Convex relaxations for mixed integer predictive control

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Abstract

The main objective in this work is to compare different convex relaxations for Model Predictive Control (MPC) problems with mixed real valued and binary valued control signals. In the problem description considered, the objective function is quadratic, the dynamics are linear, and the inequality constraints on states and control signals are all linear. The relaxations are related theoretically and the quality of the bounds and the computational complexities are compared in numerical experiments. The investigated relaxations include the Quadratic Programming (QP) relaxation, the standard Semidefinite Programming (SDP) relaxation, and an equality constrained SDP relaxation. The equality constrained SDP relaxation appears to be new in the context of hybrid MPC and the result presented in this work indicates that it can be useful as an alternative relaxation, which is less computationally demanding than the ordinary SDP relaxation and which often gives a better bound than the bound from the QP relaxation. Furthermore, it is discussed how the result from the SDP relaxations can be used to generate suboptimal solutions to the control problem. Moreover, it is also shown that the equality constrained SDP relaxation is equivalent to a QP in an important special case.

Key words: Predictive control, Hybrid systems, Finite alphabet control, Integer programming, Semidefinite programming

1 Introduction

Model Predictive Control (MPC) is a popular control strategy which is widely used in industry. In this work, the objective is control of hybrid systems in Mixed Logical Dynamical (MLD) form [5]. The work is focused on a special case of MLD systems with a mix of real valued and binary valued control signals, and the binary state case is not explicitly covered. The considered problem is similar to the finite alphabet control problem which has been considered in, *e.g.*, [10]. The linear MPC problem can be cast in the form of a Quadratic Programming (QP) problem, while in the hybrid case, binary variables are introduced and the optimization problem is changed from a QP to a Mixed Integer Quadratic Programming (MIQP) problem. This type of control problem is sometimes called a Mixed Integer Predictive Control (MIPC) problem. Unfortunately, this change to the problem implies that the convex QP problem is replaced by a nonconvex problem which is, except for certain instances, known to be $\mathcal{NP}$-hard [21].

This work summarizes and extends the work previously presented in the conference papers [2] and [3]. The objective is to compare different relaxations applicable to MIPC. Recent research has shown that it is possible to use Semidefinite Programming (SDP) in order to compute bounds with high quality for integer programming problems and how to use them to increase performance when solving binary QP problems [7,11]. The SDP relaxations have previously been considered in several contexts, and they have successfully been applied to, *e.g.*, the Max Cut problem [9] and the Multiuser Detection problem [15]. For some problems, where the Max Cut problem perhaps is the most well-known one, the quality of the solution from the SDP relaxation can be guaranteed [9]. This idea has been extended in, *e.g.*, [16,22]. SDP relaxations have previously been proposed for control of systems with binary inputs in [10,18].

There are several interesting applications related to the MIPC problem where these relaxations can be useful. Perhaps the two most important ones are to use them to

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produce good suboptimal integer feasible solutions, and potentially to use them as an alternative to QP relaxations in a branch and bound method. Their usefulness to produce suboptimal solutions for hybrid MPC problems was investigated in [3]. According to several authors, [5,8], MIQP problems are preferably solved using the branch and bound method [21]. The efficiency of the branch and bound method highly relies on the possibility to efficiently compute good bounds on the optimal objective function value, and the idea is to make use of the potentially better bounds provided by the SDP relaxation, compared to the QP relaxation, to reduce the size of the tree that has to be explored.

1.1 Notation

In this article, $\mathbb{S}^n$ denotes symmetric matrices with n columns and $\mathbb{S}^n_{++}$ ($\mathbb{S}^n_{+}$) denotes symmetric positive (semi) definite matrices with n columns. Superscript $*$ is used to denote values of variables and functions at optimum. The function $\operatorname{diag}(\cdot)$ is defined such that if its argument is a vector, it evaluates to a diagonal matrix with the argument along its diagonal, and if its argument is a matrix, it evaluates to a vector whose elements consist of the diagonal elements in the matrix. The sets $\mathcal{T} = \{0, \ldots, N-1\}$ and $\mathcal{I} = \{1, \ldots, Nm\}$ are also frequently used. A Sans Serif font is used to indicate that a matrix or a vector is, in some way, composed by stacked matrices or vectors from different time instants. The stacked matrices or vectors have a similar symbol as the composed matrix, but are typeset using a normal font. For example, $\mathsf{Q_u} = \operatorname{diag}(Q_u, \ldots, Q_u)$. A detailed description of the notation can be found in [1, pp. 65–67].

