICRU (8).

In the following, all a-particles and Auger electrons generated with initial kinetic energies above the limit \( \Delta \) are included among the primary electrons. Thus, all secondary electrons have energies below \( \Delta \). Part of the electrons with energies less than \( \Delta \) are, however, primary electrons which have slowed down to energies below \( \Delta \).

When partial a-particle (secondary electron) equilibrium exists due to the uniform liberation of secondary electrons with kinetic energies less than \( \Delta \) within a homogeneous medium, the absorbed dose can be written

\[
D = \int \frac{d\Phi_p(T)}{dT} \left( \frac{1}{\rho} \frac{dT}{dx} \right) f(T) \left[ 1 - m_{col, \Delta}(T) \right] dT \tag{35}
\]

where \( \Phi_p \) is the fluence of primary electrons and \( m_{col, \Delta} \) is that fraction of the kinetic energy lost by the primary electron which reappears as energy of characteristic roentgen rays or a-particles and Auger electrons with kinetic energies above \( \Delta \) (here regarded as primary electrons and contained in \( \Phi_p \)). Eq 35 is valid provided the conditions (1) to (3) on page 24 are fulfilled. The contribution to the absorbed dose from those primary electron interactions by which new primary electrons (a-particles and Auger electrons with energies above \( \Delta \)) are emitted are, however, included in eq 35.
The integration in eq 35 is over all kinetic energies sufficient for defining an electron as an ionizing particle (being less than $\Delta$).

If the influence of the emission of characteristic roentgen rays is negligible, $m_{col,\Delta}$ can be interpreted as that fraction of the kinetic energy lost by the primary electron which reappears as the kinetic energy of $\delta$ – particles and Auger electrons with kinetic energies above $\Delta$. The quantity $(1/\rho)(dT/dx)_{col,\Delta}[1 - m_{col,\Delta}]$ then comes close to the quantity the restricted mass collision stopping power $(1/\rho)(dT/dx)_{col}$. These quantities are, however, not identical. Not all of the kinetic energy lost by the primary electron in collisions with energy transfers above $\Delta$ reappears in the form of kinetic energy of $\delta$ – particles and Auger electrons with kinetic energy above $\Delta$. This can only be the case if the atomic electrons have no binding energy.

The expression for the absorbed dose in eq 35 resembles that formulated in the Spencer-Attix cavity theory (9). This theory treats the case with a cavity situated within a homogeneous medium and irradiated by photons such that electronic equilibrium is established at that point at which the cavity is placed.

As soon as electronic equilibrium is established, complete partial $\delta$ – particle equilibrium also exists. The absorbed dose in the homogeneous medium is given by eq 35. In the corresponding expression in the Spencer-Attix cavity theory, the integration is, however, not performed over all kinetic energies of ionizing electrons. Instead, Spencer-Attix utilize the fact that all primary electrons, and in particular those having their kinetic energies less than $\Delta$, are as well in a state of equilibrium. They need not calculate the energy fluence spectrum of primary electrons with kinetic energies less than $\Delta$. As a compensation, all the kinetic energy (and not only the energy loss) of the primary electron is included in the quantity $(1/\rho)(dT/dx)_{col,\Delta}$ in cases when primary electrons with kinetic energies above $\Delta$ lose so much energy that their residual kinetic energy is below $\Delta$ (10). This quantity is, in the Spencer-Attix theory, thus not identical to the restricted mass collision stopping power for electrons with kinetic energies between $\Delta$ and $2\Delta$.

In the Spencer-Attix cavity theory, the expression for the mean absorbed dose in the cavity is not based on the assumption of a partial $\delta$ – particle equilibrium in the cavity. No kind of equilibrium exists in such a small cavity, not even partial. $\delta$ – particle equilibrium. An approximate solution of the transport in the cavity of the $\delta$ – particles generated in the same cavity underlies the expression for the mean absorbed dose (9, 11). This point is missed by the ICRU (8) in their electron dosimetry theory. In calculating the absorbed dose to a cavity (probe) surrounded by walls of an equivalent material as to establish partial $\delta$ – particle equilibrium in the cavity, they refer to the expression for the mean absorbed dose to the cavity in the Spencer-Attix theory. When partial $\delta$ – particle equilibrium is established throughout the cavity, eq 35 is applicable for the absorbed dose to the cavity and $\Delta$ equals the maximum kinetic energy of the secondary electrons for which equilibrium is established. On the contrary, the parameter $\Delta$ is in the
Spencer-Attix theory related to the cavity size, $\Delta$ being the kinetic energy of an electron, which can just pass the cavity.

