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Tumula V. K. Chaitanya and Erik G. Larsson, Outage-Optimal Power Allocation for Hybrid ARQ with Incremental Redundancy, 2011, accepted IEEE Transactions on Wireless Communications.

Pre-print available at: Linköping University Electronic Press

<http://urn.kb.se/resolve?urn=urn:nbn:se:liu:diva-67132>

# Outage-Optimal Power Allocation for Hybrid ARQ with Incremental Redundancy

Tumula V. K. Chaitanya and Erik G. Larsson

**Abstract**—We consider the optimization of power in incremental redundancy (IR) based hybrid automatic repeat request (HARQ) schemes when the maximum number of (re)transmissions is fixed. We formulate two optimization problems: (i) minimizing the packet drop probability (PDP) under a total average transmit power constraint, and (ii) minimizing the average transmit power under a fixed PDP constraint. We consider in detail the special case of only two allowed transmissions, and we prove that the two optimization problems are equivalent. For this special case, we also provide a sub-optimal root-finding solution and compare its performance with the optimal solution obtained through an exhaustive search. The results show that the optimal power allocation can provide significant gains over the equal power solution in terms of average transmit power spent. The performance of the proposed root-finding solution is practically the same as that of the optimal solution.

**Index Terms**—Hybrid ARQ, incremental redundancy, optimal power allocation, outage probability.

## I. INTRODUCTION

Hybrid automatic repeat request (HARQ) schemes are commonly used in wireless networks to combat the loss of data packets due to channel fading. In HARQ schemes, if the receiver fails in decoding the data correctly, it asks for a retransmission. This retransmission can contain a repetition of previously sent bits, or new parity bits [1]. When the retransmission consists of repeated bits, we speak of chase-combining based HARQ (CC-HARQ). Schemes where the retransmission consists of new parity bits are referred to as incremental-redundancy HARQ (IR-HARQ) schemes. IR-HARQ schemes require code-combining at the receiver.

**Related work:** Previous works on HARQ have focused on many aspects. For example, an information theoretic throughput analysis of HARQ schemes for the Gaussian collision channel was presented in [2]. A performance analysis of IR-HARQ based on low-density parity check (LDPC) codes was presented in [3]. A fixed outage probability analysis of HARQ in block-fading channels with statistical channel state information at the transmitter (CSIT) was presented in [4]. Reference [4] also presented a rate adaptation scheme which

maintains a fixed target outage probability. Many variations of HARQ have also been developed for practical systems, e.g., [5].

In this work, we consider the problem of optimal power allocation for IR-HARQ schemes under the assumption of statistical CSIT (distribution of channel gains). This assumption is practical in the sense that it requires only a modest amount of feedback from the receiver and fits well with scenarios where instantaneous CSI feedback signaling cost would be non-negligible. The specific optimization problems we are interested in are:

- *Problem 1:* Minimizing the PDP under an average transmit power constraint.
- *Problem 2:* Minimizing the average transmit power under a specified PDP.

For CC-HARQ schemes, an ad-hoc power ramping scheme was proposed in [6] to improve the throughput. In [7], a cross-layer optimization problem to maximize the average system goodput with HARQ was studied under the assumption of outdated CSIT, and an asymptotic cross-layer policy involving power allocation, rate allocation and user selection was derived. Recently in [8], the authors considered *Problem 2* for CC-HARQ and used outage probability analysis to find the solution. For IR-HARQ schemes, in [9], the authors used a geometric programming approach to minimize the average transmitted energy under a bit-error-rate (BER) constraint, assuming BPSK modulation. However, the problem formulation there involves parameters which can only be obtained through simulations.

**Contributions:** We formulate two optimization problems (*Problem 1* and *Problem 2* above) for IR-HARQ schemes. We prove that for the special case of only two allowed transmissions, *Problem 1* and *Problem 2* are equivalent. We also provide a simple root-finding solution to the problems and compare its performance with that of the optimal solution obtained using an exhaustive search.

In previous works on this topic that we are aware of, the authors considered only optimization with an average transmit power constraint, without any limit on the peak transmit power [8], [9]. By contrast, in this paper we consider both an average power constraint and a peak power constraint. The motivation is that a peak power constraint is more relevant in practice. We use outage probability analysis to derive PDP expressions, which is more fundamental than BER analysis used in much previous work.

