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TRANSMIT BEAMFORMING TO MULTIPLE CO-CHANNEL MULTICAST GROUPS

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ABSTRACT

The problem of transmit beamforming to multiple co-channel multicast groups is considered, from the viewpoint of guaranteeing a prescribed minimum signal-to-interference-plus-noise-ratio (SINR) at each receiver. The problem is a multicast generalization of the SINR-constrained multiuser downlink beamforming problem: the difference is that each transmitted stream is directed to multiple receivers, each with its own channel. Such generalization is relevant and timely, e.g., in the context of 802.16 wireless networks. Based on earlier results for a single multicast group, the joint problem is easily shown to be NP-hard, a fact that motivates the pursuit of quasi-optimal computationally efficient solutions. It is shown that Lagrangian relaxation coupled with a randomization / co-channel multicast power control loop yields a computationally efficient high-quality approximate solution. For a significant fraction of problem instances, the solutions generated this way are exactly optimal. Carefully designed and extensive simulation results are presented to support the main findings.

1. DATA MODEL AND PROBLEM STATEMENT

Consider a wireless scenario incorporating a single transmitter with N antenna elements and M receivers, each with a single antenna. Let $\mathbf{h}_i$ denote the $N \times 1$ complex vector that models the propagation loss and phase shift of the frequency-flat quasi-static channel from each transmit antenna to the receive antenna of user $i \in \{1, \dots, M\}$. Let there be a total of $1 \leq G \leq M$ multicast groups, $\{\mathcal{G}_1, \dots, \mathcal{G}_G\}$, where $\mathcal{G}_k$ contains the indices of receivers participating in multicast group k , and $k \in \{1, \dots, G\}$. Each receiver listens to a single multicast; thus $\mathcal{G}_k \cap \mathcal{G}_l = \emptyset$, $l \neq k$, $\cup_k \mathcal{G}_k = \{1, \dots, M\}$, and, denoting $G_k := |\mathcal{G}_k|$, $\sum_{k=1}^G G_k = M$.

Let $\mathbf{w}_k^H$ denote the beamforming weight vector applied to the N transmitting antenna elements to generate the spatial channel for transmitting to group k . Then the signal transmitted by the antenna array is equal to $\sum_{k=1}^G \mathbf{w}_k^H s_k(t)$, where $s_k(t)$ is the temporal information-bearing signal directed to receivers in multicast group k . Note that the above setup includes the case of *broadcasting* (a single multicast group, $G = 1$) [6], as well as the case of individual information transmission to each receiver ($G = M$) by means of spatial multiplexing (see, e.g., [1]). If each $s_k(t)$ is zero-mean white with unit variance, and the waveforms $\{s_k(t)\}_{k=1}^G$ are mutually uncorrelated, then the total power radiated by the transmitting antenna array is equal to $\sum_{k=1}^G \|\mathbf{w}_k\|_2^2$.

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The joint design of transmit beamformers can then be posed as the problem of minimizing the total radiated power subject to meeting prescribed SINR constraints c_i at each of the M receivers

$$\begin{aligned} \mathcal{I} : \quad & \min_{\{\mathbf{w}_k \in \mathbb{C}^N\}_{k=1}^G} \sum_{k=1}^G \|\mathbf{w}_k\|_2^2 \\ \text{s.t. :} \quad & \frac{|\mathbf{w}_k^H \mathbf{h}_i|^2}{\sum_{l \neq k} |\mathbf{w}_l^H \mathbf{h}_i|^2 + \sigma_i^2} \geq c_i, \quad \forall i \in \mathcal{G}_k, \quad \forall k \in \{1, \dots, G\}. \end{aligned}$$

Problem $\mathcal{I}$ contains the associated broadcasting problem as a special case; from this and [6], it immediately follows that

Claim 1 Problem $\mathcal{I}$ is NP-hard.

This motivates (cf. [4]) the pursuit of sensible approximate solutions to problem $\mathcal{I}$.