2 Introduction to the control problem

In this work, an MIPC problem over a prediction horizon of length N for a system in the form

$$\begin{aligned} x(0) &= x_0 \\ x(t+1) &= Ax(t) + Bu(t),\ \forall t \in \mathcal{T} \end{aligned} \tag{1}$$

is considered, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $x(t) \in \mathbb{R}^n$ is the state, and where $x_0 \in \mathbb{R}^n$ is the initial state. Furthermore, to be able to keep the notation as simple as possible, an all binary control input $u(t) \in \{0,1\}^m$ is used in the presentation. However, all results presented also hold for the case with mixed real valued and binary valued control signals, and such examples are used in the numerical experiments. The quadratic objective function to be minimized is

$$\begin{aligned} &J_{\mathrm{MPC}}(x(\cdot), u(\cdot)) \\ &= \frac{1}{2}\sum_{t=0}^{N-1}\left(\|x(t)\|^2_{Q_x} + \|u(t)\|^2_{Q_u}\right) + \frac{1}{2}\|x(N)\|^2_{Q_x} \end{aligned} \tag{2}$$

where $\|v\|^2_Q = v^TQv$, $Q_x \in \mathbb{S}^n_{++}$, and where $Q_u \in \mathbb{S}^m_{++}$. The system is also at each time instant subject to c linear inequality constraints in the form

$$\begin{aligned} H_x x(t) + H_u u(t) + h \le 0,\ \forall t \in \mathcal{T} \\ H_x x(N) + h \le 0 \end{aligned} \tag{3}$$

where $H_x \in \mathbb{R}^{c\times n}$, $H_u \in \mathbb{R}^{c\times m}$, and where $h \in \mathbb{R}^c$. This MIPC problem can be written as an MIQP problem in two different equivalent forms [1,14]. First, the equality constraints representing the dynamics of the system can be kept and the result is an MIQP problem in the form

$$\begin{aligned} \underset{\mathsf{x},\mathsf{u}}{\text{minimize}}\quad & J_{\mathrm{MIQP}_1}(\mathsf{x},\mathsf{u}) && \text{(4a)}\\ \text{s.t.}\quad & [\mathsf{A}\ \mathsf{B}]\left[\mathsf{x}^T\ \mathsf{u}^T\right]^T = b, && \text{(4b)}\\ & [\mathsf{H_x}\ \mathsf{H_u}]\left[\mathsf{x}^T\ \mathsf{u}^T\right]^T + \mathsf{h} \le 0 && \text{(4c)}\\ & \mathsf{u}_i \in \{0,1\},\ \forall i \in \mathcal{I} && \text{(4d)} \end{aligned}$$

where $J_{\mathrm{MIQP}_1}(\mathsf{x},\mathsf{u}) = \frac{1}{2}\mathsf{x}^T\mathsf{Q_x}\mathsf{x} + \frac{1}{2}\mathsf{u}^T\mathsf{Q_u}\mathsf{u}$ for some suitable choice of A, B, $\mathsf{H_x}$, $\mathsf{H_u}$, h, b, and block diagonal matrices $\mathsf{Q_x}$ and $\mathsf{Q_u}$ [1,14]. The vectors $\mathsf{x} \in \mathbb{R}^{(N+1)n}$ and $\mathsf{u} \in \mathbb{R}^{Nm}$ contain stacked states and control inputs, respectively. Second, the equality constraints in (1) can be used to eliminate the states as $\mathsf{x} = S_\mathsf{x}x_0 + S_\mathsf{u}\mathsf{u}$ and the resulting optimization problem can be expressed, for some suitable choice of S_x and S_u, as an MIQP problem equivalent to the problem in (4) in the form