Our work differs from [4] in that 1) we are interested in minimizing the PDP, and 2) in that [4] assumed that the same transmission power is used in different ARQ rounds.

Manuscript received November 5, 2010; revised March 28, 2011; accepted March 28, 2011. The associate editor coordinating the review of this letter and approving it for publication was Prof. J. M. Shea.

T. V. K. Chaitanya and E. G. Larsson are with Linköping University, Dept. of Electrical Engineering (ISY), Division of Communication Systems, SE-581 83 Linköping, Sweden. (e-mail: {tvk, erik.larsson}@isy.liu.se).

This work was supported in part by the Swedish Research Council (VR), the Swedish Foundation for Strategic Research (SSF), and ELLIIT. E. G. Larsson is a Royal Swedish Academy of Sciences (KVA) Research Fellow supported by a grant from the Knut and Alice Wallenberg Foundation.

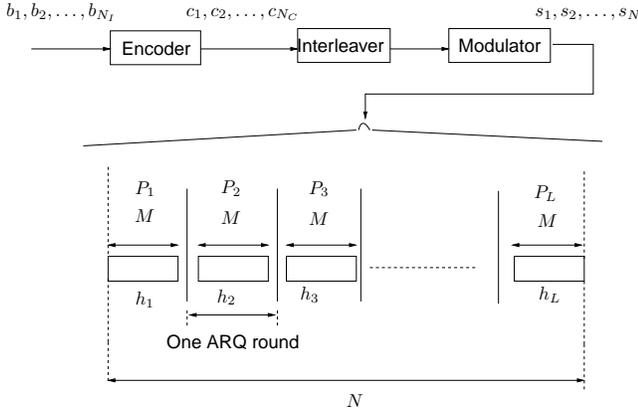


Figure 1. System model

The results from our work show that using equal transmission power in different ARQ rounds is sub-optimal in terms of minimizing the PDP for a given average transmit power. Our work also differs from [8] in that the outage probability expressions for IR-HARQ differ from those of CC-HARQ and hence they result in different optimization problems.

## II. SYSTEM MODEL AND OPTIMIZATION PROBLEMS

In this section, we describe the system model and the optimization problems considered in our paper along with the assumptions that we make. The system model is shown in Fig. 1. Let  $b_1, b_2, \dots, b_{N_I}$  represent the information bits and let  $c_1, c_2, \dots, c_{N_C}$  represent the coded output bits from the channel encoder.  $N_I$  and  $N_C$  denote the number of information bits and the number of coded bits respectively. After interleaving, the coded bits are mapped onto a fixed constellation  $\mathcal{S}$  to obtain the modulation symbols  $s_1, s_2, \dots, s_N$ , where  $N$  denotes the total number of modulation symbols. We assume that  $\mathbb{E}[|s_n|^2] = 1, 1 \leq n \leq N$ .

The modulation symbols are transmitted to the destination using an IR-HARQ scheme. We assume statistical CSIT, i.e., the transmitter has knowledge about the distribution of channel gains but not of the instantaneous gains. The number of channel uses spent on each ARQ round is fixed to  $M$  and we transmit one modulation symbol per channel use. We also assume that all symbols sent during the  $l$ th ARQ round have the same energy. After each ARQ round, the destination tries to decode the data packet based on the information it has received till then. We assume conventional acknowledgment (ACK)/negative ACK (NACK) signaling feedback from the destination and that this feedback is instantly received at the source and is error-free. In this study, we assume that the maximum number of transmissions is limited, to say  $L$  (we have  $LM = N$ ). If the destination is unable to decode the packet after  $L$  transmissions, the source drops the current packet and moves on to the next packet.