2. RELAXATION

Towards this end, define $\mathbf{Q}_i := \mathbf{h}_i \mathbf{h}_i^H$ and $\mathbf{X}_k := \mathbf{w}_k \mathbf{w}_k^H$, and note that $|\mathbf{w}_k^H \mathbf{h}_i|^2 = \mathbf{h}_i^H \mathbf{w}_k \mathbf{w}_k^H \mathbf{h}_i = \text{trace}(\mathbf{h}_i^H \mathbf{w}_k \mathbf{w}_k^H \mathbf{h}_i) = \text{trace}(\mathbf{h}_i \mathbf{h}_i^H \mathbf{w}_k \mathbf{w}_k^H) = \text{trace}(\mathbf{Q}_i \mathbf{X}_k)$. Then, problem $\mathcal{I}$ can be equivalently reformulated as

$$\begin{aligned} \min_{\{\mathbf{X}_k \in \mathbb{C}^{N \times N}\}_{k=1}^G} \quad & \sum_{k=1}^G \text{trace}(\mathbf{X}_k) \\ \text{s.t. :} \quad & \text{trace}(\mathbf{Q}_i \mathbf{X}_k) \geq c_i \sum_{l \neq k} \text{trace}(\mathbf{Q}_i \mathbf{X}_l) + c_i \sigma_i^2, \\ & \forall i \in \mathcal{G}_k, \quad \forall k \in \{1, \dots, G\}, \\ & \mathbf{X}_k \succeq \mathbf{0}, \quad \forall k \in \{1, \dots, G\}, \\ & \text{rank}(\mathbf{X}_k) = 1, \quad \forall k \in \{1, \dots, G\}, \end{aligned}$$

where the fact that the terms in the denominator are all non-negative has also been taken into account. Dropping the rank-one constraints, we arrive at the following relaxation of problem $\mathcal{I}$

$$\begin{aligned} \mathcal{R} : \quad & \min_{\{\mathbf{X}_k \in \mathbb{C}^{N \times N}\}_{k=1}^G, \{s_i \in \mathbb{R}\}_{i=1}^M} \sum_{k=1}^G \text{trace}(\mathbf{X}_k) \\ \text{s.t. :} \quad & \text{trace}(\mathbf{Q}_i \mathbf{X}_k) - c_i \sum_{l \neq k} \text{trace}(\mathbf{Q}_i \mathbf{X}_l) - s_i = c_i \sigma_i^2, \\ & \forall i \in \mathcal{G}_k, \quad \forall k \in \{1, \dots, G\}, \\ & s_i \geq 0, \quad \forall i \in \{1, \dots, M\}, \\ & \mathbf{X}_k \succeq \mathbf{0}, \quad \forall k \in \{1, \dots, G\}, \end{aligned}$$

where M non-negative real “slack” variables s_i have been introduced, in order to convert the inequality constraints to equality constraints, plus non-negativity constraints. Problem $\mathcal{R}$ is a *Semi-Definite Program* (SDP), expressed in the primal standard form used by SDP solvers, such as SeDuMi [7]. SeDuMi uses interior point methods to solve efficiently such SDP problems, at a complexity cost that is at most $O((GN^2 + M)^{3.5})$, and usually much less.

3. OBTAINING AN APPROXIMATE SOLUTION TO PROBLEM $\mathcal{I}$

Problem $\mathcal{I}$ may not admit a feasible solution (counter-examples may be easily constructed), but if it does, the aforementioned approach will yield a solution to problem $\mathcal{R}$. Due to relaxation, this solution will not, in general, consist of rank-one blocks. In order to obtain a high-quality approximate solution of problem $\mathcal{I}$, the concept of *randomization* can be employed to generate candidate beamforming vectors in the span of the respective transmit covariance matrices; see, for example, [6]. The main difference relative to the simpler broadcast case ($G = 1$) considered in [6], is that here we cannot simply “scale up” the candidate beamforming vectors generated during randomization to satisfy the hard constraints of problem $\mathcal{I}$. The reason is that, in contrast to [6], we herein deal with an interference scenario, and boosting one group’s beamforming vector also increases interference to nodes in other groups. Whether it is feasible to satisfy the constraints for a given set of candidate beamforming vectors is also an issue here. Towards resolving this situation, let $a_{k,i} := |\mathbf{w}_k^H \mathbf{h}_i|^2$ denote the signal power received at receiver i from the stream directed towards users in multicast group k . Let $\beta_k := \|\mathbf{w}_k\|^2$, and p_k denote the power boost factor for multicast group k . Then the following *Multi-Group Power Control* ($\mathcal{MGPC}$) problem emerges in converting candidate beamforming vectors to a candidate solution of problem $\mathcal{I}$