$$\begin{aligned} \underset{\mathsf{u}}{\text{minimize}}\quad & J_{\mathrm{MIQP}_2}(\mathsf{u}) && \text{(5a)}\\ \text{s.t.}\quad & (\mathsf{H_x}S_\mathsf{u} + \mathsf{H_u})\,\mathsf{u} + \mathsf{h} + \mathsf{H_x}S_\mathsf{x}x_0 \le 0, && \text{(5b)}\\ & \mathsf{u}_i \in \{0,1\},\ \forall i \in \mathcal{I} && \text{(5c)} \end{aligned}$$

where
$J_{\mathrm{MIQP}_2}(\mathsf{u}) = \frac{1}{2}\mathsf{u}^T\left(S_\mathsf{u}^T\mathsf{Q_x}S_\mathsf{u} + \mathsf{Q_u}\right)\mathsf{u} + (S_\mathsf{x}x_0)^T\,\mathsf{Q_x}S_\mathsf{u}\mathsf{u} + \kappa$ and $\kappa = \frac{1}{2}(S_\mathsf{x}x_0)^T\,\mathsf{Q_x}\,(S_\mathsf{x}x_0)$ is a constant [1,14]. The optimal objective function values of the problems in (4) and in (5) coincide, *i.e.*, $J^*_{\mathrm{MPC}} = J^*_{\mathrm{MIQP}_1} = J^*_{\mathrm{MIQP}_2}$.

3 Relaxations

The most straightforward way to relax the integer constraints is to replace these non-convex constraints with interval constraints, *i.e.*, the constraint $u_i \in \{0,1\}$ is replaced by the relaxed constraint $u_i \in [0,1]$. The resulting problem is a QP problem, and it is therefore called a QP relaxation. Following this procedure, the QP relaxation of the problem in (4) is

$$\begin{aligned} \underset{\mathsf{x},\mathsf{u}}{\text{minimize}} \quad & J_{\text{QP}_1}(\mathsf{x},\mathsf{u}) \\ \text{s.t.} \quad & \text{(4b) and (4c)} \\ & 0 \le \mathsf{u}_i \le 1,\ \forall i \in \mathcal{I} \end{aligned} \tag{6}$$

where $J_{\text{QP}_1}(\mathsf{x},\mathsf{u}) = J_{\text{MIQP}_1}(\mathsf{x},\mathsf{u})$. The QP relaxation of the problem in (5) is

$$\begin{aligned} \underset{\mathsf{u}}{\text{minimize}} \quad & J_{\text{QP}_2}(\mathsf{u}) \\ \text{s.t.} \quad & \text{(5b)} \\ & 0 \le \mathsf{u}_i \le 1,\ \forall i \in \mathcal{I} \end{aligned} \tag{7}$$

where $J_{\text{QP}_2}(\mathsf{u}) = J_{\text{MIQP}_2}(\mathsf{u})$.

In recent years, the moment relaxation [12,17,20], or SDP relaxation, of problems with binary variables has been extensively studied. This relaxation is an SDP problem and it cannot in general be written as a QP. The SDP relaxation of the problem formulation in (4) is

$$\begin{aligned} \underset{U,\mathsf{x},\mathsf{u}}{\text{minimize}} \quad & J_{\text{SDP}_1}(U,\mathsf{x},\mathsf{u}) \\ \text{s.t.} \quad & \text{(4b) and (4c)} \\ & U_{ii} = \mathsf{u}_i,\ \forall i \in \mathcal{I} \\ & U \succeq \mathsf{u}\mathsf{u}^T \end{aligned} \tag{8}$$

where $J_{\text{SDP}_1}(U,\mathsf{x},\mathsf{u}) = \frac{1}{2}\mathsf{x}^T\mathsf{Q}_\mathsf{x}\mathsf{x} + \frac{1}{2}\operatorname{tr}(\mathsf{Q}_\mathsf{u}U)$, $U \in \mathbb{S}^{Nm}$ and where U_{ii} denotes diagonal element i of the matrix U. The relaxation in (8) is in this work referred to as the equality constrained SDP relaxation. As shown in [3], this relaxation can equivalently be rewritten as