Nonequilibrium limit

In cases when no kind of radiation equilibrium is established, the absorbed dose equation can be found by extending the concept of primary electrons to include not only those $\delta$ particles and Auger electrons with initial kinetic energies above $A$ but all of them. Then, the absorbed dose equation 35 is applicable if $m_{\text{col},\Delta}$ is replaced by $m_{\text{col},0}$ where $m_{\text{col},0}$ is that fraction of the kinetic energy lost by the interacting electron which reappears as energy of characteristic roentgen rays or $\delta$ particles and Auger electrons. The resultant equation is identical to eq 10 for an electron field since $k_{\text{col}} = 1 - m_{\text{col},0}$. 
APPENDIX A

Derivation of the condition $\text{div} \bar{\psi}_c = 0$ by the use of reciprocity relations

When primary charged particles are emitted uniformly within distances equal to the maximum range of the primary charged particles and their associated secondary charged particles from a given point P in a homogeneous medium (homogeneous in both density and atomic composition), then a reciprocity relation is valid such that the absorbed dose at P can be written, cf page 15:

$$D = \frac{dN_{i,c}}{dm} \Delta \varepsilon_{i,c} + \frac{dN_{i,u}}{dm} \Delta \varepsilon_{i,u} + \frac{dN_s}{dm} \Delta \varepsilon_s \quad \ldots \quad (A1)$$

The absorbed dose at P can also be written, cf eq 5

$$D = \frac{dN_{i,c}}{dm} \Delta \varepsilon_{i,c} + \frac{dN_{i,u}}{dm} \Delta \varepsilon_{i,u} + \frac{dN_s}{dm} \Delta \varepsilon_s \quad \ldots \quad (A2)$$

where $dN_{i,c}/dm$ is the expectation value of the number of interactions by charged ionizing particles per unit mass at P and $\Delta \varepsilon_{i,c}$ is the expectation value of the $\Delta \varepsilon$ for these processes.

The expression on the right hand side of eq A2 can be expanded by substituting the expressions for $\Delta \varepsilon$, eq 6, into it. It can be further expanded by splitting $\sum T_a$ contained in $\Delta \varepsilon$, into two terms:

$$\sum T_a = \sum T_{a,u} + \sum T_{a,c} \quad \ldots \quad (A3)$$

where $\sum T_{a,u}$ is the sum of the kinetic energies of all uncharged ionizing particles appearing just after an interaction, nuclear decay or nuclear deexcitation and $\sum T_{a,c}$ is the corresponding sum of the kinetic energies of charged ionizing particles.

The fully expanded expression for the absorbed dose is

$$D = \frac{dN_{i,c}}{dm} \bar{T}_{b,i,c} + \frac{dN_{i,u}}{dm} \bar{T}_{b,i,u} - \frac{dN_{i,c}}{dm} (\sum T_{a,u})_{i,c} - \frac{dN_{i,c}}{dm} (\sum T_{a,c})_{i,c} - \frac{dN_{i,u}}{dm} (\sum T_{a,u})_{i,u} - \frac{dN_{i,u}}{dm} (\sum T_{a,c})_{i,u} + \frac{dN_s}{dm} (\sum T_{a,u})_s + \frac{dN_s}{dm} (\sum T_{a,c})_s \quad \ldots \quad (A4)$$

Due to the reciprocity properties of the case in hand one has

$$\frac{dN_{i,c}}{dm} \left[ (\sum T_u)_{T_c} + (\sum Q)_{T_c} \right] = -\frac{dN_{i,c}}{dm} (\sum T_{a,u})_{i,c} + \frac{dN_{i,c}}{dm} \bar{Q}_{i,c} \quad \ldots \quad (A5)$$
where \( \langle \sum T \rangle \) and \( \langle \sum Q \rangle \) are mean values of the quantities in eq 23 averaged over the initial kinetic energies of the primary charged particles.