$$\begin{aligned} \mathcal{MGPC} : \\ \min_{\{p_k \in \mathbb{R}\}_{k=1}^G} \quad & \sum_{k=1}^G \beta_k p_k \\ \text{s.t. :} \quad & \frac{p_k a_{k,i}}{\sum_{l \neq k} p_l a_{l,i} + \sigma_i^2} \geq c_i, \\ & \forall i \in \mathcal{G}_k, \forall k \in \{1, \dots, G\}, \\ & p_k \geq 0, \forall k \in \{1, \dots, G\}. \end{aligned}$$

As in Section 2, taking advantage of the fact that the terms in the denominator are all non-negative and introducing M non-negative real “slack” variables s_i , problem $\mathcal{MGPC}$ can be reformulated as

$$\begin{aligned} \mathcal{MGPC} : \\ \min_{\{p_k \in \mathbb{R}\}_{k=1}^G, \{s_i \in \mathbb{R}\}_{i=1}^M} \quad & \sum_{k=1}^G \beta_k p_k \\ \text{s.t. :} \quad & p_k a_{k,i} - c_i \sum_{l \neq k} p_l a_{l,i} - s_i = c_i \sigma_i^2, \\ & \forall i \in \mathcal{G}_k, \forall k \in \{1, \dots, G\}, \\ & p_k \geq 0, \forall k \in \{1, \dots, G\}, \\ & s_i \geq 0, \forall i \in \{1, \dots, M\}, \end{aligned}$$

Problem $\mathcal{MGPC}$ is a *Linear Program* (LP), since the cost function and all constraints are linear. SeDuMi can be used again to solve it efficiently. Note that SeDuMi will also yield an infeasibility certificate in case the $\mathcal{MGPC}$ problem is not solvable for a particular beamforming configuration, which is nice.

For $G = M$ (independent information transmission to each receiver), problem $\mathcal{R}$ is equivalent to and not a relaxation of $\mathcal{I}$, see [1], and problem $\mathcal{MGPC}$ reduces to the well-known multiuser downlink power control problem, which can be solved using simpler means (e.g., [3]): matrix inversion, but also iterative descent algorithms. In this special case, (in)feasibility can be determined from the spectral radius of a certain “connectivity” matrix. Similar simplifications for the general instance of $\mathcal{MGPC}$ are perhaps possible, but appear highly non-trivial. At any rate, LP routines are very efficient.

The overall algorithm for obtaining an approximate solution to problem $\mathcal{I}$ can thus be summarized as follows:

1. **Relaxation:** Solve problem $\mathcal{R}$, using SDP. Denote the solution $\{\mathbf{X}_k\}_{k=1}^G$.
2. **Randomization / Scaling Loop:** For each k , generate a vector in the span of $\mathbf{X}_k$, using the Gaussian randomization technique (randC) in [6]. If, for some k , $\text{rank}(\mathbf{X}_k) = 1$, then use the principal component instead. Next, feed the resulting set of candidate beamforming vectors $\{\mathbf{w}_k\}_{k=1}^G$ into problem $\mathcal{MGPC}$ and solve it using LP. If the particular instance of $\mathcal{MGPC}$ is infeasible, discard the proposed set of candidate beamforming vectors; else, see if it yields smaller $\mathcal{MGPC}$ objective than previously checked candidates. If so, record solution and associated objective value.

The *quality* of approximate solutions to problem $\mathcal{I}$ generated this way can be checked against the lower bound on transmit power obtained in solving problem $\mathcal{R}$. This bound can be further motivated from a duality perspective, as in [6]; that is, the aforementioned relaxation lower bound is in fact the tightest lower bound on the optimum of problem $\mathcal{I}$ attainable via Lagrangian duality [2]. This follows from arguments in [8] (see also the single-group case in [6]), due to the fact that problem $\mathcal{I}$ is a quadratically constrained quadratic program.