$$\begin{aligned} \underset{U(\cdot),x(\cdot),u(\cdot)}{\text{minimize}} \quad & J_{\text{SDP}_1'}(U(\cdot),x(\cdot),u(\cdot)) \\ \text{s.t.} \quad & \text{(1) and (3)} \\ & U_{ii}(t) = u_i(t), \forall t \in \mathcal{T}, \forall i \in \{1,\dots,m\} \\ & U(t) \succeq u(t)u(t)^T, \forall t \in \mathcal{T} \end{aligned} \tag{9}$$

where

$$\begin{aligned} & J_{\text{SDP}_1'}(U(\cdot),x(\cdot),u(\cdot)) \\ & = \frac{1}{2}\sum_{t=0}^{N-1}\Big(x(t)^TQ_xx(t) + \operatorname{tr}\left(Q_uU(t)\right)\Big) \\ & + \frac{1}{2}x(N)^TQ_xx(N) \end{aligned}$$

and where the matrix variable $U \in \mathbb{S}^{Nm}$ in (8) has been replaced by N matrix variables $U(t) \in \mathbb{S}^m$. Hence, the number of variables (matrix elements) in (9) grows linearly with the prediction horizon N, which can be compared to the quadratic growth in (8). In [2] it was shown how to exploit the structure in (9) in order to be able to solve the optimization problem efficiently.

Using similar ideas, the SDP relaxation of the problem in (5) can be found to be

$$\begin{aligned} \underset{U,\mathsf{u}}{\text{minimize}} \quad & J_{\text{SDP}_2}(U,\mathsf{u}) \\ \text{s.t.} \quad & \text{(5b)} \\ & U_{ii} = \mathsf{u}_i,\ \forall i \in \mathcal{I} \\ & U \succeq \mathsf{u}\mathsf{u}^T \end{aligned} \tag{10}$$

where $J_{\text{SDP}_2}(U,\mathsf{u}) = \frac{1}{2}\operatorname{tr}\Big(\left(S_\mathsf{u}^T\mathsf{Q}_\mathsf{x}S_\mathsf{u} + \mathsf{Q}_\mathsf{u}\right)U\Big) + \left(S_\mathsf{x}x_0\right)^T\mathsf{Q}_\mathsf{x}S_\mathsf{u}\mathsf{u} + \kappa$, and where $U \in \mathbb{S}^{Nm}$ and κ is a constant defined below (5). Note that, the number of elements in U in (10) grows quadratically with N.

3.1 Relations between the relaxations

In this subsection, the quality of the bounds provided by the relaxations in (6), (7), (8), (9) and in (10) is compared theoretically. First, it is straightforward to show that the problems in (6) and in (7) are equivalent by eliminating the equality constraints in (6). Hence, their optimal objective function values coincide.

In the first step in the analysis, the QP relaxation in (7) is related to the SDP relaxation in (10). Consider the feasible sets of the problems. By Theorem 1 in [3], the set of u feasible in (7) and the set of u feasible in (10) coincide. However, the objective functions in (7) and in (10) are different. The difference is that $\mathsf{u}^T\left(S_\mathsf{u}^T\mathsf{Q}_\mathsf{x}S_\mathsf{u} + \mathsf{Q}_\mathsf{u}\right)\mathsf{u} = \operatorname{tr}\Big(\left(S_\mathsf{u}^T\mathsf{Q}_\mathsf{x}S_\mathsf{u} + \mathsf{Q}_\mathsf{u}\right)\mathsf{u}\mathsf{u}^T\Big)$ in the problem in (7) has been replaced by $\operatorname{tr}\Big(\left(S_\mathsf{u}^T\mathsf{Q}_\mathsf{x}S_\mathsf{u} + \mathsf{Q}_\mathsf{u}\right)U\Big)$ in the problem in (10). The relations between the optimal objective function values are therefore