Furthermore, the \( \text{div} \vec{E} \) is given by, eq 7.

\[
- \frac{1}{\rho} \text{div} \vec{\psi} = \frac{dN_{i,c}}{dm} T_{b,i,c} + \frac{dN_{i,c}}{dm} (\langle \sum T \rangle)_{i,c} - \frac{dN_{i,u}}{dm} (\langle \sum T \rangle)_{i,u} - \frac{dN_{s}}{dm} (\langle \sum T \rangle)_{s} \quad \text{... (A6)}
\]

Substituting the eqs A5 and A6 into eq A4 yields

\[
D = - \frac{1}{\rho} \text{div} \vec{\psi} + \frac{dN_{i,c}}{dm} \left[ - \langle \sum T \rangle_{i,c} - \langle \sum Q \rangle_{i,c} \right] + \frac{dN_{i,u}}{dm} T_{b,i,u} - \frac{dN_{i,u}}{dm} (\langle \sum T \rangle)_{i,u} + \frac{dN_{s}}{dm} (\langle \sum T \rangle)_{s} \quad \text{......... (A7)}
\]

By reintroducing the quantities \( \Delta E_{i,u} \) and \( \Delta E_s \) into eq A7 one gets

\[
D = - \frac{1}{\rho} \text{div} \vec{\psi} + \frac{dN_{i,c}}{dm} \left[ - \langle \sum T \rangle_{i,c} - \langle \sum Q \rangle \right] + \frac{dN_{i,u}}{dm} (\langle \sum T \rangle)_{i,u} + \frac{dN_{s}}{dm} (\langle \sum T \rangle)_{s} + \frac{dN_{i,u}}{dm} \Delta E_{i,u} + \frac{dN_{s}}{dm} \Delta E_s \quad \text{......... (A8)}
\]

However, since in eq A8

\[
\frac{dN_{i,u}}{dm} (\langle \sum T \rangle)_{i,u} + \frac{dN_{s}}{dm} (\langle \sum T \rangle)_{s} = \frac{dN_{i,c}}{dm} T_c \quad \text{............... (A9)}
\]

one finally has

\[
D = - \frac{1}{\rho} \text{div} \vec{\psi} + \frac{dN_{i,c}}{dm} T_c + \frac{dN_{i,u}}{dm} \Delta E_{i,u} + \frac{dN_{s}}{dm} \Delta E_s \quad \text{............... (A10)}
\]

This expression for the absorbed dose is identical to that in eq A1 only if \( \text{div} \vec{\psi} = 0 \). Since both eq A1 and eq A10 give the absorbed dose at the same point P under the same conditions, identity between the equations must be the case, and it follows that under these particular conditions \( \text{div} \vec{\psi} \) must necessarily be zero.

The condition \( \text{div} \vec{\psi} = 0 \) is, however, not sufficient for deriving the absorbed dose equation A1. The reciprocity relation in eq A5 must be used together with \( \text{div} \vec{\psi} = 0 \) in order to arrive at this expression, eq A1, for the absorbed dose.
APPENDIX B

The relation between the kerma and the absorbed dose under conditions of charged particle equilibrium

The absorbed dose, $D$, can generally be written, cf eq 5.

$$ D = \frac{dN_{i,c}}{dm} \overline{\Delta e_{i,c}} + \frac{dN_{i,u}}{dm} \overline{\Delta e_{i,u}} + \frac{dN_s}{dm} \overline{\Delta e_s} \quad \ldots \quad \text{(B1)} $$

where $dN_{i,c}/dm$ is the expectation value of the number of interactions by charged ionizing particles per unit mass, $dN_{i,u}/dm$ and $dN_s/dm$ are the corresponding numbers of interactions by uncharged ionizing particles and nuclear decays (deexcitations) and $\overline{\Delta e_{i,c}}$, $\overline{\Delta e_{i,u}}$, $\overline{\Delta e_s}$ are the expectation values of the $\Delta e$ of these basic processes.

