

Evaluation of Janson's constant for the variance in
the random minimum spanning tree problem

JOHAN WÄSTLUND

Linköping studies in Mathematics, No. 7, 2005
Series editor: Bengt Ove Turesson

The publishers will keep this document on-line on the Internet (or its possible replacement network in the future) for a period of 25 years from the date of publication barring exceptional circumstances as described separately.

The on-line availability of the document implies a permanent permission for anyone to read, to print out single copies and to use it unchanged for any non-commercial research and educational purpose. Subsequent transfers of copyright cannot revoke this permission. All other uses of the document are conditional on the consent of the copyright owner. The publication also includes production of a number of copies on paper archived in Swedish University libraries and by the copyright holder(s). The publisher has taken technical and administrative measures to assure that the on-line version will be permanently accessible and unchanged at least until the expiration of the publication period.

For additional information about the Linköping University Electronic Press and its procedures for publication and for assurance of document integrity, please refer to its WWW home page: <http://www.ep.liu.se>.

Linköping studies in mathematics, No. 7 (2005)

Series editor: Bengt Ove Turesson

Department of Mathematics (MAI)

<http://math.liu.se/index-e.html>

Linköping University Electronic Press

Linköping, Sweden, 2005

ISSN 0348-2960 (print)

www.ep.liu.se/ea/lsm/2005/007/ (www)

ISSN 1652-4454 (on line)

© Johan Wästlund.

**EVALUATION OF JANSON'S CONSTANT FOR THE
VARIANCE IN THE RANDOM MINIMUM SPANNING
TREE PROBLEM**

JOHAN WÄSTLUND

ABSTRACT. We complete a result of Svante Janson on the asymptotic normal distribution of the cost W_n of the minimum spanning tree in a complete graph on n vertices with independent uniform $(0,1)$ edge costs. By evaluating a triple sum given by Janson, we show that the variance of W_n is asymptotically $n^{-1}(6\zeta(4) - 4\zeta(3))$, where ζ is the Riemann zeta function.

Suppose that the edges of the complete graph K_n on n vertices are given independent random costs taken from uniform distribution on $[0, 1]$. We let W_n denote the cost of the minimum spanning tree in K_n , that is, the minimum sum of the edge costs in a spanning tree. A famous result of Alan Frieze [3] states that as $n \rightarrow \infty$,

$$\mathbb{E}(W_n) \rightarrow \zeta(3) = \sum_{n=1}^{\infty} n^{-3} \approx 1.2021.$$

This theorem has been generalized in various directions, see for instance [1, 2, 4]. In [5], Svante Janson showed that, suitably normalized, the distribution of W_n is asymptotically normal. Janson proved that

$$\sqrt{n}(W_n - \zeta(3)) \xrightarrow{d} N(0, \sigma^2),$$

where

$$(1) \quad \sigma^2 = \frac{\pi^4}{45} - 2 \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{(i+k-1)! k^k (i+j)^{i-2} j}{i! k! (i+j+k)^{i+k+2}}.$$

Janson was unable to simplify the triple sum, but gave the approximate value $\sigma^2 \approx 1.6857$, obtained by numerical summation.

In view of some analogous results for the related minimum assignment problem, we had reason to believe that this constant can be expressed as a linear combination of values of the Riemann zeta function. Clearly the first term $\pi^4/45$ is equal to $2\zeta(4)$, and we also expect $\zeta(3)$ to occur. By evaluating combinations of $\zeta(3)$ and $\zeta(4)$ numerically, we found that

$$6\zeta(4) - 4\zeta(3) = 1.685711794\dots$$

To show that indeed $\sigma^2 = 6\zeta(4) - 4\zeta(3)$, we have to show that the triple sum evaluates to $2\zeta(3) - 2\zeta(4)$. We use the following identity:

Date: 22nd August 2005.

Lemma 1.

$$(2) \quad \sum_{i=0}^m \binom{m}{i} (x-i)^{m-i} (y+i)^{i-2} = \frac{(x+y)^m}{y^2} - \frac{m(x+y)^{m-1}}{y(y+1)}.$$

Proof. We start by deriving the simpler identity

$$(3) \quad \sum_{i=0}^m \binom{m}{i} (x-i)^{m-i} (y+i)^{i-1} = \frac{(x+y)^m}{y}.$$

The term corresponding to $i = 0$ is x^m/y , and the other terms are polynomials in x and y . If we expand the left hand side, then clearly the terms of degree $m - 1$ have the correct coefficients. We have to show that all the terms of smaller degree cancel. Let p and q be nonnegative integers such that $p + q < m - 1$. The coefficient of $x^p y^q$ in the left hand side of (3) is

$$\sum_{i=0}^m \binom{m}{i} \binom{m-i}{p} \binom{i-1}{q} i^{m-1-p-q} (-1)^{m-i-p},$$

which simplifies to

$$(q+1) \binom{m}{p, q+1, m-p-q-1} \sum_{i=0}^m (-1)^{m-i-p} \binom{m-1-p-q}{i-1-q} i^{m-2-p-q}.$$

By changing the summation variable to $j = i - 1 - q$, and putting $N = m - 1 - p - q$, we can identify this sum as a constant times

$$(4) \quad \sum_{j=0}^N (-1)^j \binom{N}{j} P(j),$$

where $P(j) = i^{m-2-p-q} = (j+1+q)^{m-2-p-q}$. The sum (4) is known to be zero whenever P is a polynomial of degree smaller than N . This establishes the identity (3), and we turn to (2).

