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the proof of Lemma 5(b) can be shortened by using stochastic calculus arguments in [15, Section IV.3] differing from those in this technical note.

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Global Robust Output Regulation by State Feedback for Strict Feedforward Systems

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Abstract—This note studies the global robust output regulation problem by state feedback for strict feedforward systems. By utilizing the general framework for tackling the output regulation problem [10], the output regulation problem is converted into a global robust stabilization problem for a class of feedforward systems that is subject to both time-varying static and dynamic uncertainties. Then the stabilization problem is solved by using a small gain based bottom-up recursive design procedure.

Index Terms—Nonlinear systems, output regulation, robust control.

I. INTRODUCTION

Output regulation problem of nonlinear systems has been one of the central control problems for nearly two decades [3], [6]–[16], [20], [21], [23], [24]. The research was first focused on the local version of the problem where all the initial conditions and uncertain parameters are assumed to be sufficiently small [3], [8], [11], [12], [14], [20]. The research on the nonlocal version of the problem started in the late 1990s [6], [7], [10], [13], [15], [16], [21], [23], [24]. It is now well known (see, e.g., [10]) that the robust output regulation problem can be approached in two steps. In the first step, the problem is converted into a robust stabilization problem of a so-called augmented system which consists of the original plant and a suitably defined dynamic system called an internal model candidate, and in the second step, the robust stabilization problem of the augmented system is further pursued. The success of the first step depends on whether or not an internal model candidate exists which can usually be ascertained by the property of the solution of the regulator equations. Even though the first step can be accomplished, the success of the second step is by no means guaranteed due to at least two obstacles. First, the stabilizability of the augmented system is dictated not only by the given plant but also by the particular internal model candidate employed. An internal model candidate can be chosen from an infinite set of dynamic systems and a suitable internal model candidate is usually obtained from the past experience and some trial and error. Second, the structure of the augmented system may be much more complex than that of the original plant. Therefore, even though the stabilization of the original plant with the exogenous signal set to 0 is solvable, the stabilization of the augmented system may still be intractable. Perhaps, it is because of these difficulties, so far almost all papers on semi-global or global robust output regulation problem are focused on the lower triangular systems [6], [10], [13], [24], feedback linearizable systems [15], [16], and output feedback systems [7], [23].

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In this note, we study the global robust output regulation problem by state feedback for the following strict feedforward systems

$$\begin{aligned}\dot{x}_i &= f_i(x_{i-1}, \dots, x_1, u, v, w), \quad i = n, \dots, 2 \\ \dot{x}_1 &= cu + f_1(v, w) \\ e &= x_1 - q_d(v, w)\end{aligned}\quad (1)$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ the control, $e \in \mathbb{R}$ the tracking error, $w \in \mathbb{R}^{n_w}$ the uncertain constant parameter, $v \in \mathbb{R}^q$ the state of the exosystem

$$\dot{v} = Sv \quad (2)$$

where all eigenvalues of the matrix S are simple with zero real parts, c is a known nonzero real number, and for $i = 1, \dots, n$, the functions f_i and q_d are smooth functions satisfying $f_i(0, \dots, 0, w) = 0$ and $q_d(0, w) = 0$ for all $w \in \mathbb{R}^{n_w}$.

Global Robust Output Regulation Problem (GRORP)

For any compact set $V_0 \subset \mathbb{R}^q$ with a known bound and any compact set $W \subset \mathbb{R}^{n_w}$ with a known bound, design for system (1) a dynamic state feedback controller in the following form:

$$u = \mathcal{K}(\eta, x, e), \quad \dot{\eta} = \mathcal{F}(\eta, x, e) \quad (3)$$

where η is the compensator state and $\mathcal{K}, \mathcal{F}$ are locally Lipschitz functions vanishing at the origin, such that the closed-loop system composed of (1) and (3) has the following properties:

- for all $v(0) \in V_0$, $w \in W$ and for all initial state $x(0), \eta(0)$, the trajectory of the closed-loop system exists and is bounded for all $t \geq 0$;
- the tracking error converges to zero as t tends to infinity, i.e., $\lim_{t \rightarrow \infty} e(t) = 0$.

To our knowledge, the only papers that are relevant to the problem described above are [2] and [19]. An approximate and restricted tracking problem for a class of block feedforward systems was studied in [2] via dynamic output feedback control. The term approximate refers to the approximate regulation which is achieved by utilizing the k -fold internal model [8]. The term restricted refers to the fact that the state of the exosystem should be sufficiently small. In [19], the authors studied the input disturbance suppression problem (IDSP) via dynamic state feedback control for the following system:

$$\begin{aligned}\dot{x}_i &= w_{i-1}x_{i-1} + g_i(\dot{x}_{i-1}, \dots, \dot{x}_1, w), \quad i = n, \dots, 2 \\ \dot{x}_1 &= u - g_1(v)\end{aligned}\quad (4)$$

where $w = (w_1, \dots, w_{n-1})$ is the uncertain constant parameter and the functions $g_i, i = 2, \dots, n$, vanish at $(0, \dots, 0, w)$. The goal of IDSP is to achieve property a) of GRORP and $\lim_{t \rightarrow \infty} x(t) = 0$. There is distinct difference between IDSP and GRORP. Roughly speaking (see Remark 3.3 for more specific comparison between IDSP and GRORP), for the IDSP, the internal model consists of only one dynamic system associated with the input u . The IDSP of system (4) can be converted into a global robust stabilization problem for a class of feedforward systems subject to input unmodeled dynamics. Several results about this robust stabilization problem have been reported, see e.g., [1], [17], [18], [22]. In contrast, for the GRORP, the internal model in general consists of n dynamic systems associated with $x_2, \dots, x_n$ and the input u , respectively. The GRORP of system (1) can be converted into a global robust stabilization problem for a class of feedforward systems subject to both time-varying static and dynamic uncertainties described by (5) below. The stabilization problem of a system of the form (5) is itself interesting and is worth an independent study.