The right hand side of eq B1 can be evaluated by introducing the expression in eq 6 for the $\Delta e$ of the basic processes. Furthermore, $\sum T_c$ can as in Appendix A, eq A3, be split into two terms. The fully expanded expression for the absorbed dose is then

$$ D = \frac{dN_{i,c}}{dm} \overline{T_{b,i,c}} + \frac{dN_{i,u}}{dm} \overline{T_{b,i,u}} - \frac{dN_{i,c}}{dm} \left( \sum \overline{T_{a,c}} \right)_{i,c} - \frac{dN_{i,u}}{dm} \left( \sum \overline{T_{a,u}} \right)_{i,u} - \frac{dN_s}{dm} \left( \sum \overline{T_{a,s}} \right)_{i,s} + \frac{dN_{i,c}}{dm} \overline{\Omega}_{i,c} + \frac{dN_{i,u}}{dm} \overline{\Omega}_{i,u} + \frac{dN_s}{dm} \overline{\Omega}_{i,s} $$

$$ \ldots \quad \text{(B2)} $$

Here, the sum of the first, the third, the fifth and the seventh terms on the right hand side of eq B2 represents the $-\frac{1}{\rho} \text{div} \overline{\Psi}_c$. The kerma, $K$, is represented by the fifth term with a negative sign. By reintroducing the quantities $\overline{\Delta e_{i,u}}$ and $\overline{\Delta e_s}$ into eq B2 together with $-\frac{1}{\rho} \text{div} \overline{\Psi}_c$ and $K$ one gets

$$ D = -\frac{1}{\rho} \text{div} \overline{\Psi}_c + K - \frac{dN_{i,c}}{dm} \left( \sum \overline{T_{a,c}} \right)_{i,c} + \frac{dN_{i,c}}{dm} \overline{\Omega}_{i,c} $$

$$ + \frac{dN_s}{dm} \left( \sum \overline{T_{a,s}} \right)_{i,s} + \frac{dN_{i,u}}{dm} \overline{\Delta e_{i,u}} + \frac{dN_s}{dm} \overline{\Delta e_s} \quad \ldots \quad \text{(B3)} $$

In eq B3, the third and the fourth terms represent together a quantity called $B$ in the papers by Rossi and Roesch (4) and Roesch (5). Introducing the quantity $B$ into eq B3 this reduces to

$$ D = K - B - \frac{1}{\rho} \text{div} \overline{\Psi}_c + \frac{dN_s}{dm} \left( \sum \overline{T_{a,c}} \right)_{i,s} + \frac{dN_{i,u}}{dm} \overline{\Delta e_{i,u}} + \frac{dN_s}{dm} \overline{\Delta e_s} \quad \ldots \quad \text{(B4)} $$
When charged particle equilibrium prevails, $\text{div } \Psi$ is zero and the absorbed dose equation reduces to

$$D = K - B + \frac{dn_s}{dm} \left( \sum T_{a,c} \right)_s + \frac{dN_{i,a}}{dm} \Delta e_{i,a} + \frac{dN_s}{dm} \Delta e_s$$ ........................................ (B6)

Provided that radioactive and excited nuclei are neither present in the medium at the point considered before the dose measurement nor are created through the interactions by ionizing particles and provided that the contribution to the absorbed dose from the interactions by the uncharged ionizing particles is neglected, the absorbed dose under charged particle equilibrium conditions can be written

$$D = K - B$$ ...........................................................................................................................................................................(B7)

The absorbed dose equation (B7) resembles that in eq 27. However, in the general case, $B$ refers to those interactions by charged particles, which occur per unit mass at the point considered. Only in that particular case when the charged particle equilibrium is established due to the uniform liberation of charged particles within a homogeneous medium, $B$ is identical to

$$B = \frac{dn_e}{dm} \left[ \left( \sum T_{a,c} \right)_e - \left( \sum Q \right)_e \right]$$ .......................................................... (B8)

Finally, the equations B5 and B6 demonstrate the approximations involved in the statement that "when charged particle equilibrium exists in the material at the point of interest, and bremsstrahlung losses are negligible, the kerma equals the absorbed dose at that point".
References