If we set $x' = x - 1$, $y' = y + 1$, $m' = m - 1$ and $i' = i - 1$, then we have

$$(5) \quad \binom{m}{i} (x-i)^{m-i} (y+i)^{i-2} = \frac{1}{y} \cdot \binom{m}{i} (x-i)^{m-i} (y+i)^{i-1} - \frac{m}{y} \binom{m'}{i'} (x'-i')^{m'-i'} (y'+i')^{i'-1},$$

from which it follows that the left hand side of (2) equals

$$\frac{1}{y} \cdot \frac{(x+y)^m}{y} - \frac{m}{y} \cdot \frac{(x'+y')^{m'}}{y'}.$$

□

We now go back to Janson's triple sum. Let

$$T(i, j, k) = \frac{(i+k-1)! k^k (i+j)^{i-2} j}{i! k! (i+j+k)^{i+k+2}}.$$

We wish to find the sum of $T(i, j, k)$ over all triples (i, j, k) of integers with $i \geq 0$, $j \geq 1$ and $k \geq 1$.

Lemma 2. *If we keep j fixed, and sum $T(i, j, k)$ over those i and k for which $i + j + k = n$, we get*

$$(6) \quad \sum_{i=0}^{n-j-1} T(i, j, n-i-j) = \frac{n+j+j^2}{j(j+1)n^4} = \frac{1}{j(j+1)n^3} + \frac{1}{n^4}.$$

Proof. The left hand side of (6) is equal to

$$\sum_{i=0}^{n-j-1} \frac{(n-j-1)!(n-i-j)^{n-i-j}(i+j)^{i-2}j}{i!(n-i-j)!n^{n-j+2}},$$

which we can write as

$$(7) \quad \frac{j}{(n-j)n^{n-j+2}} \sum_{i=0}^{n-j-1} \binom{n-j}{i} (n-j-i)^{n-j-i}(i+j)^{i-2}.$$

The sum in (7) takes the form of (2) if we set $x = n-j$, $y = j$, and $m = n-j$. However, since the range of summation in (7) only goes to $n-j-1$, we have to subtract the value of the last term, which is n^{n-j-2} , to obtain the correct result. We find by applying (2) that (7) is equal to

$$\frac{j}{(n-j)n^{n-j+2}} \left[\frac{n^{n-j}}{j^2} - \frac{(n-j)n^{n-j-1}}{j(j+1)} - n^{n-j-2} \right],$$

which simplifies to

$$\frac{1}{j(j+1)n^3} + \frac{1}{n^4}.$$

□

If we now sum the right hand side of (6) over $j = 1, \dots, n-1$, we obtain the sum of $T(i, j, k)$ over all i, j, k within the range of summation such that $i + j + k = n$:

$$(8) \quad \sum_{i+j+k=n} T(i, j, k) = \sum_{j=1}^{n-1} \left(\frac{1}{j(j+1)n^3} + \frac{1}{n^4} \right) = \frac{1}{n^3} \sum_{j=1}^{n-1} \frac{1}{j(j+1)} + \frac{n-1}{n^4} = \frac{1}{n^3} \left(1 - \frac{1}{n} \right) + \frac{n-1}{n^4} = \frac{2}{n^3} - \frac{2}{n^4}$$

The entire triple sum occurring in (1) is therefore equal to

$$2 \sum_{n=2}^{\infty} \frac{1}{n^3} - 2 \sum_{n=2}^{\infty} \frac{1}{n^4} = 2\zeta(3) - 2\zeta(4),$$

and consequently, Janson's constant is

$$\sigma^2 = 6\zeta(4) - 4\zeta(3).$$

REFERENCES

- [1] Aldous, David, Steele, Michael J., *The Objective Method: Probabilistic Combinatorial Optimization and Local Weak Convergence* (2003).
- [2] Beveridge, A., Frieze, A. and McDiarmid, C., *Random minimum length spanning trees in regular graphs*, *Combinatorica* **18** (1998), 311–333.
- [3] Frieze, A. M., *On the value of a random minimum spanning tree problem*, *Discrete Appl. Math.* **10** (1985), 47–56.
- [4] Frieze, A. and McDiarmid, C., *On random minimum length spanning trees*, *Combinatorica* **9** (1989) 363–374.
- [5] Janson, Svante, *The minimal spanning tree in a complete graph and a functional limit theorem for trees in a random graph*, *Random Struct. Alg.* **7** (1995), 337–355.

JOHAN WÄSTLUND, DEPARTMENT OF MATHEMATICS, LINKÖPING UNIVERSITY, S-581
83 LINKÖPING, SWEDEN
E-mail address: `jowas@mai.liu.se`