We consider a block fading channel in which the channel gains are constant during one ARQ round and change independently between the ARQ rounds. Under these assumptions, we write the received signal for the  $m$ th channel use of the

$l$ th transmission as:

$$y_{l,m} = \sqrt{P_l} h_l s_{l,m} + e_{l,m} \text{ for } 1 \leq m \leq M \text{ and } 1 \leq l \leq L \quad (1)$$

where  $s_{l,m} \in \{s_1, s_2, \dots, s_N\}$  and  $P_l$  and  $h_l$  are the power used and the Rayleigh fading channel gain in the  $l$ th ARQ round, respectively. We define  $\alpha_l \triangleq |h_l|^2$ , and we denote the average squared channel gains by:

$$\lambda \triangleq \mathbb{E}[|h_l|^2], \forall l.$$

$e_{l,m}$  for  $1 \leq m \leq M$  and  $1 \leq l \leq L$  are independent additive Gaussian noise samples, with  $e_{l,m} \sim \mathcal{CN}(0, 1)$ .

We use outage events to characterize whether the receiver sends an ACK or a NACK signal after an ARQ round. Let  $I_l$  denote the total amount of mutual information accumulated till the  $l$ th ARQ round. We can write<sup>1</sup>:

$$I_l = M \sum_{i=1}^l \log_2(1 + P_i \alpha_i) \quad l = 1, 2, \dots, L. \quad (2)$$

Since the mutual information accumulated at the receiver is a non-decreasing quantity, the outage probability after  $l$  transmissions, which is same as the probability that the receiver sends a NACK signal after the  $l$ th ARQ round, is given by [3]:

$$\begin{aligned} p_{\text{out},l} &= \Pr(I_l < N_I \cap I_{l-1} < N_I \cap \dots \cap I_1 < N_I) \\ &= \Pr(I_l < N_I). \end{aligned} \quad (3)$$

Note that the packet drop probability  $p_{\text{drop}} = p_{\text{out},L}$ . Let  $p_{\text{out},0} = 1$ , by convention. We define the average power as:

$$P_{\text{avg}} = \sum_{l=1}^L P_l p_{\text{out},l-1}.$$

In practice, the transmitter will have a limit on the maximum peak power with which it can transmit, say  $P_{\text{max}}$ . We assume that  $P_{\text{max}} \geq P_{\text{avg}}$ . Now we formulate the two optimization problems of interest.

**Problem 1:** Here, the objective is to minimize the packet drop probability under an average transmit power constraint. This problem can be formulated as follows:

$$\begin{aligned} &\min_{(P_1, P_2, \dots, P_L)} p_{\text{drop}} \\ \text{Subject to } &0 \leq P_l \leq P_{\text{max}}, \quad \text{for } 1 \leq l \leq L \\ &P_{\text{avg}} = \sum_{l=1}^L P_l p_{\text{out},l-1} \leq \bar{P}_{\text{given}} \end{aligned} \quad (4)$$

**Problem 2:** Here, the objective is to minimize the average transmit power under a fixed packet drop probability constraint. This can be formulated as:

$$\begin{aligned} &\min_{(P_1, P_2, \dots, P_L)} P_{\text{avg}} \\ \text{Subject to } &0 \leq P_l \leq P_{\text{max}}, \quad \text{for } 1 \leq l \leq L \\ &p_{\text{drop}} \leq p_{\text{drop}}^{\text{target}} \end{aligned} \quad (5)$$

<sup>1</sup>This assumes an infinite block length and a Gaussian code book. However, for many practical schemes with adaptive modulation and coding, we can write  $I_l \approx M \sum_{i=1}^l \log_2(1 + \zeta_i P_i \alpha_i)$  for  $l = 1, 2, \dots, L$ , where  $\zeta_i$  are penalty factors that represent the distance from the Shannon capacity [11].

### III. PACKET DROP PROBABILITY ANALYSIS

In this section, we compute  $p_{\text{out},l}$ ,  $1 \leq l \leq L$ . From (3), we can write:

$$\begin{aligned} p_{\text{out},l} &= \Pr(I_l < N_I) = \Pr\left(\sum_{i=1}^l \log_2(1 + P_i \alpha_i) < \delta\right) \\ &= \Pr\left(\prod_{i=1}^l (1 + P_i \alpha_i) < 2^\delta\right) \end{aligned} \quad (6)$$

where  $\delta = \frac{N_I}{M}$  is the spectral efficiency in bits per channel use (bpcu) in one ARQ round.  $p_{\text{out},l}$  in (6) can be expressed in terms of the generalized upper incomplete Fox's H-function [10]. However these closed-form expressions are lengthy and not tractable for further analysis. Because of this, the packet drop probability  $p_{\text{drop}} = p_{\text{out},L}$  in the optimization problems (4) and (5) cannot be expressed as a tractable mathematical function of  $P_1, P_2, \dots, P_L$ . Hence finding solutions to (4) and (5) is difficult in general. Next, we consider the special case of  $L = 2$  and solve for  $p_{\text{drop}}$  in closed form, using a different approach.