In order to solve the GRORP of strict feedforward system (1), we have to overcome the difficulties mentioned above. First, we need to identify the structural properties of the functions q_d and f_i in (1) so that an internal model candidate exists. Then, by looking for a suitable internal model and performing appropriate transformations on the augmented system consisting of (1) and the internal model, we can convert the GRORP of system (1) to a global robust stabilization problem of the system

$$\begin{aligned}\dot{\bar{x}}_i &= \bar{f}_i(\xi_i, \bar{x}_{i-1}, \dots, \bar{x}_1, \xi_1, \bar{u}, d) \\ \dot{\xi}_i &= \bar{g}_i(\xi_i, \bar{x}_{i-1}, \dots, \bar{x}_1, \xi_1, \bar{u}, d), \quad i = n, \dots, 2 \\ \dot{\bar{x}}_1 &= \bar{f}_1(\xi_1, \bar{u}, d) \\ \dot{\xi}_1 &= \bar{g}_1(\xi_1, \bar{u}, d)\end{aligned}\quad (5)$$

where for $i = 1, \dots, n$, $\bar{x}_i \in \mathbb{R}$, $\xi_i \in \mathbb{R}^{n_{\xi_i}}$, $d \in \mathbb{R}^{n_d}$, $\bar{u} \in \mathbb{R}$, $\bar{f}_i, \bar{g}_i$ are smooth functions vanishing at $(0, \dots, 0, d)$ for all $d \in \mathcal{D}$, n_{ξ_i} and n_d are dimensions of ξ_i and d respectively. System (5) contains two types of uncertainties, i.e., time-varying static uncertainty represented by the external disturbance d where $d : [0, \infty) \rightarrow \mathcal{D}$ is a continuous function with its range $\mathcal{D}$ a compact subset having a known bound, and dynamic uncertainty represented by dynamics governing $\xi_1, \xi_2, \dots, \xi_n$. The dynamics governing $\xi_1, \xi_2, \dots, \xi_n$ is called dynamic uncertainty because $\xi_1, \xi_2, \dots, \xi_n$ are not allowed for feedback. The global robust stabilization problem of system (5) had not been studied until recently [4] in which, a bottom-up recursive design procedure is presented to deal with the problem. Two types of the small gain theorem with restrictions adapted from [25] is applied to establish the local stability and global attractiveness of the closed-loop system at the origin respectively.

The rest of the note is organized as follows. Section II presents some definitions and preliminary results, and the result of the global robust stabilization problem for system (5). The main result of this note is contained in Section III. In Section IV, an illustrative example is elaborated. Finally, Section V concludes this note.

Like [25], we let (x_1, x_2) with $x_i \in \mathbb{R}^{n_i}, i = 1, 2$, denote the vector $(x_1^T, x_2^T)^T \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, and let $\mathcal{L}_\infty^m$ be the set of all piecewise continuous functions $u : [0, \infty) \rightarrow \mathbb{R}^m$ with a finite supremum norm $\|u\|_\infty = \sup_{t \geq 0} \|u(t)\|$, and let $\|u\|_a = \limsup_{t \rightarrow \infty} \|u(t)\|$ denote the asymptotic $\mathcal{L}_\infty$ norm of u , where $\|\cdot\|$ denotes the standard Euclidean norm. A function $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is called a gain function if it is continuous, nondecreasing, and satisfies $\gamma(0) = 0$. Finally, let I_n be n dimensional identity matrix.

II. PRELIMINARY

We first review some terminologies in [5], [25] for nonlinear systems of the following form:

$$\dot{x} = f(x, u, d), \quad y = h(x, u, d) \quad (6)$$

where $x \in \mathbb{R}^n$ is the plant state, $y \in \mathbb{R}^p$ the output, $u \in \mathbb{R}^m$ the piecewise continuous input, $f(x, u, d)$ and $h(x, u, d)$ are locally Lipschitz functions vanishing at $(0, 0, d)$ for all $d \in \mathcal{D}$, and $d : [0, \infty) \rightarrow \mathcal{D}$ is a continuous function with its range $\mathcal{D}$ a compact subset of $\mathbb{R}^{n_d}$. Let $x(t)$ denote the solution of system (6) with initial state $x(0)$, input u , and d .

Definition 2.1: [5] The output y of system (6) is said to satisfy a robust $\mathcal{L}_\infty$ stability bound (RLB) with restrictions X, Δ on $x(0), u$ and gains γ^0, γ respectively, if there exist an open subset X of $\mathbb{R}^n$ containing the origin, a positive real number Δ , gain functions γ^0, γ , all independent of d , such that, for each $x(0) \in X, d \in \mathcal{D}, \|u\|_\infty < \Delta$, the solution of (6) exists for all $t \geq 0$ and

$$\|y\|_\infty \leq \max\{\gamma^0(\|x(0)\|), \gamma(\|u\|_\infty)\}. \quad (7)$$

Definition 2.2: [25] The output y of system (6) is said to satisfy a robust a - $\mathcal{L}_\infty$ stability bound (Ra-LB) with restrictions X, Δ on $x(0)$, u and gains γ^0, γ respectively, if y satisfies RLB with restrictions X, Δ on $x(0)$, u and gains γ^0, γ respectively, and for each $x(0) \in X, d \in \mathcal{D}$, $\|u\|_\infty < \Delta$,

$$\|y\|_a \leq \gamma(\|u\|_a). \quad (8)$$

The output y of system (6) is said to satisfy a robust asymptotic bound (RAB) with restriction X on $x(0)$, restriction Δ on u and gain γ , if there exist an open subset X of $\mathbb{R}^n$ containing the origin, a non-negative real number Δ , a gain function γ , all independent of d , such that, for each $x(0) \in X, d \in \mathcal{D}$ and piecewise continuous u satisfying $\|u\|_a \leq \Delta$, the solution of (6) exists for all $t \geq 0$ and

$$\|y\|_a \leq \gamma(\|u\|_a). \quad (9)$$

Remark 2.1: In both Definitions 2.1 and 2.2, the word *robust* is used to emphasize that the inequalities (7)–(9) hold regardless of the presence of the disturbance d in (6). For convenience, we will simply use LB, AB and a-LB to mean RLB, RAB and Ra-LB respectively. The combination of LB and AB can be used to study the asymptotic stability of system (6) with $u = 0$. More specifically, if the state x of system (6) satisfies LB and AB with restrictions X_s and X_a on $x(0)$ respectively, then the equilibrium point $x = 0$ of system (6) with $u = 0$ is locally asymptotically stable, and if, in addition, $X_a = \mathbb{R}^n$, then it is globally asymptotically stable. As for the relationship between a-LB and the combination of LB and AB, we refer the reader to [5]. $\diamond$

For simplicity, if the output y of system (6) satisfies LB with restriction on $x(0)$, restriction Δ on u and gain γ , and satisfies AB with no restriction on $x(0)$, restriction Δ on u and gain γ , then we will say y satisfies LB with restriction and AB with no restriction on $x(0)$, both with restriction Δ on u and gain γ .

Like [1], [25], our approach will utilize saturation functions characterized as follows.

Definition 2.3: A locally Lipschitz function $\sigma(\cdot) : \mathbb{R} \rightarrow [-\lambda, \lambda]$ is said to be a saturation function with saturation level $\lambda > 0$, if $\sigma(s) = s$ when $|s| \leq \lambda/2$, and $\lambda/2 \leq \text{sgn}(s)\sigma(s) \leq \min\{|s|, \lambda\}$ when $|s| \geq \lambda/2$.