$$J_{\text{QP}_2}(\mathsf{u}^*_{\text{QP}_2}) \le J_{\text{QP}_2}(\mathsf{u}^*_{\text{SDP}_2}) \le J_{\text{SDP}_2}(U^*_{\text{SDP}_2},\mathsf{u}^*_{\text{SDP}_2}) \tag{11}$$

where $\mathsf{u}^*_{\text{QP}_2}$ denotes the optimal solution to the QP relaxation in (7), and where $\mathsf{u}^*_{\text{SDP}_2}$ and $U^*_{\text{SDP}_2}$ denote the optimal solution to the SDP relaxation in (10). The first inequality follows from that $\mathsf{u}^*_{\text{SDP}_2}$ is not necessarily optimal in the QP relaxation. The second inequality follows from that the positive semidefinite constraint in (10) can be written as $U - \mathsf{u}\mathsf{u}^T \succeq 0$, and that $\operatorname{tr}\left(\left(S_\mathsf{u}^T\mathsf{Q}_\mathsf{x}S_\mathsf{u} + \mathsf{Q}_\mathsf{u}\right)\left(U - \mathsf{u}\mathsf{u}^T\right)\right) \ge 0$ for positive semidefinite matrices $S_\mathsf{u}^T\mathsf{Q}_\mathsf{x}S_\mathsf{u} + \mathsf{Q}_\mathsf{u} \succeq 0$ and $U - \mathsf{u}\mathsf{u}^T \succeq 0$. Furthermore, $S_\mathsf{u}^T\mathsf{Q}_\mathsf{x}S_\mathsf{u} + \mathsf{Q}_\mathsf{u} \succ 0$ since $\mathsf{Q}_\mathsf{u} \succ 0$. This implies that the second inequality in (11) holds with equality only if $U = \mathsf{u}\mathsf{u}^T$. It can be shown that if $U = \mathsf{u}\mathsf{u}^T$, the optimal solution to the SDP relaxation is feasible in the original non-convex problem, and hence, in that case the SDP relaxation is tight [20]. Since this is not true in general, the inequality does not hold with equality in general either. As a result, the following relation between the problems in (7) and in (10) holds

$$J^*_{\mathrm{QP}_2} \leq J^*_{\mathrm{SDP}_2}. \tag{12}$$

Using similar arguments, it can also be shown that

$$J^*_{\mathrm{QP}_1} \leq J^*_{\mathrm{SDP}_1}. \tag{13}$$

The conclusion is that the considered QP relaxations do not in general give a bound with as high quality as the considered SDP relaxations. This result is general and holds also for non-MPC MIQP problems.

The next step in the analysis is to relate the optimal objective function values of the problem in (8) and the problem in (10). Note that the problems in (8) and in (9) are equivalent, hence $J^*_{\mathrm{SDP}_1} = J^*_{\mathrm{SDP}_{1'}}$. If the equality constraints representing the dynamics in (8) are eliminated, the result is an equivalent problem in the form

$$\begin{aligned} \underset{U,\mathsf{u}}{\text{minimize}} \quad & J_{\mathrm{SDP}_{12}}(U,\mathsf{u}) \\ \text{s.t.} \quad & \text{(5b)} \\ & U_{ii} = \mathsf{u}_i, \ \forall i \in \mathcal{I} \\ & U \succeq \mathsf{u}\mathsf{u}^T \end{aligned} \tag{14}$$