#### A. The special case of $L = 2$

When only two transmissions are allowed for a data packet, we can compute  $p_{\text{out},1}$  and  $p_{\text{out},2} = p_{\text{drop}}$  in closed form. The expressions are given below, and their derivations are given in Appendix A:

$$p_{\text{out},1} = 1 - \exp\left(-\frac{2^\delta - 1}{P_1 \lambda}\right) = 1 - \sum_{i=0}^{\infty} \frac{\left(\frac{1-2^\delta}{P_1 \lambda}\right)^i}{i!}, \quad (7)$$

$$p_{\text{drop}} = p_{\text{out},2} = \frac{\delta 2^\delta \ln 2 - 2^\delta + 1}{P_1 P_2 \lambda^2} + O\left(\frac{1}{P^3}\right). \quad (8)$$

We neglect the  $O\left(\frac{1}{P^3}\right)$  term of (8) in the further analysis. With this approximation, we see that  $p_{\text{drop}} \propto \frac{1}{P_1 P_2}$ . Hence we write the optimization problems in (4) and (5) for  $L = 2$  as

$$\begin{aligned} \min_{(P_1, P_2)} X &= \frac{1}{P_1 P_2} \\ \text{Subject to } C_{11} &: 0 \leq P_1 \leq P_{\max}, \\ C_{12} &: 0 \leq P_2 \leq P_{\max}, \\ C_{13} &: P_1 + P_2 p_{\text{out},1} \leq \bar{P}_{\text{given}}, \end{aligned} \quad (9)$$

and

$$\begin{aligned} \min_{(P_1, P_2)} P_1 + P_2 p_{\text{out},1} \\ \text{Subject to } C_{21} &: 0 \leq P_1 \leq P_{\max}, \\ C_{22} &: 0 \leq P_2 \leq P_{\max}, \\ C_{23} &: \frac{1}{P_1 P_2} \leq K \end{aligned} \quad (10)$$

where  $K$  is a constant, which is a function of target packet drop probability  $p_{\text{drop}}^{\text{target}}$ . To solve the optimization problems, the transmitter only needs the feedback about  $\lambda$ . The optimization problems in (9) and (10) are non-linear. The feasible set of these problems is compact. Hence we can always find the global optimum by an exhaustive search.

*Remark 1:* Let  $(P_1^*, P_2^*)$  be the optimal solution to (9). Then  $C_{13}$  is active at  $(P_1^*, P_2^*)$ .

*Proof:* We write the Lagrangian function for (9) as:

$$\begin{aligned} \mathcal{L}_1(P_1, P_2, \gamma_1, \gamma_2, \mu) &= \frac{1}{P_1 P_2} + \gamma_1 (P_1 - P_{\max}) + \gamma_2 (P_2 - P_{\max}) \\ &\quad + \mu (P_1 + P_2 p_{\text{out},1} - \bar{P}_{\text{given}}). \end{aligned} \quad (11)$$

From the Karush-Kuhn-Tucker (KKT) necessary conditions [12], we require  $\mu^* (P_1^* + P_2^* p_{\text{out},1} - \bar{P}_{\text{given}}) = 0$  and  $\mu^* \geq 0$ .<sup>2</sup> Now to prove that  $C_{13}$  is active, we show that  $\mu^* > 0$  at  $(P_1^*, P_2^*)$ . Considering  $\frac{\partial \mathcal{L}_1}{\partial P_1} \Big|_{(P_1^*, P_2^*, \gamma_1^*, \gamma_2^*, \mu^*)} = 0$ , we have

$$\begin{aligned} \frac{\partial \mathcal{L}_1}{\partial P_1} \Big|_{(P_1^*, P_2^*, \gamma_1^*, \gamma_2^*, \mu^*)} &= \frac{-1}{P_1^{*2} P_2^*} + \gamma_1^* + \mu^* \left(1 - \frac{P_2^* (2^\delta - 1) (1 - p_{\text{out},1}^*)}{\lambda P_1^{*2}}\right) = 0 \end{aligned} \quad (12)$$