In the rest of this section, we restate the stabilization result obtained in [4] for system (5). For this purpose, we first make two assumptions on system (5).

Define $A_1(d) = \partial \bar{g}_1 / \partial \xi_1|_{(0,0,d)}$, $B_1(d) = \partial \bar{g}_1 / \partial \bar{u}|_{(0,0,d)}$, $c_1(d) = \partial \bar{f}_1 / \partial \bar{u}|_{(0,0,d)}$, $D_1(d) = \partial \bar{f}_1 / \partial \xi_1|_{(0,0,d)}$ and for $i = 2, \dots, n$, $A_i(d) = \partial \bar{g}_i / \partial \xi_i|_{(0,\dots,0,d)}$, $B_i(d) = \partial \bar{g}_i / \partial \bar{x}_{i-1}|_{(0,\dots,0,d)}$, $c_i(d) = \partial \bar{f}_i / \partial \bar{x}_{i-1}|_{(0,\dots,0,d)}$, $D_i(d) = \partial \bar{f}_i / \partial \xi_i|_{(0,\dots,0,d)}$. To simplify the notation, we drop the argument d in the matrices defined above, then system (5) can be rewritten in the following form:

$$\begin{aligned} \dot{\bar{x}}_i &= D_i \xi_i + c_i \bar{x}_{i-1} + \bar{f}_i^r(\xi_i, \bar{x}_{i-1}, \dots, \bar{x}_1, \bar{u}, d) \\ \dot{\xi}_i &= A_i \xi_i + B_i \bar{x}_{i-1} + \bar{g}_i^r(\xi_i, \bar{x}_{i-1}, \dots, \bar{x}_1, \bar{u}, d), \\ i &= n, \dots, 2 \\ \dot{\bar{x}}_1 &= D_1 \xi_1 + c_1 \bar{u} + \bar{f}_1^r(\xi_1, \bar{u}, d) \\ \dot{\xi}_1 &= A_1 \xi_1 + B_1 \bar{u} + \bar{g}_1^r(\xi_1, \bar{u}, d) \end{aligned} \quad (10)$$

where $\bar{f}_i^r, \bar{g}_i^r$ are suitably defined smooth functions.

Assumption 2.1: For $i = 1, \dots, n$, $D_i A_i^{-1}$ is a constant matrix and $\mu_i = c_i - D_i A_i^{-1} B_i$ is a positive (or alternatively negative) constant.

Assumption 2.2: ξ_1 satisfies LB and AB with no restriction on $\xi_1(0)$, both with restriction Δ_1 on $\bar{u}$ and gain $\bar{N}_1 s$, and for $i = 2, \dots, n$, ξ_i satisfies LB and AB with no restriction on $\xi_i(0)$, both with restriction Δ_i on $(\bar{x}_{i-1}, \xi_{i-1}, \dots, \bar{x}_1, \bar{u})$ and gain $\bar{N}_i s$.

Theorem 2.1: [4] Consider system (5). Under Assumptions 2.1–2.2, there exist $\lambda_i > 0$ and nonzero k_i with the same sign as θ_i where $\theta_1 = \mu_1$ and $\theta_i = \mu_i / k_{i-1}$, $i = 2, \dots, n$, such that, under the control

$$\bar{u} = -\sigma_1(k_1 \bar{x}_1 + \sigma_2(k_2 \bar{x}_2 + \dots + \sigma_n(k_n \bar{x}_n))) \quad (11)$$

where for $i = 1, \dots, n$, σ_i is a saturation function with level λ_i , the closed-loop system at $(0, \dots, 0)$ is globally asymptotically stable for all $d \in \mathcal{D}$.

III. MAIN RESULT

In this section, we will first give conditions under which the GRORP of system (1) can be converted into a global robust stabilization problem of a well defined augmented system (19) which takes the form (5) satisfying all assumptions of Theorem 2.1. Thus, we can further conclude the solvability of the GRORP of system (1) by Theorem 2.1. The first step of our approach is to find an appropriate internal model. For this purpose, we make the following assumptions.

Assumption 3.1: There exist smooth functions $\mathbf{x}(v, w) = (\mathbf{x}_1(v, w), \dots, \mathbf{x}_n(v, w))$ and $\mathbf{u}(v, w)$ with $\mathbf{x}(0, 0) = 0$ and $\mathbf{u}(0, 0) = 0$ satisfying for all $v \in \mathbb{R}^q, w \in \mathbb{R}^{n_w}$

$$\begin{aligned} \dot{\mathbf{x}}_i(v, w) &= f_i(\mathbf{x}_{i-1}(v, w), \dots, \mathbf{x}_1(v, w), \mathbf{u}(v, w), v, w), \\ i &= n, \dots, 2 \\ \dot{\mathbf{x}}_1(v, w) &= c\mathbf{u}(v, w) + f_1(v, w) \\ \mathbf{x}_1(v, w) &= q_d(v, w). \end{aligned} \quad (12)$$

Assumption 3.2: Let $\pi_1(v, w) = \mathbf{u}(v, w)$, $\pi_i(v, w) = \mathbf{x}_i(v, w)$, $i = 2, \dots, n$. For each $1 \leq i \leq n$ such that $\pi_i(v, w)$ is not identically zero, there exist sufficiently smooth functions $\tau_i : \mathbb{R}^q \times \mathbb{R}^{n_w} \rightarrow \mathbb{R}^{r_i}$, $i = 1, \dots, n$, vanishing at $(0, 0)$, matrix Φ_i , and column vector Ψ_i , such that

$$\dot{\tau}_i(v, w) = \Phi_i \tau_i(v, w), \quad \pi_i(v, w) = \Psi_i \tau_i(v, w) \quad (13)$$

where the pair (Ψ_i, Φ_i) is observable and all eigenvalues of Φ_i are simple with zero real parts

Remark 3.1: Equation (12) is called regulator equations and the solvability of these equations is necessary but not sufficient for the solvability of the robust output regulation problem [3], [9], [10], [14]. Assumption 3.2 is made for the existence of appropriate linear internal models. Both Assumption 3.1 and 3.2 are quite standard in literature. In particular, if $\mathbf{x}(v, w)$ and $\mathbf{u}(v, w)$ are polynomial in v , Assumption 3.2 is satisfied automatically [9], [10]. Under Assumption 3.2, for each $1 \leq i \leq n$ such that $\pi_i(v, w)$ is not identically zero, given a pair of controllable matrices (M_i, N_i) with $M_i \in \mathbb{R}^{i \times r_i}$ Hurwitz and $N_i \in \mathbb{R}^{r_i}$, there exists a unique and nonsingular matrix $T_i \in \mathbb{R}^{i \times r_i}$ satisfying the Sylvester equation

$$T_i \Phi_i - M_i T_i = N_i \Psi_i \quad (14)$$

since the spectra of M_i and Φ_i are disjoint and the pair (Ψ_i, Φ_i) is observable. $\diamond$

We define the following system:

$$\begin{aligned} \dot{\eta}_1 &= M_1 \eta_1 + N_1 u - \frac{M_1 N_1 e}{c} \\ \dot{\eta}_i &= M_i \eta_i + N_i x_i, \quad i = 2, \dots, n \end{aligned} \quad (15)$$

as the internal model of (1) with output $(u, x_2, \dots, x_n)$. Note that the dimension of η_i is understood to be zero if $\pi_i(v, w)$ is identically zero.