where $J_{\mathrm{SDP}_{12}}(U,\mathsf{u}) = \frac{1}{2}\mathsf{u}^T S_\mathsf{u}^T \mathsf{Q}_\mathsf{x} S_\mathsf{u} \mathsf{u} + \frac{1}{2}\operatorname{tr}(\mathsf{Q}_\mathsf{u} U) + (S_\mathsf{x} x_0)^T \mathsf{Q}_\mathsf{x} S_\mathsf{u} \mathsf{u} + \kappa$. The difference between the problem in (10) and the problem in (14) is that the term $\operatorname{tr}\left(S_\mathsf{u}^T \mathsf{Q}_\mathsf{x} S_\mathsf{u} U\right)$ in (10) has been replaced by the term $\mathsf{u}^T S_\mathsf{u}^T \mathsf{Q}_\mathsf{x} S_\mathsf{u} \mathsf{u} = \operatorname{tr}\left(S_\mathsf{u}^T \mathsf{Q}_\mathsf{x} S_\mathsf{u} \mathsf{u}\mathsf{u}^T\right)$ in (14). Since $U - \mathsf{u}\mathsf{u}^T \succeq 0$, and $S_\mathsf{u}^T \mathsf{Q}_\mathsf{x} S_\mathsf{u} \succeq 0$, a similar reasoning as previously shows that

$$J^*_{\mathrm{SDP}_1} \leq J^*_{\mathrm{SDP}_2} \tag{15}$$

i.e., the bound from the standard SDP relaxation is at least as good as the equality constrained SDP relaxation. Note that, since $S_\mathsf{u}^T \mathsf{Q}_\mathsf{x} S_\mathsf{u}$ is not in general positive definite (only semidefinite), the inequality in (15) might hold with equality even though $U \neq \mathsf{u}\mathsf{u}^T$. The complete relation between the different problems is therefore

$$J^*_{\mathrm{QP}_1} = J^*_{\mathrm{QP}_2} \leq J^*_{\mathrm{SDP}_1} = J^*_{\mathrm{SDP}_{1'}} \leq J^*_{\mathrm{SDP}_2} \leq J^*_{\mathrm{MPC}} \tag{16}$$

where the last inequality follows from the fact that the SDP relaxation is not tight in general. It is also clear that the inequalities do not in general hold with equalities. Even though the result is shown for a problem with a binary only input, it is straightforward to generalize it to the case when only a part of u is binary valued (the "mixed case").

3.2 Diagonal cost matrix Q_u

In this subsection, it will be shown that if Q_u is diagonal, then the SDP relaxation in (9) is equivalent to a QP. It is well-known, see, *e.g.*, [7], that a family of equivalent MIQP problems is obtained by introducing a parameter $\gamma(t) \in \mathbb{R}^m$ into the objective function in (4) as has been done in

$$\begin{aligned} \frac{1}{2}\sum_{t=0}^{N-1} \Big\{ & x(t)^T Q_x x(t) + u(t)^T \Big(Q_u - \operatorname{diag}\big(\gamma(t)\big)\Big) u(t) \\ & + \gamma(t)^T u(t) \Big\} + \frac{1}{2} x(N)^T Q_x x(N). \end{aligned} \tag{17}$$

This can easily be verified by observing that for any choice of $\gamma(t)$ the objective function value is unchanged on the feasible set, *i.e.*, for all binary $u(t)$. Now, assume that Q_u is diagonal, perturb the original MIQP problem by choosing $\gamma(t) = \operatorname{diag}(Q_u)$ according to the discussion above, and QP relax this new MIQP problem. The results is then a QP problem in the form

$$\begin{aligned} \underset{x(\cdot),u(\cdot)}{\text{minimize}} \quad & J_{\mathrm{QP}_3}(x(\cdot),u(\cdot)) \\ \text{s.t.} \quad & \text{(1) and (3)} \\ & 0 \leq u_i(t) \leq 1, \forall t \in \mathcal{T}, \forall i \in \{1,\ldots,m\} \end{aligned} \tag{18}$$

where $J_{\mathrm{QP}_3}(x(\cdot),u(\cdot)) = \frac{1}{2}\sum_{t=0}^{N-1}\big(x(t)^T Q_x x(t) + \operatorname{diag}(Q_u)^T u(t)\big) + \frac{1}{2}x(N)^T Q_x x(N)$. It will now be shown that this problem is equivalent to the SDP relaxation in (9) when Q_u is diagonal. First, the objective function in (9) can equivalently be written as $\frac{1}{2}\sum_{t=0}^{N-1}\big(x(t)^T Q_x x(t) + \operatorname{diag}(Q_u)^T u(t)\big) + \frac{1}{2}x(N)^T Q_x x(N)$ since $U_{ii}(t) = u_i(t)$. Then, by Theorem 1 in [3], the positive semidefinite constraint and the constraint $U_{ii}(t) = u_i(t)$ can equivalently be represented by a constraint in the form $0 \leq u(t) \leq 1$. The resulting problem is a QP in the form in (18), and the desired result follows. Apart from being useful for on-line MPC, this formulation also seems valuable in off-line computations for explicit MIPC and can potentially help to efficiently cut away more redundant binary sequences from further consideration.