4. SIMULATION RESULTS

The first step of the proposed algorithm consists of a relaxation of the original QoS beamforming problem $\mathcal{I}$ to problem $\mathcal{R}$. The original problem $\mathcal{I}$ may or may not be feasible; if it is, then so is problem $\mathcal{R}$. If $\mathcal{R}$ is infeasible, then so is $\mathcal{I}$. The converse is generally not true; i.e., if $\mathcal{R}$ is feasible, $\mathcal{I}$ need not be feasible. In order to establish feasibility of $\mathcal{I}$ in this case, the randomization - $\mathcal{MGPC}$ loop should yield at least one feasible solution. This is most often the case, as will be verified in the sequel. If the randomization - $\mathcal{MGPC}$ loop fails to return at least one feasible solution, then the (in)feasibility of $\mathcal{I}$ cannot be determined. There is, therefore, a relatively small proportion of problem instances for which (in)feasibility of $\mathcal{I}$ cannot be decided using the proposed approach.

It is evident from the above discussion that feasibility is a key aspect of problem $\mathcal{I}$ and its proposed solution via problem $\mathcal{R}$ and the randomization - $\mathcal{MGPC}$ loop. Feasibility depends on a number of factors; namely, the number of transmit antenna elements N , the number and the populations of the multicast groups, G and

G_k respectively, the channel characteristics $\mathbf{h}_i$, the channel noise variances σ_i^2 , and finally the desired receive SINR constraints c_i .

Beyond feasibility, there are two key issues of interest. The first has to do with cases for which the solution to problem $\mathcal{R}$ yields an exact optimum of the original problem $\mathcal{I}$. This happens when the $N \times N$ blocks $\mathbf{X}_k$, $k \in \{1, \dots, G\}$ turn out all being rank-one. In this case, the associated principal components solve optimally the original problem $\mathcal{I}$, i.e., in such a case $\mathcal{R}$ is not a relaxation after all.¹ The second issue has to do with the quality of the final approximate solution to problem $\mathcal{I}$ in those cases where a feasible solution can be found using the proposed two-step algorithm. As in [6], a practical figure of merit for the quality of the final approximate solution (set of beamforming vectors and power scaling factors) is the ratio of the total transmitted power corresponding to the approximate solution over $\sum_{k=1}^G \text{trace}(\mathbf{X}_k)$ - the lower bound generated from the solution of $\mathcal{R}$.

We consider the standard i.i.d. Rayleigh fading model, i.e., the elements of the channel vectors $\mathbf{h}_i$, $\forall i \in \{1, \dots, M\}$ are i.i.d. circularly symmetric complex Gaussian random variables of variance 1. Tables 1 and 2 summarize the results obtained using the proposed algorithm for 300 Monte-Carlo runs² and 1000 Gaussian randomization samples each. The simulations are repeated for a variety of choices for N, M (see column 1). The users are considered to be evenly distributed among the multicast groups, i.e., $G_k = M/G$, $\forall k \in \{1, \dots, G\}$. For each such configuration, the problem is solved for increasing values (in dB, column 2) of the received SINR constraints (same for all users), until problem $\mathcal{R}$ becomes infeasible. The noise variance is set to $\sigma^2 = 1$ for all channels. The percentage of the 300 Monte-Carlo runs for which $\mathcal{R}$ is feasible is shown in column 3. Columns 4 and 5 report the percentage of $\mathcal{R}$ feasible solutions which yield exact solutions to problem $\mathcal{I}$ (i.e., when all $\mathbf{X}_k$'s are rank-one), and for which the ensuing randomization - $\mathcal{MGPC}$ loop yields at least one feasible solution, respectively. Finally, the last column holds the average value of the ratio of transmitted power corresponding to the final approximate solution over the lower bound obtained from the SDR solution.