Next, we will convert the GRORP for system (1) into a global robust stabilization problem for the augmented system composed of the original plant (1) and the internal model (15). Performing the following coordinate and input transformation

$$\bar{x}_1 = x_1 - \mathbf{x}_1(v, w) = e$$

$$\begin{aligned}\bar{x}_i &= x_i - \Psi_i T_i^{-1} \eta_i, \quad i = 2, \dots, n \\ \bar{\eta}_i &= \eta_i - T_i \tau_i, \quad i = 1, \dots, n \\ \hat{u} &= u - \Psi_1 T_1^{-1} \eta_1\end{aligned}\quad (16)$$

on the augmented system gives

$$\begin{aligned}\dot{\bar{x}}_i &= -\Psi_i T_i^{-1} [(M_i + N_i \Psi_i T_i^{-1}) \bar{\eta}_i + N_i \bar{x}_i] \\ &\quad + \hat{f}_i(\bar{x}_{i-1}, \bar{\eta}_{i-1}, \dots, \bar{x}_1, \bar{\eta}_1, \hat{u}, v, w) \\ \dot{\bar{\eta}}_i &= (M_i + N_i \Psi_i T_i^{-1}) \bar{\eta}_i + N_i \bar{x}_i, \quad i = n, \dots, 2 \\ \dot{\bar{x}}_1 &= c \Psi_1 T_1^{-1} \bar{\eta}_1 + c \hat{u} \\ \dot{\bar{\eta}}_1 &= (M_1 + N_1 \Psi_1 T_1^{-1}) \bar{\eta}_1 + N_1 \hat{u} - \frac{M_1 N_1 \bar{x}_1}{c}\end{aligned}\quad (17)$$

where

$$\begin{aligned}\hat{f}_2(\bar{x}_1, \bar{\eta}_1, \hat{u}, v, w) \\ = -f_2(\mathbf{x}_1, \mathbf{u}, v, w) + f_2(\bar{x}_1 + \mathbf{x}_1, \hat{u} + \Psi_1 T_1^{-1} \bar{\eta}_1 + \mathbf{u}, v, w)\end{aligned}$$

and

$$\begin{aligned}\hat{f}_i(\bar{x}_{i-1}, \bar{\eta}_{i-1}, \dots, \bar{x}_1, \bar{\eta}_1, \hat{u}, v, w) \\ = -f_i(\mathbf{x}_{i-1}, \dots, \mathbf{x}_1, \mathbf{u}, v, w) \\ + f_i(\bar{x}_{i-1} + \Psi_{i-1} T_{i-1}^{-1} \bar{\eta}_{i-1} + \mathbf{x}_{i-1}, \dots, \bar{x}_1 \\ + \mathbf{x}_1, \hat{u} + \Psi_1 T_1^{-1} \bar{\eta}_1 + \mathbf{u}, v, w), \quad i = 3, \dots, n.\end{aligned}$$

It is known from [9] and [10] that the GRORP of system (1) will be solved if we can make the equilibrium point $(\bar{x}, \bar{\eta}) = (0, 0)$ of system (17) globally asymptotically stable for any trajectories $v(t)$ starting from V_0 and any $w \in W$, where $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$ and $\bar{\eta} = (\bar{\eta}_1, \dots, \bar{\eta}_n)$. A system of the form (17) has never been encountered and there is no clue whether or not this system is stabilizable at the equilibrium point. Nevertheless, by performing some further coordinate and input transformations on (17), it is possible to convert (17) to the form of (5) with all desirable properties. For this purpose, we introduce two more assumptions.

Assumption 3.3: For each $2 \leq i \leq n$ such that $\pi_i(v, w)$ is not identically zero, Φ_i is invertible.

Assumption 3.4: For $i = 2, \dots, n$

$$\frac{\partial f_i}{\partial x_{i-1}} \Big|_{(x_{i-1}, \dots, x_1, u) = (\mathbf{x}_{i-1}(v, w), \dots, \mathbf{x}_1(v, w), \mathbf{u}(v, w))}$$

is a positive (or alternatively negative) constant.

Now define the following coordinate and input transformation:

$$\begin{aligned}\xi_1 &= c \bar{\eta}_1 - N_1 \bar{x}_1, \\ \xi_i &= (M_i + N_i \Psi_i T_i^{-1}) \bar{\eta}_i + N_i \bar{x}_i, \quad i = 2, \dots, n \\ \bar{u} &= \hat{u} + \frac{\Psi_1 T_1^{-1} N_1 \bar{x}_1}{c}.\end{aligned}\quad (18)$$

From (14), $M_i + N_i \Psi_i T_i^{-1} = T_i \Phi_i T_i^{-1}$, and then from Assumption 3.3 and $c \neq 0$, the transformation (18) is globally invertible. Performing the transformation (18) on (17) yields

$$\begin{aligned}\dot{\xi}_i &= -\Psi_i T_i^{-1} \xi_i + \tilde{f}_i(\bar{x}_{i-1}, \xi_{i-1}, \dots, \bar{x}_1, \xi_1, \bar{u}, d) \\ \dot{\xi}_i &= M_i \xi_i + N_i \tilde{f}_i(\bar{x}_{i-1}, \xi_{i-1}, \dots, \bar{x}_1, \xi_1, \bar{u}, d), \quad i = n, \dots, 2 \\ \dot{\bar{x}}_1 &= \Psi_1 T_1^{-1} \xi_1 + c \bar{u} \\ \dot{\xi}_1 &= M_1 \xi_1\end{aligned}\quad (19)$$

where $d = (v, w)$

$$\begin{aligned}\tilde{f}_2(\bar{x}_1, \xi_1, \bar{u}, d) \\ = -f_2(\mathbf{x}_1, \mathbf{u}, d) + f_2\left(\bar{x}_1 + \mathbf{x}_1, \bar{u} + \frac{\Psi_1 T_1^{-1} \xi_1}{c} + \mathbf{u}, d\right)\end{aligned}$$

and

$$\begin{aligned}\tilde{f}_i(\bar{x}_{i-1}, \xi_{i-1}, \dots, \bar{x}_1, \xi_1, \bar{u}, d) \\ = -f_i(\mathbf{x}_{i-1}, \dots, \mathbf{x}_1, \mathbf{u}, d) \\ + f_i\left((1 - \Psi_{i-1} \Phi_{i-1}^{-1} T_{i-1}^{-1} N_{i-1}) \bar{x}_{i-1} + \Psi_{i-1} \Phi_{i-1}^{-1} T_{i-1}^{-1} \xi_{i-1} \right. \\ \left. + \mathbf{x}_{i-1}, \dots, \bar{x}_1 + \mathbf{x}_1, \bar{u} + \frac{\Psi_1 T_1^{-1} \xi_1}{c} + \mathbf{u}, d\right), \quad i = 3, \dots, n.\end{aligned}$$

