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Solving nonlinear covering problems arising in WLAN design

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Wireless Local Area Networks (WLANs) are widely used for cable replacement and wireless Internet access. Since the medium access control (MAC) scheme of WLANs has a strong influence on network performance, it should be accounted for in WLAN design. This paper presents AP location models that optimize a network performance measure specific for the MAC scheme of WLANs, which represents the efficiency in sharing the wireless medium. For these models, we propose a solution framework based on an effective integer-linear programming Dantzig-Wolfe reformulation. This framework is applicable to any nonlinear covering problem where the objective function is a sum of contributions over the groundset elements (users in WLANs). Extensive computational results show that our solution strategy quickly yields optimal or near-optimal solutions for WLAN design instances of realistic size.

Subject classifications: Integer Programming, Networks, Telecommunications Area of review: Telecommunications and Networking

1. Introduction

Wireless Local Area Networks (WLANs) have achieved a tremendous popularity in providing cable replacement and Internet connection to companies, organizations, and public areas. A WLAN consists of a set of Access Points (APs) connected to a wired network. Each AP is able to serve users located within its radio coverage area. APs are cheap, and installation cost is typically not an issue. The focus in WLAN planning is on network performance optimization. Unlike cellular networks, where users obtain a dedicated resource in terms of frequency, time slot, or channelization code, WLAN applies a randomized medium access control (MAC) scheme. As this scheme has a strong influence on network performance, WLAN planning models should account for its behavior.

In this paper we consider a performance measure for AP location that is specific for the MAC protocol of WLANs. The AP location problem amounts to decide, given a set of *candidate sites* (CSs), where to install APs. Service coverage is defined by site measurements or signal propagation models (Hills and Schlegel 2004, Eisenblätter et al. 2007) for a set of *test points* (TPs). TPs are locations where the presence of a WLAN user device is expected. Throughout the paper we will equivalently refer to "TPs" or "users", whichever is more convenient. A common requirement in WLAN planning is that each TP has to be covered by at least one installed AP. In some cases,

however, coverage is not strictly required. If the objective function used for AP location favors networks with good coverage, the covering constraints can then be omitted.

We focus here on single-frequency WLANs. Besides providing a basis for the general case, singlefrequency is relevant for several reasons. Unlike in cellular networks, the small number of frequencies available in WLANs allows for only three mutually non-interfering frequencies (IEEE 802.11 1999). Their practical availability is further restricted by national regulations, as well as by external interference from other devices operating in the same (license-free) spectrum. Moreover, while the AP locations are rarely modified after network deployment, the frequency assignment has to be occasionally re-optimized to account for changes in the user distribution or in the external interference. In common two-step planning approaches, where AP location is followed by frequency assignment, the conservative use of one frequency in the location phase may avoid unexpected performance degradations when the number of available frequencies decreases. For a frequency assignment approach based on the performance measure considered here, see Bosio and Yuan 2009.

The paper is organized as follows. In the remainder of Section 1 we discuss the MAC protocol and related work on WLAN planning. In Section 2 we describe our WLAN design problems, present 0-1 hyperbolic programming formulations, and discuss complexity and approximability issues. In Section 3 we propose an Integer Linear Programming (ILP) formulation, based on Dantzig-Wolfe reformulation, that is applicable to a quite general class of nonlinear set covering problems. Section 4 describes a solution approach based on this formulation, and reports extensive computational results for our WLAN design problems. Some concluding remarks are given in Section 5.

1.1. Medium Access Control

WLANs use a Carrier Sense Multiple Access with Collision Avoidance (CSMA/CA) MAC protocol, based a "listen before talk" approach. A device can start a transmission only after sensing the channel as idle (see the IEEE 802.11 1999 standard series for details). As channel sensing does not completely eliminate collisions, an acknowledgment mechanism is used to certify reception. To discuss transmission scenarios in WLANs we adopt the term interference from cellular networks. However, whereas in cellular networks interference generally leads to lower signal quality, in WLANs it results in transmission blocking due to channel sensing, or in an unsuccessful transmission due to collision.

There are two main types of interference in WLAN. In *direct interference* two user devices cannot access the medium simultaneously because they are within each other's sensing range. This occurs even if the two devices wish to communicate to different APs, as in the so-called *exposed terminal* scenario illustrated in Figure 1(a). *Indirect interference* involves transmission collision at APs, as in the *hidden terminal* scenario depicted in Figure 1(b): Two user devices with no direct interference attempt to transmit to the same AP. Carrier sensing allows them to transmit simultaneously, which results in a collision at the AP and in two unsuccessful transmissions. Indirect interference occurs also if the two user devices communicate with different APs and one user is covered by both APs, see Figure 1(c). In general, if a user device lies in the coverage area of a set of APs, transmission from the device to its AP prohibits the other APs from being accessed by other user devices. Note that two users can be both direct and indirect interferences to each other, and that users accessing the same AP are always indirect interferences.