By assumption, we have  $P_{\max} \geq \bar{P}_{\text{given}}$ , hence  $P_1^* < P_{\max}$ . This implies that  $C_{11}$  is inactive at  $(P_1^*, P_2^*)$ , which further implies that  $\gamma_1^* = 0$ . So, from (12) we see that  $\mu^* > 0$ . ■

*Remark 2:* Let  $(\tilde{P}_1, \tilde{P}_2)$  be the optimal solution to (10). Then  $C_{23}$  is active at  $(\tilde{P}_1, \tilde{P}_2)$ .

*Proof:* We write the Lagrangian function for (10) as:

$$\begin{aligned} \mathcal{L}_2(P_1, P_2, \gamma_1, \gamma_2, \mu) &= P_1 + P_2 p_{\text{out},1} + \gamma_1 (P_1 - P_{\max}) + \\ &\quad \gamma_2 (P_2 - P_{\max}) + \mu \left(\frac{1}{P_1 P_2} - K\right). \end{aligned} \quad (13)$$

From the KKT necessary conditions, we require  $\tilde{\mu} \left(\frac{1}{\tilde{P}_1 \tilde{P}_2} - K\right) = 0$ .<sup>3</sup> Considering  $\frac{\partial \mathcal{L}_2}{\partial P_2} \Big|_{(\tilde{P}_1, \tilde{P}_2, \tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\mu})} = 0$ , we have

$$\frac{\partial \mathcal{L}_2}{\partial P_2} \Big|_{(\tilde{P}_1, \tilde{P}_2, \tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\mu})} = \tilde{p}_{\text{out},1} + \tilde{\gamma}_2 - \frac{\tilde{\mu}}{\tilde{P}_1 \tilde{P}_2^2} = 0 \quad (14)$$

From the KKT conditions we have  $\tilde{\gamma}_2 \geq 0$ . Hence from (14), we see that  $\tilde{\mu} > 0$  and hence  $C_{23}$  must be active at  $(\tilde{P}_1, \tilde{P}_2)$ . ■

*Remark 3:* The optimization problems in (9) and (10) are equivalent.

*Proof:* In Remarks 1 and 2, we proved that the constraints  $C_{13}$  and  $C_{23}$  are active at the optimal solutions for (9) and (10) respectively. Suppose  $(P_1^*, P_2^*)$  is the optimal solution to (9) for a given  $\bar{P}_{\text{given}}$ , and that the objective function value is  $X^* = \frac{1}{P_1^* P_2^*}$ . Then if we use  $K = X^*$  in  $C_{23}$  of (10), the objective function value of (10) can never be smaller than  $\bar{P}_{\text{given}}$  as this would violate Remark 1 (stating that  $C_{13}$  is active at the optimal solution in (9)). Hence  $(P_1^*, P_2^*)$  is also the optimal solution to (10) with  $K = X^*$ . We can similarly prove the converse, i.e., the optimal solution to (10) is also the optimal solution to (9). ■

<sup>2</sup>We denote the optimal values of Lagrangian coefficients in (9) by  $\gamma_1^*, \gamma_2^*, \mu^*$ .

<sup>3</sup>We denote the optimal values of Lagrangian coefficients in (10) by  $\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\mu}$ .

$$P_1^{\text{opt}} = -\frac{b}{3a} - \frac{1}{3a} \sqrt[3]{\frac{1}{2} \left[ 2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3} \right]} - \frac{1}{3a} \sqrt[3]{\frac{1}{2} \left[ 2b^3 - 9abc + 27a^2d - \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3} \right]}. \quad (18)$$

$$P_2^{\text{opt}} = \min \left( \frac{P_{\text{out},1}}{P_1^{\text{opt}} (\bar{P}_{\text{given}} - P_1^{\text{opt}})}, P_{\text{max}} \right). \quad (19)$$