4 Efficient generation of suboptimal solutions

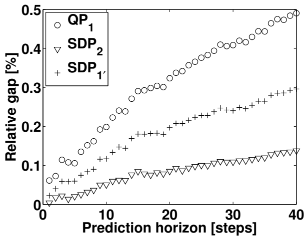
In this section, it is shown how the optimal solution to the relaxation in (9) can be used in a method called randomized rounding to generate suboptimal integer feasible solutions to the optimization problems in (4) and in (5). The result in this section relies on the assumption that the constraints are such that the problem is feasible for all integer sequences. Otherwise there is a risk that the found suboptimal binary sequence is infeasible. In problems with state constraints or mixed integer constraints, it might be necessary to solve an additional QP problem after the rounding process where the suboptimal binary sequence is kept fixed in order to get a feasible solution for the continuous variables. The method proposed originates from [6], which is a variant of the one presented in [9]. The idea is to generate random variables from a Gaussian distribution with mean $u(t)$ and variance $U(t) - u(t)u(t)^T$, where $U(t)$ and $u(t)$ is the solution to the SDP relaxation in (9). The generated random variables are rounded to either 0 or to 1. Preferably, several realizations $\bar{u}(t)$ are generated from the distribution and the sample that gives the lowest objective function value is kept as the suboptimal solution. Note that, the block structure in the problem in (9) enables a computational complexity that grows as $\mathcal{O}(N)$.

5 Numerical experiments

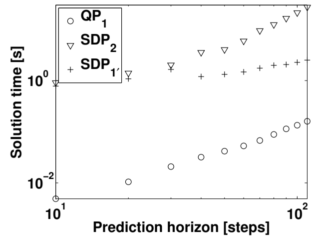
In this section, the quality of the bounds and the computational performance of the relaxations in (6), (9) and in (10) are compared in numerical experiments. Furthermore, the use of the SDP relaxation is investigated in the case when Q_u is diagonal. All experiments were performed on a computer with two processors of the type Dual Core AMD Opteron 270 sharing 4 GB RAM (only one core was used) running CentOS release 5.3 Kernel 2.6.18 (64 bit) and MATLAB 7.8. The solvers used were SDPT3 version 4.0 [19], and CPLEX version 11, and they were called using YALMIP [13].

5.1 Comparisons of relaxations

In this experiment, the relative gaps of the different relaxations are compared for different prediction horizons. The relative gap is here defined as $\frac{J^* - J_R^*}{J^*}$, where J_R^* represents the optimal objective function value of the relaxation of interest. The results are presented in Figure 1a and are for each prediction horizon found as the average of 50 problems generated by the MATLAB function `drss` with 4 states, 2 real valued control signals, 2 binary valued control signals and full cost matrices Q_e and Q_u. The cost matrices are created as $Q_e = W_e^T W_e$ and $Q_u = W_u^T W_u + I$, where W_e and W_u are created using the MATLAB command `randn` and I is the identity matrix. The two real valued control signals were constrained by random upper and lower bounds, chosen such that the problems are feasible and such that the constraints are "reasonably active" along the prediction horizon. The result from the experiment confirms the theoretical result in (16), which states that the SDP relaxation in (10) provides the best bound, the QP relaxation in (6) the worst bound, and the SDP relaxation in (9) provides a bound somewhere in between the other two. A comparison of the computational times is presented in Figure 1b.



(a) Average relative gaps.