The $\mathcal{R}$ feasibility percentage, and the percentage of cases where $\mathcal{R}$ is equivalent to $\mathcal{I}$, listed in columns 3 and 4, are also plotted in Figures 1 and 2, versus the requested SINR values, for most of the scenarios under consideration. It is observed that $\mathcal{R}$ is getting more difficult to solve (for increasing values of the SINR constraints) as the number G and/or the population G_k of the multicast groups increases and/or the number N of available transmit antenna elements decreases. In all configurations considered, the higher the target SINR, the less likely it is that problem $\mathcal{R}$ is feasible, which is intuitive. Interestingly though, the percentage of exact solutions to $\mathcal{I}$ generated via $\mathcal{R}$ also increases with target SINR. It seems as if rank-one solutions are more likely when operating close to the infeasibility boundary. Furthermore, if the same number of users is distributed over more multicast groups (thus, the number G_k of users per group drops) the attainable common SINR is reduced, as is perhaps intuitive. On the other hand, when the target SINR is

¹It is interesting to find the frequency of occurrence of such an event, whose benefit is twofold: not only the problem is solved optimally, but also at smaller complexity, since the randomization step and the repeated solution of the ensuing $\mathcal{MGPC}$ problem is avoided.

²3000 Monte-Carlo runs were employed in cases where $\mathcal{R}$ was feasible in less than 10% of the 300 problem instances initially considered. This was done to improve the estimation accuracy for quantities conditioned on the feasibility of $\mathcal{R}$.

on the relatively low side, optimum solutions are more frequently encountered in this case (e.g. see the case of 12 users distributed in 2, 3, and 4 groups for SINR of 6dB), since it is more likely for the fewer users of any group to be spatially close (the respective probability is approximately $1/G^{G_k}$). Last but not least, the randomization - $\mathcal{MGPC}$ loop yields a feasible solution with a probability higher than 90% in most cases where $\mathcal{R}$ is feasible; this solution entails transmission power that is under two times (3 dB from) the possibly unattainable lower bound, on average.

In some scenarios, $\mathcal{R}$ consistently yields an exact solution of $\mathcal{I}$. That is, the $\mathbf{X}_k$ blocks are all consistently rank-one. In this case, no further randomization is needed - the principal components of the extracted blocks are the optimal beamformers. More on this will be included in [5].

5. CONCLUSIONS

Transmit beamformer design was considered in the context of co-channel multicast transmission to multiple groups of users. The problem is a generalization of downlink transmit beamforming of independent information streams to individual users ([1] and references therein); and the single-group multicast beamforming in [6]. Using [6], the general instance of the problem is easily shown to be NP-hard. A two-step approach comprising semidefinite relaxation and a randomization - multicast power control loop was proposed and shown to yield high-quality approximate solutions, plus means of testing feasibility, at manageable complexity cost.

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Fig. 1. $\mathcal{R}$ feasibility percentages

Fig. 2. $\mathcal{R}$ equivalence to $\mathcal{I}$ percentages,

Table 1. MC simulation results for QoS Beamforming (Rayleigh)

$N/G \times G_k$	SINR	$\mathcal{R} \%$	$\mathcal{R} \equiv \mathcal{I} \%$	$\mathcal{MGPC} \%$	mean
$8/2 \times 8$	6	100	9.33	99.67	1.57
$8/2 \times 6$	6	100	34.33	100	1.17
$8/3 \times 4$	6	100	76.67	100	1.04
$8/4 \times 3$	6	100	92.67	99.67	1.01
$6/2 \times 8$	6	96.33	13.49	83.74	2.74
$6/2 \times 6$	6	100	37.67	100	1.39
$6/2 \times 4$	6	100	84	99.67	1.02
$4/2 \times 8$	6	4.57	35.77	68.61	1.86
$4/2 \times 6$	6	46.67	48.57	88.57	1.64
$4/2 \times 4$	6	97.67	74.40	100	1.07
$8/2 \times 8$	8	100	13	99.33	1.85
$8/2 \times 6$	8	100	34.67	100	1.16
$8/3 \times 4$	8	100	79.67	100	1.04
$8/4 \times 3$	8	83	95.18	100	1.01
$6/2 \times 8$	8	70.33	21.33	79.62	2.05
$6/2 \times 6$	8	99.67	38.80	99.67	1.26
$6/2 \times 4$	8	100	83.33	100	1.02
$4/2 \times 6$	8	12.67	60.53	92.11	2.24
$4/2 \times 4$	8	90	80.37	100	1.05

Table 2. MC simulation results for QoS Beamforming (Rayleigh)