### B. Root-Finding solution to (9):

In this section, we provide a sub-optimal root-finding solution to the optimization problem in (9) (or equivalently, to (10)). Since we are interested in solutions which satisfy  $C_{13}$  with equality, we can write  $P_2 = \frac{\bar{P}_{\text{given}} - P_1}{P_{\text{out},1}}$ . Using this, the objective in (9) is now to minimize

$$f(P_1) = \frac{P_{\text{out},1}}{P_1 (\bar{P}_{\text{given}} - P_1)} \quad (15)$$

Note that while doing this, we relaxed the constraint that  $P_2 \leq P_{\text{max}}$ . Now  $P_1^{\text{opt}}$  can be obtained by finding all the saddle points of  $f(P_1)$  and checking the Hessian at these saddle points.  $p_{\text{out},1}$  is an infinite series as seen from (7), and we approximate it as:

$$P_{\text{out},1} \approx \frac{2^\delta - 1}{P_1 \lambda} - \frac{(2^\delta - 1)^2}{2P_1^2 \lambda^2} \quad (16)$$

Using (16) in (15), the saddle points are the solutions to the following cubic equation:

$$aP_1^3 + bP_1^2 + cP_1 + d = 0 \quad (17)$$

where  $a = 6z\lambda^2$ ,  $b = -4\bar{P}_{\text{given}}z\lambda^2 - 4z^2\lambda$ ,  $c = 3\bar{P}_{\text{given}}z^2\lambda + z^3$ ,  $d = -z^3\bar{P}_{\text{given}}$  and  $z = 2^\delta - 1$ . Since the constant term  $d \leq 0$ , this cubic equation has at least one positive real root. We can prove that there is only one positive root to the cubic equation for  $\bar{P}_{\text{given}} > 0$ , and  $\lambda > 0$  by considering the discriminant of the cubic equation and showing that it is always smaller than zero (see Appendix B). Furthermore, we can prove that there is a feasible positive root to (17) only if

$$\delta \leq \log_2 (1 + 2\bar{P}_{\text{given}}\lambda).$$

It can also be shown that this is the condition for the feasible set of (9) to be convex. The optimal solutions for  $P_1$  and  $P_2$  are given by (18) and (19) shown on top of this page. In (19),  $p_{\text{out},1}$  is given by (16). If  $\zeta$  denotes the desired accuracy of the optimal solution, then finding the optimal solution by exhaustive search has a complexity of  $O\left(\frac{1}{\zeta^2}\right)$ , whereas the root-finding solutions can be directly computed from (18) and (19).

## IV. NUMERICAL RESULTS

In this section, we present results for the case of  $L = 2$ . We define the average signal-to-noise-ratio (SNR) per transmission in dB as  $10 \log_{10} \left( \frac{P_{\text{avg}} \lambda}{2} \right)$ . Fig. 2 shows an analytical performance comparison of the packet drop probabilities for

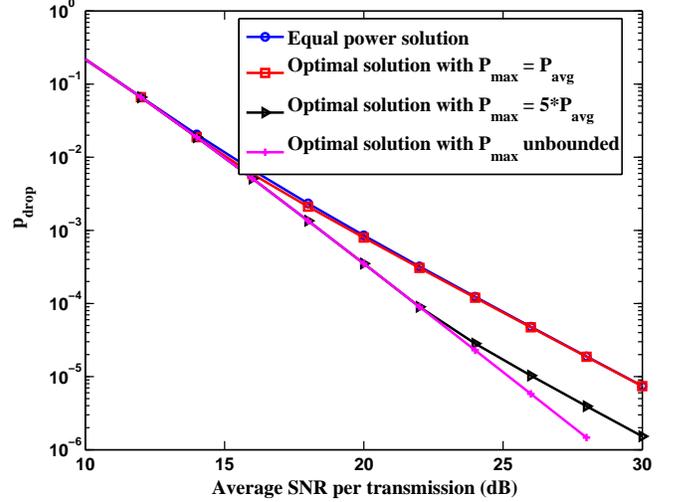


Figure 2. Comparison of the packet drop probability for different values of  $P_{\text{max}}$ . Optimal power values are obtained by an exhaustive search. For comparison purposes, the performance with equal power solution is also plotted.  $P_{\text{avg}}$  is set to 2 and  $\delta = 4$  bits per channel use [bpcu]. The average SNR per transmission is varied by varying  $\lambda$ .

