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EVALUATION OF JANSON'S CONSTANT FOR THE VARIANCE IN THE RANDOM MINIMUM SPANNING TREE PROBLEM

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ABSTRACT. We complete a result of Svante Janson on the asymptotic normal distribution of the cost W_n of the minimum spanning tree in a complete graph on n vertices with independent uniform (0,1) edge costs. By evaluating a triple sum given by Janson, we show that the variance of W_n is asymptotically $n^{-1}(6\zeta(4)-4\zeta(3))$, where ζ is the Riemann zeta function.

Suppose that the edges of the complete graph K_n on n vertices are given independent random costs taken from uniform distribution on $[0,1]$. We let W_n denote the cost of the minimum spanning tree in K_n, that is, the minimum sum of the edge costs in a spanning tree. A famous result of Alan Frieze [3] states that as $n \to \infty$,

$$\mathbb{E}\,(W_n) \to \zeta(3) = \sum_{n=1}^{\infty} n^{-3} \approx 1.2021.$$

This theorem has been generalized in various directions, see for instance [1, 2, 4]. In [5], Svante Janson showed that, suitably normalized, the distribution of W_n is asymptotically normal. Janson proved that

$$\sqrt{n}\,(W_n - \zeta(3)) \xrightarrow{\text{d}} N(0,\sigma^2),$$

where

$$\sigma^2 = \frac{\pi^4}{45} - 2\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{(i+k-1)!k^k(i+j)^{i-2}j}{i!k!(i+j+k)^{i+k+2}}. \tag{1}$$

Janson was unable to simplify the triple sum, but gave the approximate value $\sigma^2 \approx 1.6857$, obtained by numerical summation.

In view of some analogous results for the related minimum assignment problem, we had reason to believe that this constant can be expressed as a linear combination of values of the Riemann zeta function. Clearly the first term $\pi^4/45$ is equal to $2\zeta(4)$, and we also expect $\zeta(3)$ to occur. By evaluating combinations of $\zeta(3)$ and $\zeta(4)$ numerically, we found that

$$6\zeta(4) - 4\zeta(3) = 1.685711794...$$

To show that indeed $\sigma^2 = 6\zeta(4) - 4\zeta(3)$, we have to show that the triple sum evaluates to $2\zeta(3) - 2\zeta(4)$. We use the following identity:

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Lemma 1.

$$\sum_{i=0}^{m}\binom{m}{i}(x-i)^{m-i}(y+i)^{i-2}=\frac{(x+y)^m}{y^2}-\frac{m(x+y)^{m-1}}{y(y+1)}. \tag{2}$$

Proof. We start by deriving the simpler identity

$$\sum_{i=0}^{m}\binom{m}{i}(x-i)^{m-i}(y+i)^{i-1}=\frac{(x+y)^m}{y}. \tag{3}$$

The term corresponding to $i=0$ is x^m/y, and the other terms are polynomials in x and y. If we expand the left hand side, then clearly the terms of degree $m-1$ have the correct coefficients. We have to show that all the terms of smaller degree cancel. Let p and q be nonnegative integers such that $p+q<m-1$. The coefficient of x^py^q in the left hand side of (3) is

$$\sum_{i=0}^{m}\binom{m}{i}\binom{m-i}{p}\binom{i-1}{q}i^{m-1-p-q}(-1)^{m-i-p},$$

which simplifies to

$$(q+1)\binom{m}{p,q+1,m-p-q-1}\sum_{i=0}^{m}(-1)^{m-i-p}\binom{m-1-p-q}{i-1-q}i^{m-2-p-q}.$$

By changing the summation variable to $j=i-1-q$, and putting $N=m-1-p-q$, we can identify this sum as a constant times

$$\sum_{j=0}^{N}(-1)^j\binom{N}{j}P(j), \tag{4}$$

where $P(j)=i^{m-2-p-q}=(j+1+q)^{m-2-p-q}$. The sum (4) is known to be zero whenever P is a polynomial of degree smaller than N. This establishes the identity (3), and we turn to (2).

If we set $x'=x-1$, $y'=y+1$, $m'=m-1$ and $i'=i-1$, then we have

$$\binom{m}{i}(x-i)^{m-i}(y+i)^{i-2}= \tag{5}$$
$$\frac{1}{y}\cdot\binom{m}{i}(x-i)^{m-i}(y+i)^{i-1}-\frac{m}{y}\binom{m'}{i'}(x'-i')^{m'-i'}(y'+i')^{i'-1},$$

from which it follows that the left hand side of (2) equals

$$\frac{1}{y}\cdot\frac{(x+y)^m}{y}-\frac{m}{y}\cdot\frac{(x'+y')^{m'}}{y'}.$$

□

We now go back to Janson's triple sum. Let

$$T(i,j,k)=\frac{(i+k-1)!k^k(i+j)^{i-2}j}{i!k!(i+j+k)^{i+k+2}}.$$

We wish to find the sum of $T(i,j,k)$ over all triples (i,j,k) of integers with $i\geq 0$, $j\geq 1$ and $k\geq 1$.

Lemma 2. *If we keep j fixed, and sum $T(i,j,k)$ over those i and k for which $i+j+k=n$, we get*

$$\sum_{i=0}^{n-j-1} T(i,j,n-i-j) = \frac{n+j+j^2}{j(j+1)n^4} = \frac{1}{j(j+1)n^3} + \frac{1}{n^4}. \tag{6}$$

Proof. The left hand side of (6) is equal to

$$\sum_{i=0}^{n-j-1} \frac{(n-j-1)!(n-i-j)^{n-i-j}(i+j)^{i-2}j}{i!(n-i-j)!n^{n-j+2}},$$

which we can write as

$$\frac{j}{(n-j)n^{n-j+2}} \sum_{i=0}^{n-j-1} \binom{n-j}{i} (n-j-i)^{n-j-i}(i+j)^{i-2}. \tag{7}$$

The sum in (7) takes the form of (2) if we set $x=n-j$, $y=j$, and $m=n-j$. However, since the range of summation in (7) only goes to $n-j-1$, we have to subtract the value of the last term, which is n^{n-j-2}, to obtain the correct result. We find by applying (2) that (7) is equal to

$$\frac{j}{(n-j)n^{n-j+2}} \left[\frac{n^{n-j}}{j^2} - \frac{(n-j)n^{n-j-1}}{j(j+1)} - n^{n-j-2} \right],$$

which simplifies to

$$\frac{1}{j(j+1)n^3} + \frac{1}{n^4}.$$

□

If we now sum the right hand side of (6) over $j=1,\dots,n-1$, we obtain the sum of $T(i,j,k)$ over all i,j,k within the range of summation such that $i+j+k=n$:

$$\sum_{i+j+k=n} T(i,j,k) = \sum_{j=1}^{n-1} \left(\frac{1}{j(j+1)n^3} + \frac{1}{n^4} \right) = \tag{8}$$
$$\frac{1}{n^3} \sum_{j=1}^{n-1} \frac{1}{j(j+1)} + \frac{n-1}{n^4} = \frac{1}{n^3}\left(1-\frac{1}{n}\right) + \frac{n-1}{n^4} = \frac{2}{n^3} - \frac{2}{n^4}$$

The entire triple sum occurring in (1) is therefore equal to

$$2\sum_{n=2}^{\infty} \frac{1}{n^3} - 2\sum_{n=2}^{\infty} \frac{1}{n^4} = 2\zeta(3) - 2\zeta(4),$$

and consequently, Janson's constant is

$$\sigma^2 = 6\zeta(4) - 4\zeta(3).$$

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