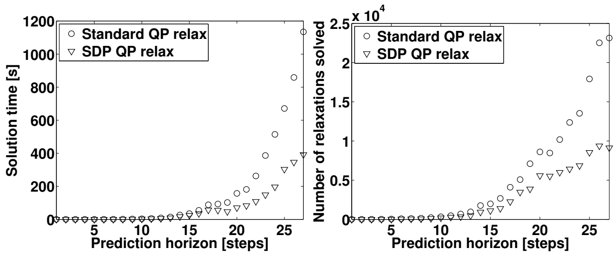
(b) Average computational times.

Fig. 1. The numerical results illustrating the relative quality of the bounds of the relaxations in (6), (9) and (10) and the corresponding computational time.

The conclusion from this experiment is that the QP relaxation is the least computationally demanding relaxation to compute. The standard SDP relaxation is the one that requires most computational time, while the equality constrained SDP relaxation requires a computational effort in between the other two. Furthermore, in these examples, the computational complexity for the latter grows as $\mathcal{O}(N^{0.8})$ which is the slowest growth of all compared relaxations and is similar to what is expected from the tailored algorithm described in [2].

5.2 Diagonal cost matrix Q_u

To minimize the number of nodes to explore in the branch and bound tree, it would have been desirable to use SDP relaxations in the nodes in branch and bound. This is usually not computationally beneficial in practice due to the relatively high computational effort required to compute the SDP relaxations. However, if the cost matrix Q_u is diagonal, relaxations of the type in (18) can be used. As a result, it is possible to use the equality constrained SDP relaxation in the nodes and this is the topic of this section. In order to be able to use this relaxation in an off-the-shelf QP solver, *e.g.*, *miqp.m* [4] or CPLEX, the interpretation from Section 3.2 is useful. That is, the equality constrained SDP relaxation can in this case be interpreted as the QP relaxation of an alternative equivalent MIQP formulation. When solving this alternative MIQP formulation using branch and bound, the QP relaxations will automatically be QP problems which are equivalent to the equality constrained SDP relaxation. In this experiment, the solver *miqp.m* is used as the MIQP solver. This MATLAB code implements a basic branch and bound method without advanced heuristics and preprocessing. The result is shown in Figure 2, and is an average of the result from 50 random unconstrained MIPC problems for each prediction horizon length, with 2 states and 2 binary control signals. For practical reasons, the experiments were aborted after 10^5 explored nodes, and only experiments where both tested approaches terminated within this bound are shown in the result. No more than 10 % of the realizations were aborted for this reason for any prediction horizon length. In Figure 2a it is shown that



(a) Average computational times. (b) Average number of explored nodes.

Fig. 2. Numerical results when the cost matrix Q_u is a diagonal matrix. The relaxations used in branch and bound in the comparison are the ones in (6) and in (18).

the computational time can be significantly reduced if the SDP relaxation is used. This improvement is a result of the decrease in the number of nodes necessary to explore in the branch and bound tree, which is illustrated in Figure 2b. An important conclusion in this section is that even though the improvement of the bounds seems small, it is actually useful in branch and bound, and can be used to speed up the solution process.

6 Conclusions

In this article, the QP relaxation, the standard SDP relaxation and an equality constrained SDP relaxation have been applied to an MIPC problem with mixed real valued and binary valued control signals subject to linear inequality constraints on states and control signals. It has been shown, both theoretically and numerically, that the standard SDP relaxation gives the best lower bound. Furthermore, the QP relaxation gives the worst lower bound. The equality constrained SDP relaxation is able to produce lower bounds of a quality that is in between the previous two relaxations. Moreover, the computational complexity is worst for the standard SDP relaxation, it is considerably lower for the equality constrained SDP relaxation, and it is lowest for the QP relaxation. It has also been shown that very good results can be obtained in the case when the cost matrix Q_u is diagonal. Furthermore, it has been shown how the SDP relaxations can be used to generate suboptimal solutions to the control problem.

Acknowledgments

The research has been supported by the Swedish Research Council for Engineering Sciences under contract Nr. 621-2002-3822.

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