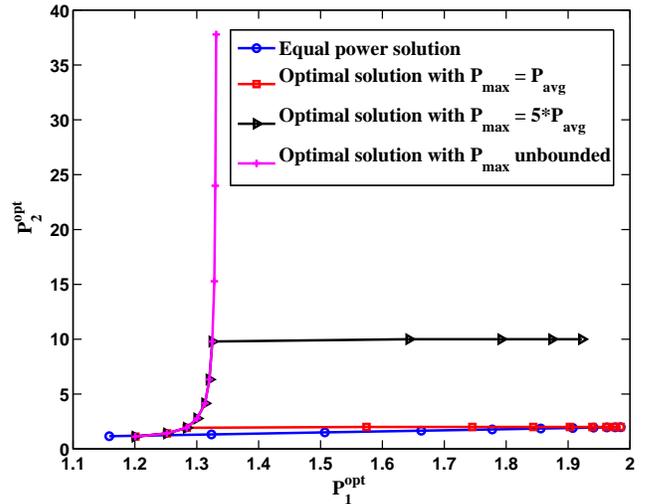


Figure 3. Optimal power values corresponding to the packet drop probability curves in Fig. 2.

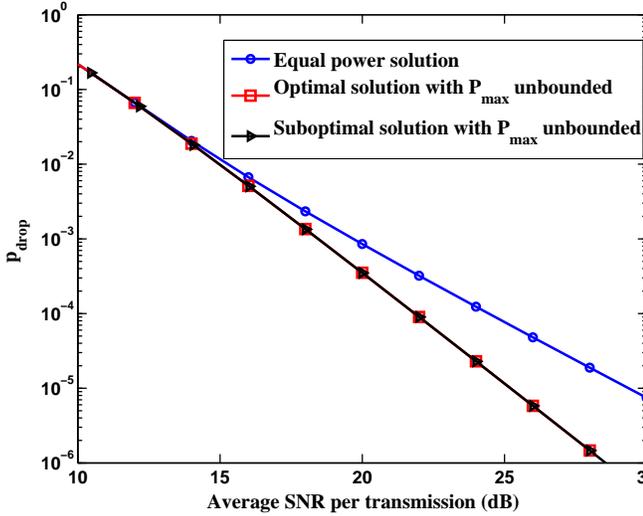


Figure 4. Comparison of the root-finding solution given in Section III-B with the optimal and equal power solution.  $P_{\text{avg}}$  is set to 2 and  $\delta = 4$  bpcu.

the optimization problem in (9) for different values of  $P_{\text{max}}$ . For comparison purposes, we also plotted the equal power solution (obtained by adding the additional constraint  $P_1 = P_2$  in (9)). From the figure, we see that when  $P_{\text{max}} = P_{\text{avg}}$ , the optimal solution has similar performance as the equal power solution. The reason for this is that at high SNR, since  $P_2$  is bounded from above by  $P_{\text{avg}}$ ,  $P_1$  will also approach  $P_{\text{avg}}$  in order to minimize  $P_{\text{drop}}$ . Hence the optimal solution  $(P_1^{\text{opt}}, P_2^{\text{opt}}) \approx (P_{\text{avg}}, P_{\text{avg}})$  will be close to the equal power solution. This can also be observed from Fig. 3, which shows the optimal power values corresponding to the packet drop probability curves in Fig. 2. Note that the curves corresponding to the equal power solution and to the optimal solution with  $P_{\text{max}} = P_{\text{avg}}$  in Fig. 2 have a slope of -2. This is so because

$$P_{\text{drop}} \propto \frac{1}{P_1^{\text{opt}} P_2^{\text{opt}}} \approx \frac{1}{P_{\text{avg}}^2}.$$

For the other curves, we have  $P_2^{\text{opt}} > P_{\text{avg}}$ , hence the slope will be smaller than -2. However the exact relation between  $P_{\text{drop}}$  and  $P_{\text{avg}}$  is not known, so we cannot easily determine the slope. When  $P_{\text{max}}$  is greater than  $P_{\text{avg}}$ , the optimal power solution offers better performance than the equal power solution. At a packet drop probability of  $10^{-4}$ , the optimal solution with  $P_{\text{max}} = 5P_{\text{avg}}$  gives a gain of 3 dB in average SNR per transmission over the equal power solution.

