An elementary proof of Wallis' product formula for pi

Johan Wästlund

Linköping studies in Mathematics, No. 2, February 21, 2005
Series editor: Bengt Ove Turesson

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Linköping studies in mathematics, No. 2 (2005)
Series editor: Bengt Ove Turesson
Department of Mathematics (MAI)
http://math.liu.se/index-e.html
Linköping University Electronic Press
Linköping, Sweden, 2005

ISSN 0348-2960 (print)
www.ep.liu.se/ea/lsm/2005/002/ (www)
ISSN 1652-4454 (on line)

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AN ELEMENTARY PROOF OF WALLIS' PRODUCT FORMULA FOR PI

JOHAN WÄSTLUND

ABSTRACT. We give an elementary proof of the Wallis product formula for pi. The proof does not require any integration or trigonometric functions.

1. The Wallis product formula

In 1655, John Wallis wrote down the celebrated formula

$$\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdots=\frac{\pi}{2}. \tag{1}$$

Most textbook proofs of (1) rely on evaluation of some definite integral like

$$\int_0^{\pi/2}(\sin x)^n\,dx$$

by repeated partial integration. The topic is usually reserved for more advanced calculus courses. The purpose of this note is to show that (1) can be derived using only the mathematics taught in elementary school, that is, basic algebra, the Pythagorean theorem, and the formula $\pi\cdot r^2$ for the area of a circle of radius r.

Viggo Brun gives an account of Wallis' method in [1] (in Norwegian). Yaglom and Yaglom [2] give a beautiful proof of (1) which avoids integration but uses some quite sophisticated trigonometric identities.

2. A number sequence

We denote the Wallis product by

$$W=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdots. \tag{2}$$

The partial products involving an even number of factors form an increasing sequence, while those involving an odd number of factors form a decreasing sequence. We let $s_0=0$, $s_1=1$, and in general,

$$s_n=\frac{3}{2}\cdot\frac{5}{4}\cdots\frac{2n-1}{2n-2}.$$

The partial products of (2) with an odd number of factors can be written as

$$\frac{2n}{s_n^2}=\frac{2^2\cdot4^2\cdots(2n)}{1\cdot3^2\cdots(2n-1)^2}>W,$$

while the partial products with an even number of factors are of the form

$$\frac{2n-1}{s_n^2}=\frac{2^2\cdot4^2\cdots(2n-2)^2}{1\cdot3^2\cdots(2n-3)^2\cdot(2n-1)}<W.$$

It follows that

$$\frac{2n-1}{W}<s_n^2<\frac{2n}{W}. \tag{3}$$

Date: February 21, 2005.

We denote the difference $s_{n+1} - s_n$ by a_n, and observe that

$$a_n = s_{n+1} - s_n = s_n \left(\frac{2n+1}{2n} - 1 \right) = \frac{s_n}{2n} = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n}.$$

We first derive the identity

$$a_i a_j = \frac{j+1}{i+j+1} a_i a_{j+1} + \frac{i+1}{i+j+1} a_{i+1} a_j. \tag{4}$$

Proof. After the substitutions

$$a_{i+1} = \frac{2i+1}{2(i+1)} a_i$$

and

$$a_{j+1} = \frac{2j+1}{2(j+1)} a_j,$$

the right hand side of (4) becomes

$$a_i a_j \left(\frac{2j+1}{2(j+1)} \cdot \frac{j+1}{i+j+1} + \frac{2i+1}{2(i+1)} \cdot \frac{i+1}{i+j+1} \right) = a_i a_j.$$

□

If we start from a_0^2 and repeatedly apply (4), we obtain the identities

$$1 = a_0^2 = a_0 a_1 + a_1 a_0 = a_0 a_2 + a_1^2 + a_2 a_0 = \dots$$
$$\dots = a_0 a_n + a_1 a_{n-1} + \dots + a_n a_0. \tag{5}$$