Without loss of generality, assume $d \in \mathcal{D} = V \times W$ where V is a compact set containing all trajectories of (2) starting from V_0 . Since V_0 is compact, and all eigenvalues of the matrix S in (2) are simple with zero real parts, V exists. Thus, $\mathcal{D}$ is compact. Clearly, the GRORP of system (1) will be solved, if we can globally asymptotically stabilize the origin of system (19) for all $d \in \mathcal{D}$.

Theorem 3.1: Suppose system (1) satisfies Assumptions 3.1 to 3.4. Then, the GRORP can be solved by a dynamic state feedback controller of the form

$$\begin{aligned}u &= \Psi_1 T_1^{-1} \left(\eta_1 - \frac{N_1 e}{c} \right) \\ &\quad - \sigma_1(k_1 e + \dots + \sigma_n(k_n x_n - k_n \Psi_n T_n^{-1} \eta_n)) \\ \dot{\eta}_1 &= M_1 \eta_1 + N_1 u - \frac{M_1 N_1 e}{c} \\ \dot{\eta}_i &= M_i \eta_i + N_i x_i, \quad i = 2, \dots, n.\end{aligned}\quad (20)$$

Proof: Since system (19) is in the form of (5), by Theorem 2.1, it suffices to show that, system (19) satisfies Assumptions 2.1 and 2.2.

Let us first verify that system (19) satisfies Assumption 2.1. Rewrite (19) in the form of (10) as follows:

$$\begin{aligned}\dot{\bar{x}}_i &= -\Psi_i T_i^{-1} \xi_i + \bar{c}_i \bar{x}_{i-1} + \bar{f}_i^r(\bar{x}_{i-1}, \xi_{i-1}, \dots, \bar{x}_1, \xi_1, \bar{u}, d) \\ \dot{\xi}_i &= M_i \xi_i + N_i \bar{c}_i \bar{x}_{i-1} + N_i \bar{f}_i^r(\bar{x}_{i-1}, \xi_{i-1}, \dots, \bar{x}_1, \xi_1, \bar{u}, d) \\ i &= n, \dots, 2 \\ \dot{\bar{x}}_1 &= \Psi_1 T_1^{-1} \xi_1 + \bar{c}_1 \bar{u} \\ \dot{\xi}_1 &= M_1 \xi_1\end{aligned}\quad (21)$$

where $\bar{f}_i^r, i = 2, \dots, n$ are suitably defined smooth functions and $\bar{c}_1 = c, \bar{c}_2 = \partial \tilde{f}_2 / \partial \bar{x}_1|_{(0,0,0,d)} = \partial f_2 / \partial x_1|_{(x_1,u)=(\mathbf{x}_1,u)}$

$$\begin{aligned}\bar{c}_i &= \frac{\partial \tilde{f}_i}{\partial \bar{x}_{i-1}} \Big|_{(0, \dots, 0, d)} \\ &= (1 - \Psi_{i-1} \Phi_{i-1}^{-1} T_{i-1}^{-1} N_{i-1}) \\ &\quad \cdot \frac{\partial f_i}{\partial x_{i-1}} \Big|_{(x_{i-1}, \dots, x_1, u) = (\mathbf{x}_{i-1}, \dots, \mathbf{x}_1, \mathbf{u})}, \quad i = 3, \dots, n.\end{aligned}$$

Then noting the form of (21) yields

$$\begin{aligned}\mu_1 &= c, \quad \mu_2 = (1 + \Psi_2 T_2^{-1} M_2^{-1} N_2) \frac{\partial f_2}{\partial x_1} \Big|_{(x_1,u)=(\mathbf{x}_1,u)}, \\ \mu_i &= (1 + \Psi_i T_i^{-1} M_i^{-1} N_i) (1 - \Psi_{i-1} \Phi_{i-1}^{-1} T_{i-1}^{-1} N_{i-1}) \\ &\quad \cdot \frac{\partial f_i}{\partial x_{i-1}} \Big|_{(x_{i-1}, \dots, x_1, u) = (\mathbf{x}_{i-1}, \dots, \mathbf{x}_1, \mathbf{u})}, \quad i = 3, \dots, n.\end{aligned}$$

We claim that $1 + \Psi_i T_i^{-1} M_i^{-1} N_i \neq 0, i = 2, \dots, n$, and $1 - \Psi_{i-1} \Phi_{i-1}^{-1} T_{i-1}^{-1} N_{i-1} \neq 0, i = 3, \dots, n$. From (14), Assumption 3.3 and the identity $\det(I_n - PQ) = \det(I_m - QP)$ where P, Q are $n \times m$ and $m \times n$ matrices respectively, we have

$$\begin{aligned}1 + \Psi_i T_i^{-1} M_i^{-1} N_i &= \det(I_{r_i} + N_i \Psi_i T_i^{-1} M_i^{-1}) \\ &= \det(T_i \Phi_i T_i^{-1} M_i^{-1}) \neq 0, \\ 1 - \Psi_{i-1} \Phi_{i-1}^{-1} T_{i-1}^{-1} N_{i-1} &= \det(I_{r_{i-1}} - N_{i-1} \Psi_{i-1} \Phi_{i-1}^{-1} T_{i-1}^{-1}) \\ &= \det(M_{i-1} T_{i-1} \Phi_{i-1}^{-1} T_{i-1}^{-1}) \neq 0.\end{aligned}$$

Then noting the form of (21) and Assumption 3.4 shows that Assumption 2.1 is satisfied.