$N/G \times G_k$	SINR	$\mathcal{R} \%$	$\mathcal{R} \equiv \mathcal{I} \%$	$\mathcal{MGPC} \%$	mean
$8/2 \times 8$	10	100	13	99.67	1.92
$8/2 \times 6$	10	100	37	99.67	1.17
$8/3 \times 4$	10	99	80.81	99.33	1.04
$8/4 \times 3$	10	43.4	97.31	98.92	1.00
$6/2 \times 8$	10	30.67	36.96	84.78	1.64
$6/2 \times 6$	10	98	44.90	96.94	1.46
$6/2 \times 4$	10	100	82.67	100	1.02
$4/2 \times 6$	10	1.97	74.58	93.22	1.39
$4/2 \times 4$	10	74	82.43	99.10	1.04
$8/2 \times 8$	12	97.67	17.41	96.93	1.75
$8/2 \times 6$	12	100	37.33	100	1.15
$8/3 \times 4$	12	91.67	87.64	100	1.04
$8/4 \times 3$	12	11.73	97.44	99.72	1.00
$6/2 \times 8$	12	5.1	49.02	84.31	1.99
$6/2 \times 6$	12	86.33	52.51	98.07	1.37
$6/2 \times 4$	12	100	86	99	1.02
$4/2 \times 4$	12	51.33	86.36	99.35	1.14
$8/2 \times 8$	14	90.33	32.84	95.94	2.11
$8/2 \times 6$	14	100	40.67	100	1.13
$8/3 \times 4$	14	73.33	92.27	100	1.04
$8/4 \times 3$	14	1.93	96.55	100	1.10
$6/2 \times 6$	14	68.67	64.08	97.09	1.21
$6/2 \times 4$	14	100	87	100	1.01
$4/2 \times 4$	14	32.33	90.72	97.94	1.04
$8/2 \times 8$	16	70.67	48.11	95.28	1.63
$8/2 \times 6$	16	100	48	100	1.11
$8/3 \times 4$	16	51.33	92.86	100	1.03
$6/2 \times 6$	16	49	68.71	92.28	1.15
$6/2 \times 4$	16	100	88.33	99.33	1.01
$4/2 \times 4$	16	18.33	90.91	100	1.01
$8/2 \times 8$	18	48.67	57.53	94.52	1.28
$8/2 \times 6$	18	100	55	100	1.10
$8/3 \times 4$	18	31	93.55	100	1.02
$6/2 \times 6$	18	33.67	79.21	98.02	1.13
$6/2 \times 4$	18	100	87.67	99.33	1.01
$4/2 \times 4$	18	8.53	95.70	98.83	1.02
$8/2 \times 8$	20	30	64.44	97.78	1.29
$8/2 \times 6$	20	100	57.33	100	1.08
$8/3 \times 4$	20	19	92.98	98.25	1.01
$6/2 \times 6$	20	17	78.43	96.08	1.15
$6/2 \times 4$	20	100	89	100	1.01
$4/2 \times 4$	20	4.37	96.95	98.47	1.02
$8/2 \times 8$	22	15.67	72.34	95.74	1.29
$8/2 \times 6$	22	100	61	100	1.08
$8/3 \times 4$	22	6.93	95.19	99.04	1.02
$6/2 \times 6$	22	10	80	96.67	1.37
$6/2 \times 4$	22	100	91	100	1.01
$4/2 \times 4$	22	1.83	98.18	98.18	1.00
$8/2 \times 8$	24	6.33	78.95	94.74	1.39
$8/2 \times 6$	24	100	64	100	1.07
$8/3 \times 4$	24	2.76	96.39	98.80	1.02
$6/2 \times 6$	24	4.37	90.84	96.95	1.12
$6/2 \times 4$	24	100	91	98.33	1.01
$8/2 \times 8$	26	2	83.33	83.33	1.00
$8/2 \times 6$	26	99	65.66	99.63	1.07
$8/3 \times 4$	26	1.37	95.12	100	1.01
$6/2 \times 6$	26	1.9	96.49	100	1.03
$6/2 \times 4$	26	100	91.33	99	1.01
$8/2 \times 6$	28	100	65.67	98.67	1.07
$6/2 \times 4$	28	98.33	91.28	99.33	1.01
$8/2 \times 6$	30	98.67	66.55	99.32	1.07