Proof. By applying (4) to every term, the sum $a_0 a_{n-1} + \dots + a_{n-1} a_0$ becomes

$$\left(a_0 a_n + \frac{1}{n} a_1 a_{n-1} \right) + \left(\frac{n-1}{n} a_1 a_{n-1} + \frac{2}{n} a_2 a_{n-2} \right) + \dots + \left(\frac{1}{n} a_{n-1} a_1 + a_n a_0 \right).$$

After collecting terms, this simplifies to $a_0 a_n + \dots + a_n a_0$. □

3. A geometric construction

We divide the positive quarter of the x-y-plane into rectangles by drawing the straight lines $x = s_n$ and $y = s_n$ for all n. Let $R_{i,j}$ be the rectangle with lower left corner (s_i, s_j) and upper right corner (s_{i+1}, s_{j+1}). The area of $R_{i,j}$ is $a_i a_j$. Therefore the identity (5) states that the total area of the rectangles $R_{i,j}$ for which $i + j = n$ is 1. We let P_n be the polygonal region consisting of all rectangles $R_{i,j}$ for which $i + j < n$. Hence the area of P_n is n (see Figure 1).

The outer corners of P_n are the points (s_i, s_j) for which $i + j = n + 1$. By the Pythagorean theorem, the distance of such a point to the origin is

$$\sqrt{s_i^2 + s_j^2}.$$

By (3), this is bounded from above by

$$\sqrt{\frac{2(i+j)}{W}} = \sqrt{\frac{2(n+1)}{W}}.$$

Similarly, the inner corners of P_n are the points $(s_i + s_j)$ for which $i + j = n$. The distance of such a point to the origin is bounded from below by

$$\sqrt{\frac{2(i+j-1)}{W}} = \sqrt{\frac{2(n-1)}{W}}.$$

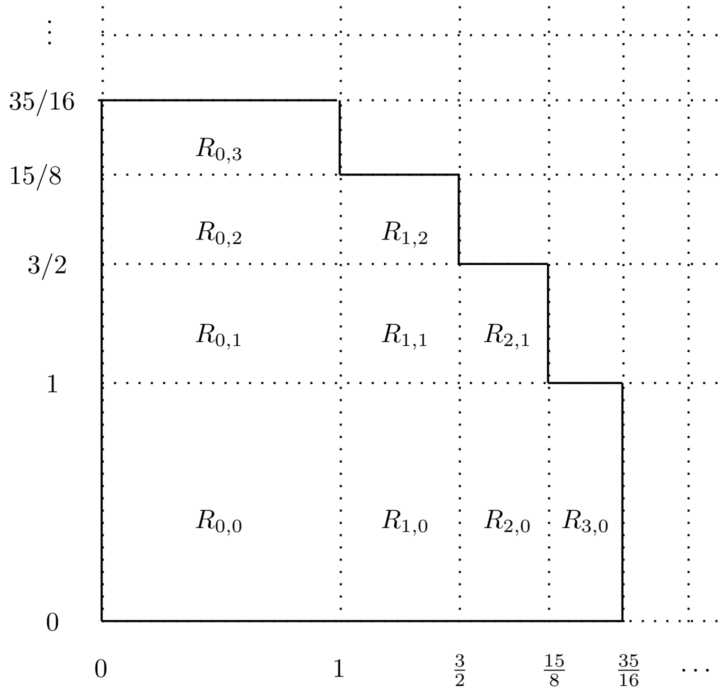

FIGURE 1. The region P_4 of area 4.

Therefore P_n contains a quarter circle of radius $\sqrt{2(n-1)/W}$, and is contained in a quarter circle of radius $\sqrt{2(n+1)/W}$. Since the area of a quarter circle of radius r is equal to $\pi r^2/4$, we obtain the following bounds for the area of P_n:

$$\frac{\pi(n-1)}{2W} < n < \frac{\pi(n+1)}{2W}.$$

Since this holds for every n, we conclude that

$$W = \frac{\pi}{2}.$$

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