Next, we show that system (19) also satisfies Assumption 2.2. For $i = 1$, the specific form of the last equation of (19) immediately implies that ξ_1 satisfies Assumption 2.2 with $\bar{N}_1 = 0$ and $\Delta_1 = \infty$. For $i = 2, \dots, n$, let $\tilde{u}_i = \tilde{f}_i(\bar{x}_{i-1}, \xi_{i-1}, \dots, \bar{x}_1, \xi_1, \bar{u}, d)$. Then ξ_i subsystem in (19) is rewritten as $\dot{\xi}_i = M_i \xi_i + N_i \tilde{u}_i$. Since M_i is Hurwitz, ξ_i satisfies a-LB with no restriction on $\xi_i(0)$, no restriction on $\tilde{u}_i$ and linear gain $J_i s$, where J_i is an appropriate nonnegative constant. Since $\tilde{f}_i(\bar{x}_{i-1}, \xi_{i-1}, \dots, \bar{x}_1, \xi_1, \bar{u}, d)$ is smooth and $\tilde{f}_i(0, \dots, 0, d) = 0$ for $d \in \mathcal{D}$, there exist positive constants $\bar{L}_i, \bar{\delta}_i$ independent of d such that $\|\tilde{u}_i\| \leq \bar{L}_i \|\bar{x}_{i-1}, \xi_{i-1}, \dots, \bar{x}_1, \xi_1, \bar{u}\|$ for $\|\bar{x}_{i-1}, \xi_{i-1}, \dots, \bar{x}_1, \xi_1, \bar{u}\| \leq \bar{\delta}_i$ and $d \in \mathcal{D}$. Thus, Assumption 2.2 is satisfied with $\bar{N}_i = J_i \bar{L}_i$ and $\Delta_i = \bar{\delta}_i, i = 2, \dots, n$.

By Theorem 2.1, there exist $\lambda_i > 0$ and nonzero k_i such that, the following control

$$\bar{u} = -\sigma_1(k_1 \bar{x}_1 + \sigma_2(k_2 \bar{x}_2 + \dots + \sigma_n(k_n \bar{x}_n))) \quad (22)$$

can globally asymptotically stabilize the origin of system (19) for all $d \in \mathcal{D}$. Noting (15), (16), (18) and (22) yields the controller (20), which solves the GRORP of system (1). ■

For the class of strict feedforward systems which only involve polynomial nonlinearities, Assumptions 3.1 to 3.3 can be easily testified. To this end, let $v^{[1]} = v = (v_1, \dots, v_q) \in \mathbb{R}^q$ and for $l \geq 2$

$$v^{[l]} = (v_1^l, v_1^{l-1} v_2, \dots, v_1^{l-1} v_q, v_1^{l-2} v_2^2, v_1^{l-2} v_2 v_3, \dots, v_1^{l-2} v_2 v_q, \dots, v_q^l).$$

Proposition 3.1: Assume q is even, and $f(v, w)$ is an odd polynomial in v , i.e., there exist nonnegative integer κ and row vectors $F_{2i+1}(w), i = 0, \dots, \kappa$, with suitable dimensions such that

$$f(v, w) = \sum_{i=0}^{\kappa} F_{2i+1}(w) v^{[2i+1]}. \quad (23)$$

Then there exists an odd polynomial $x(v, w)$ in v such that $\dot{x}(v, w) = f(v, w)$ for all trajectories $v(t)$ of the exosystem and $w \in \mathbb{R}^n$. Moreover, there exist an integer r and $\Phi \in \mathbb{R}^{r \times r}, \Psi \in \mathbb{R}^{1 \times r}$, where Φ is nonsingular with all its eigenvalues simple and on the imaginary axis and the pair (Ψ, Φ) is observable, such that $\tau(v, w) = (x(v, w), \dot{x}(v, w), \dots, d^{(r-1)}x(v, w)/dt^{(r-1)})$ satisfies $\dot{\tau}(v, w) = \Phi \tau(v, w)$ and $x(v, w) = \Psi \tau(v, w)$.

Proof: First, note from [9, p. 123] that, there exists a matrix $S^{[l]}$ such that $dv^{[l]}/dt = S^{[l]} v^{[l]}$ and moreover, all the eigenvalues of $S^{[l]}$ are given by

$$\lambda = l_1 \lambda_1 + \dots + l_q \lambda_q, l_1 + \dots + l_q = l, l_1, \dots, l_q = 0, 1, \dots, l \quad (24)$$

where $\lambda_1, \dots, \lambda_q$ are eigenvalues of S . Thus, if all eigenvalues of S are simple with zero real parts, then $S^{[l]}$ is nonsingular if and only if q is even and l is odd. Then it is ready to derive that

$$x(v, w) = \sum_{i=0}^{\kappa} F_{2i+1}(w) (S^{[2i+1]})^{-1} v^{[2i+1]}$$

is a solution of $\dot{x}(v, w) = f(v, w)$ for all trajectories $v(t)$ of the exosystem and $w \in \mathbb{R}^n$.

Since all eigenvalues of S are simple with zero real parts, from [9, Theor. 5.16], the roots of the minimal polynomial of $S^{[l]}$ coincide with all distinct eigenvalues of $S^{[l]}$. Thus, from (24), the minimal polynomial of $S^{[2i+1]}$ divides the minimal polynomial of $S^{[2j+1]}$ whenever $i \leq j$. Denote the minimal polynomial of the matrix $S^{[2\kappa+1]}$ by $P(\lambda) = \lambda^r - a_1 - a_2 \lambda - \dots - a_r \lambda^{r-1}$ for some real numbers $a_1, \dots, a_r$. Then, the roots of $P(\lambda)$ are non-repeated with zero

real parts. By the Cayley-Hamilton Theorem, $P(S^{[2i+1]}) = 0, i = 0, \dots, \kappa$. Thus

$$\frac{d^r x(v(t), w)}{dt^r} - a_1 x(v(t), w) - a_2 \frac{dx(v(t), w)}{dt} - \dots - a_r \frac{d^{(r-1)} x(v(t), w)}{dt^{(r-1)}} = 0$$

for all trajectories $v(t)$ of the exosystem and $w \in \mathbb{R}^n$. From the above equation and by Proposition 6.12 of [9], $\Phi \in \mathbb{R}^{r \times r}, \Psi \in \mathbb{R}^{1 \times r}$ exist and the pair (Ψ, Φ) is observable. Moreover, since the characteristic polynomial of Φ is the minimal polynomial of $S^{[2\kappa+1]}$, Φ is nonsingular, and all its eigenvalues are simple and on the imaginary axis. ■

Remark 3.2: If q is even and $q_d(v, w)$ is an odd polynomial in v , then by Proposition 3.1, it can be concluded that, Assumptions 3.1 to 3.3 are satisfied if $f_1(v, w)$ is an odd polynomial in v and for $i = 2, \dots, n$, $f_i(x_{i-1}, \dots, x_1, u, v, w)$ is an odd polynomial in $(x_{i-1}, \dots, x_1, u, v)$. ◇