Fig. 4 shows an analytical performance comparison of the root-finding solution given in Section III-B with the optimal solution and the equal power solution. We see that the root-finding solution has practically the same performance as that of the optimal solution.

## V. CONCLUSIONS

Using equal powers for the transmissions in IR-HARQ schemes is strictly sub-optimal in terms of minimizing the packet drop probability under an average transmit power constraint. However, the difference between optimal power allocation and equal power allocation becomes substantial only

for rather small packet drop probabilities. One may extend this work to the case of an unequal number of channel uses per ARQ round and to the case of correlated channel gains.

## APPENDIX A DERIVATION OF $P_{\text{out},1}$ AND $P_{\text{out},2}$

From (6), we have

$$P_{\text{out},1} = \int_{t_1=0}^{\frac{2^\delta-1}{P_1}} \frac{1}{\lambda} \exp\left(-\frac{t_1}{\lambda}\right) dt_1 = 1 - \exp\left(-\frac{2^\delta-1}{P_1\lambda}\right). \quad (20)$$

$$\begin{aligned} P_{\text{out},2} &= \int_{t_1=0}^{\frac{2^\delta-1}{P_1}} \int_{t_2=0}^{\frac{2^\delta}{P_2(1+P_1t_1)} - \frac{1}{P_2}} \frac{1}{\lambda^2} \exp\left(-\frac{t_1+t_2}{\lambda}\right) dt_1 dt_2 \\ &= \frac{1}{\lambda} \int_0^{\frac{2^\delta-1}{P_1}} \exp\left(-\frac{t_1}{\lambda}\right) \times \\ &\quad \left(1 - \exp\left(-\frac{1}{\lambda} \left(\frac{2^\delta}{P_2(1+P_1t_1)} - \frac{1}{P_2}\right)\right)\right) dt_1 \\ &= \left(1 - \exp\left(\frac{1-2^\delta}{P_1\lambda}\right)\right) - \left(\frac{1}{\lambda} + \frac{1}{\lambda^2 P_2} + \frac{1}{2\lambda^3 P_2^2} + \right. \\ &\quad \left. O\left(\frac{1}{P^3}\right)\right) \left[\int_0^{\frac{2^\delta-1}{P_1}} \left(1 - \frac{2^\delta}{P_2\lambda(1+P_1t_1)} - \frac{t_1}{\lambda}\right) dt_1\right] \end{aligned} \quad (21)$$

In (21), we have used  $\exp(-x) = 1 - x + O(x^2)$ . Simplifying further, we arrive at (8).

## APPENDIX B PROOF THAT THE DISCRIMINANT OF (17) IS NEGATIVE FOR $\bar{P}_{\text{given}} > 0$ AND $\lambda > 0$

The discriminant of (17) is given by  $\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2$ . Substituting the values for  $a, b, c$  and  $d$  and simplifying, we get

$$\Delta = -8z^6\lambda^2 \left( \underbrace{z^4 - 11z^3\mu + 38.5z^2\mu^2 - 33z\mu^3 + 14\mu^4}_{\triangleq \Psi} \right) \quad (22)$$

where  $\mu = \bar{P}_{\text{given}}\lambda > 0$ . Since  $-8z^6\lambda^2 < 0$ , we can prove that the discriminant  $\Delta < 0$  by considering  $\Psi$  of (22) and showing that its minimum value is positive. Letting  $z = \rho\mu$  for some  $\rho > 0$  and considering the minimum value as an optimization problem shown below:

$$\min_{\rho>0} (\rho^4 - 11\rho^3 + 38.5\rho^2 - 33\rho + 14)$$

The saddle points for this problem are the solutions to the equation  $4\rho^3 - 33\rho^2 + 77\rho - 33 = 0$ . There is only one real root for this solution with  $\rho = 0.54925$  and the Hessian at the saddle point is positive. The minimum of  $\Psi$  is always greater than zero and hence  $\Delta < 0$ .

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