Remark 3.3: When $q_d(v, w) = 0$ and $f_i, i = 2, \dots, n$, in (1) are independent of u and vanish at $(0, \dots, 0, v, w)$, the GRORP of system (1) reduces to the IDSP studied in [19]. For this special case, $\mathbf{u}(v, w) = -f_1(v, w)/c, \mathbf{x}(v, w) = 0$ and thus Assumption 3.1 is satisfied automatically. Moreover, since $\mathbf{x}(v, w) = 0$, there is no need to estimate $\mathbf{x}(v, w)$. It suffices to use one single system $\dot{\eta}_1 = M_1 \eta_1 + N_1 u - M_1 N_1 x_1/c$ to define the internal model which essentially reduces to the same case as what has been done in [19]. Also, Assumption 3.3 is not needed anymore and thus Assumption 3.2 with $i = 1$ and Assumption 3.4 become the assumptions to the IDSP of system (1). The IDSP of system (1) can be converted into a global robust stabilization problem for a class of feedforward systems with input unmodeled dynamics. On the other hand, when $q_d(v, w) \neq 0, \mathbf{x}(v, w) \neq 0$ in general. To estimate $\mathbf{u}(v, w)$ and $\mathbf{x}_2(v, w), \dots, \mathbf{x}_n(v, w)$, we define the internal model (15). If Assumptions 3.1-3.4 are satisfied, then the GRORP of system (1) can be solved by converting it into a global robust stabilization problem for a class of feedforward systems with both time-varying static and dynamic uncertainties. Thus, there is distinct difference between IDSP and GRORP. ◇

IV. AN ILLUSTRATIVE EXAMPLE

We study the global robust output regulation problem of the following system:

$$\begin{aligned} \dot{x}_2 &= (1 + 0.05 w v_1^2 v_2) x_1 + 0.05 x_1 u + w(v_1 - v_1^3) \\ \dot{x}_1 &= 10u + 7w v_1^2 v_2 \\ \dot{v}_1 &= -v_2, \dot{v}_2 = v_1 \\ e &= x_1 - w v_1^3 \end{aligned} \quad (25)$$

where for illustration, we assume $|w| \leq 1$ is the uncertain constant parameter and $\|v(0)\| \leq 1$.

System (25) is in the form of (1). Let us verify that (25) satisfies Assumptions 3.1 to 3.4. Firstly, the regulator equations of (25) have a global solution as follows:

$$\mathbf{x}_1(v, w) = w v_1^3, \mathbf{x}_2(v, w) = w v_2, \mathbf{u}(v, w) = -w v_1^2 v_2 \quad (26)$$

which implies that Assumption 3.2 is satisfied. Simple calculation shows that

$$\begin{aligned} \Phi_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9 & 0 & -10 & 0 \end{bmatrix}, \Phi_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ \Psi_1 &= [1 \ 0 \ 0 \ 0], \Psi_2 = [1 \ 0]. \end{aligned} \quad (27)$$

From (27) and $\partial f_2 / \partial x_1|_{(x_1, u) = (\mathbf{x}_1(v, w), \mathbf{u}(v, w))} = 1$, Assumptions 3.3 and 3.4 are satisfied. Thus, Theorem 3.1 can be applied to solve the global robust output regulation problem of system (25).

To design the internal model, let

$$M_1 = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, M_2 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$N_1 = [1 \ 1 \ 1 \ 1]^T, N_2 = [1 \ 1]^T.$$

Solving the Sylvester equation (14) gives

$$T_1 = \begin{bmatrix} 0.2447 & -0.0612 & 0.0094 & -0.0024 \\ 0.3167 & -0.1056 & 0.0167 & -0.0056 \\ 0.4308 & -0.2154 & 0.0308 & -0.0154 \\ 0.5500 & -0.5500 & 0.0500 & -0.0500 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 0.4 & -0.2 \\ 0.5 & -0.5 \end{bmatrix}.$$

Then the internal model takes the following form:

$$\dot{\eta}_1 = M_1\eta_1 + N_1u - 0.1M_1N_1e, \dot{\eta}_2 = M_2\eta_2 + N_2x_2. \quad (28)$$

Using the coordinate and input transformations (16) and (18), the augmented system consisting of (25) and (28) is put into the following form

$$\begin{aligned} \dot{\bar{x}}_2 &= -\Psi_2T_2^{-1}\xi_2 + \bar{x}_1 + 0.05(\bar{x}_1 + wv_1^3)(\bar{u} + 0.1\Psi_1T_1^{-1}\xi_1) \\ \dot{\xi}_2 &= M_2\xi_2 + N_2\bar{x}_1 + 0.05N_2(\bar{x}_1 + wv_1^3)(\bar{u} + 0.1\Psi_1T_1^{-1}\xi_1) \\ \dot{\bar{x}}_1 &= \Psi_1T_1^{-1}\xi_1 + 10\bar{u} \\ \dot{\xi}_1 &= M_1\xi_1. \end{aligned} \quad (29)$$

In the following, Theorem 2.1 will be used to design the stabilizing control $\bar{u} = -\sigma_1(-k_1\bar{x}_1 + \sigma_2(k_2\bar{x}_2))$ for system (29). The design procedure follows the proof of [4, Theor. 3.1].

Performing the coordinate transformation

$$z_1 = \bar{x}_1 - \Psi_1T_1^{-1}M_1^{-1}\xi_1, z_2 = \bar{x}_2 + \Psi_2T_2^{-1}M_2^{-1}\xi_2 + \frac{\theta_2}{\theta_1}z_1$$

on (29) gives

$$\begin{aligned} \dot{z}_2 &= \theta_2\bar{u} + 0.025(\bar{x}_1 + wv_1^3)(\bar{u} + 0.1\Psi_1T_1^{-1}\xi_1) + \theta_2k_1\bar{x}_1 \\ \dot{\xi}_2 &= M_2\xi_2 + N_2\bar{x}_1 + 0.05N_2(\bar{x}_1 + wv_1^3)(\bar{u} + 0.1\Psi_1T_1^{-1}\xi_1) \\ \dot{z}_1 &= \theta_1\bar{u} \\ \dot{\xi}_1 &= M_1\xi_1 \end{aligned} \quad (30)$$

where $\theta_1 = \mu_1 = 10, \theta_2 = \mu_2/k_1 = 0.5/k_1$. Since k_i has the same sign with θ_i, k_1, k_2 are both positive in this case.

First, consider z_1, ξ_1 dynamics. Since $\bar{N}_1 = 0, \Delta_1 = \infty$, for arbitrarily positive λ_1, k_1 , under the control $\bar{u} = -\sigma_1(k_1z_1 + k_1\Psi_1T_1^{-1}M_1^{-1}\xi_1 - u_1)$, $z_1, \bar{x}_1, \bar{u}$ satisfy LB with restriction and AB with no restriction on $(z_1(0), \xi_1(0))$, both with restriction $\lambda_1/3$ on u_1 and gains $3s/k_1, 6s/k_1, 6s$, respectively. Then consider z_2, ξ_2 dynamics. We first calculate the gain from u_1 to ξ_2 . Let P_2 be a positive definite and symmetric matrix such that $M_2^TP_2 + P_2M_2 = -2I$, and $\bar{u}_2 = \bar{x}_1 + 0.05(\bar{x}_1 + wv_1^3)(\bar{u} + 0.1\Psi_1T_1^{-1}\xi_1)$. It can be verified that $V(\xi_2) = \xi_2^TP_2\xi_2/2$ satisfies the assumptions of Lemma 3.3 of [25] and ξ_2 subsystem satisfies a-LB with no restriction on $\xi_2(0)$, no restriction on $\bar{u}_2$ and gain $\lambda_{\max}(P_2)\|P_2N_2\|s/\lambda_{\min}(P_2)$, where $\lambda_{\max}(P_2), \lambda_{\min}(P_2)$ are maximal and minimal eigenvalues of P_2 , respectively. Then note that

$$\begin{aligned} &(\bar{x}_1 + wv_1^3)(\bar{u} + 0.1\Psi_1T_1^{-1}\xi_1) \\ &= \bar{x}_1\bar{u} + wv_1^3\bar{u} + 0.1\bar{x}_1\Psi_1T_1^{-1}\xi_1 + 0.1wv_1^3\Psi_1T_1^{-1}\xi_1 \\ &\leq 0.50005\bar{x}_1^2 + 0.5\bar{u}^2 + |\bar{u}| \\ &\quad + 50\|\Psi_1T_1^{-1}\|^2\|\xi_1\|^2 + 0.1\|\Psi_1T_1^{-1}\|\|\xi_1\| \end{aligned} \quad (31)$$

which implies

$$\begin{aligned} &(\bar{x}_1 + wv_1^3)(\bar{u} + 0.1\Psi_1T_1^{-1}\xi_1) \\ &\leq 0.50005|x_1| + 1.5|\bar{u}| + (50\|\Psi_1T_1^{-1}\| + 0.1)\|\Psi_1T_1^{-1}\|\|\xi_1\| \end{aligned}$$

for $|\bar{x}_1| \leq 1, |\bar{u}| \leq 1$ and $\|\xi_1\| \leq 1$. Then we have

$$|\bar{u}_2| \leq 1.026|\bar{x}_1| + 0.075|\bar{u}| + 35462\|\xi_1\|$$

for $|\bar{x}_1| \leq 1, |\bar{u}| \leq 1$ and $\|\xi_1\| \leq 1$. Thus, ξ_2 satisfies LB with restriction and AB with no restriction on $\xi_2(0)$, both with restriction $\min\{\lambda_1/3, k_1/6, 1/6\}$ on u_1 and gain $N_{\xi_2 u_1}s$, where $N_{\xi_2 u_1} = 2(1.026 \times 6/k_1 + 0.075 \times 6)$.

Now let $\zeta_1 = (\xi_2, z_1, \xi_1)$. Then (30) can be written in the form

$$\begin{aligned} \dot{\zeta}_2 &= \theta_2 u_1 + \tilde{F}_2(\zeta_1, u_1, d) \\ \dot{\zeta}_1 &= G_1(\zeta_1, u_1, d) \end{aligned}$$

where

$$\begin{aligned} \tilde{F}_2(\zeta_1, u_1, d) &= 0.025(\bar{x}_1 + wv_1^3)(\bar{u} + 0.1\Psi_1T_1^{-1}\xi_1) \\ &\quad + \theta_2 h_1(\bar{x}_1, u_1) \end{aligned}$$

$h_1(\bar{x}_1, u_1) = k_1\bar{x}_1 - u_1 - \sigma_1(k_1\bar{x}_1 - u_1)$, and G_1 is a suitably defined function.

Let $u_1 = -\sigma_2(k_2z_2 - k_2\Psi_2T_2^{-1}M_2^{-1}\xi_2 - k_2\theta_2z_1/\theta_1)$. Clearly, $\bar{\gamma}_{11}^{(2)}(s) \leq 2 \max\{\|\Psi_2T_2^{-1}M_2^{-1}\|N_{\xi_2 u_1}, 3\theta_2/(\theta_1k_1)\}k_2s$. Note that $h_1(\bar{x}_1, u_1)$ has no contribution to $\bar{\gamma}_{12}^{(2)}(s)$ when $\lambda_2 < \min\{\lambda_1/12, k_1/6, 1/6\}$, then from (31), we obtain

$$\begin{aligned} \bar{\gamma}_{12}^{(2)}(s) &\leq 0.1k_1 \\ &\quad \cdot \frac{(6 \min\{k_2s, \lambda_2\} + (0.50005 \times \frac{36}{k_1^2} + 18) \min\{k_2s, \lambda_2\}^2)}{k_2}. \end{aligned}$$

Solving $6 \max\{\bar{\gamma}_{11}^{(2)}(s), \bar{\gamma}_{12}^{(2)}(s)\} < s$ for $s > 0$, we choose $k_1 = 0.2, k_2 = 0.00041, \lambda_1 = 10$ and $\lambda_2 = 0.0049$.

As a result, the global robust output regulation problem of system (25) is solved by the following dynamic state feedback control:

$$\begin{aligned} u &= \Psi_1T_1^{-1}(\eta_1 - 0.1N_1e) \\ &\quad - \sigma_1(0.2e + \sigma_2(0.00041(x_2 - \Psi_2T_2^{-1}\eta_2))) \\ \dot{\eta}_1 &= M_1\eta_1 + N_1u - 0.1M_1N_1e, \dot{\eta}_2 = M_2\eta_2 + N_2x_2 \end{aligned} \quad (32)$$

where σ_1, σ_2 are saturation functions with level 10 and 0.0049 respectively.

For illustration, Figs. 1 and 2 show the simulation result of system (25) under the control (32) with initial state

$$\begin{aligned} (x_1(0), x_2(0), v(0), \eta_1(0), \eta_2(0)) \\ = (5, -0.7, (0.5, -0.6), (0.5, 1, 1.5, 1), (5, 5)) \end{aligned}$$

and $w = 0.5$.

V. CONCLUSION

In this note, we have presented the solvability conditions of the GRORP by state feedback for strict feedforward systems. The problem is approached in two steps. In the first step, the GRORP of the system is converted into a global robust stabilization problem of an augmented system. In the second step, the stabilization problem of the augmented system is further addressed. For the success of the first step, a suitable internal model and appropriate transformations have to be found so that the augmented system takes a suitable form and is stabilizable. For the success of the second step, we need to globally robustly stabilize a class of feedforward systems subject to both time-varying static

Fig. 1. State trajectory.

Fig. 2. Tracking error.

and dynamic uncertainties, which is solved by using the bottom-up recursive design procedure recently developed in [4].

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