The above discussion suggests that the probability of successful transmission of a user is inversely proportional to the number of its direct and indirect interferers. To account for this relation, we consider a network performance measure that extends the one proposed in Amaldi et al. (2004). The rationale behind this measure, which we refer to as *network efficiency*, is illustrated by two extreme cases. In the first one, each user accesses a distinct AP without direct interference from other users, while in the second one all users access the same AP, or are all within direct interference. These two cases correspond respectively to the maximum and minimum network efficiency. The

Figure 1 WLAN interference scenarios.



Note. Example scenarios for (a) exposed terminal, (b) hidden terminal, and (c) general indirect interference. The coverage range of the APs is shown with a solid circle, while the range of direct interference of the users (black points) is shown with a dashed circle.

applicability of the network efficiency measure is further supported by a recent work on WLAN engineering (Bosio et al. 2007), where simulation results show that it provides a good estimate of WLAN performance, and that optimizing network efficiency in AP location leads to higher data throughput with respect to other objectives aiming at reducing AP coverage overlap.

1.2. Related Work on WLAN planning

Wireless network optimization models are often based on the Set Covering Problem (SCP, see Ceria et al. 1997) or on the Facility Location Problem (FLP, see Labbé and Louveaux 1997). SCP problems are used for selecting a subset of CSs to install radio antennas so as to cover all the TPs (see e.g. Tutschku 1998). In FLP problems, an explicit assignment of the TPs to the antennas has also to be decided (see e.g. Amaldi et al. 2006). Early work on WLAN planning (see e.g. Hills 2001, Prasad 2000) qualitatively investigates the impact of various planning choices on the performance. Some mathematical programming models for WLAN design deal with signal quality without explicitly accounting for interference. Rodrigues et al. (2000) and Mateus et al. (2001) propose an ILP model to maximize the signal quality at the TPs. Kamenetsky and Unbehaun (2002) and Unbehaun and Kamenetsky (2003) present a heuristic aimed at minimizing a convex combination of the average and maximum path loss. Lee et al. (2002) propose a facility location-based model to balance the load among APs.

Interference aspects in WLAN planning have been considered in several papers. Prommak et al. (2002) present a constraint satisfaction model to maximize WLAN capacity subject to constraints specifying received power, perceived interference, and achieved data rate at the user locations. The interference constraint used in the model derives from performance considerations in cellular networks. For WLANs, however, the constraint is less appropriate because interference blocks a transmission or makes it fail, rather than degrading the signal quality. Lu et al. (2006) propose to measure the performance of a WLAN cell (the service area of an AP) with a Markovian process. The process, which is embedded into a Tabu search heuristic for WLAN planning, accurately takes into account intra-cell interference but ignores inter-cell interference. An attempt to deal with intercell interference is presented in Ling and Yeung (2006). The authors propose a model to address the performance impact of overlapping cells operating on the same frequency, and a measure of the total throughput. The resulting optimization problem, which combines AP location and frequency assignment, is very challenging and out of the reach of exact methods. A simple heuristic algorithm is presented and applied to small instances.

Eisenblätter et al. (2007) consider an FLP model for AP location that neglects interferences and maximizes user throughput subject to a budget constraint on the number of installed APs. Frequency assignment is addressed by a second model, which resembles cell overlap minimization for channel allocation in cellular networks. An integrated model optimizing a convex combination of the objective functions used in AP location and frequency assignment is also presented. In Siomina and Yuan (2007), the frequency assignment model is extended to account for AP transmission power, and refined to consider both direct and indirect interference. The optimization models are solved in both references with a standard solver.

The network efficiency considered here extends the measure defined in Amaldi et al. (2004), where only indirect interference was taken into account, coverage was explicitly required, and the resulting problem was solved by simple greedy heuristics. In this paper we refine the aforementioned measure to account for direct interference, and propose efficient optimization models and methods for finding optimal or near-optimal solutions, both with or without coverage requirements.

2. Problem definition

Let I be the set of test points (TPs), and J the set of candidate sites (CSs). We denote by $I_j \subseteq I$ the subset of TPs that are covered if an AP is installed at CS j, and by $J_i = \{j \in J : i \in I_j\}$ the set of CSs from which TP i can be covered. For simplicity, we assume that the AP devices have fixed transmission power, and comment later on the extension to multiple power levels. A solution to the single-frequency WLAN design problem is a subset $S \subseteq J$ of CSs where APs will be installed. We denote the subset of TPs covered by S by $I(S) = \bigcup_{j \in S} I_j$. A solution S is a *cover* if I(S) = I, and it is a *partial cover* otherwise.

Given a partial cover S and a user $i \in I(S)$, let $N_i(S) = I(J_i \cap S) \setminus \{i\}$ denote the set of indirect interferers, i.e., the set of users covered by some AP covering also user i, and let N_i be a short-hand notation for the set $N_i(J_i)$ of all potential indirect interferers of user i. Moreover, given for each user i the set D_i of its potential direct interferers, let $D_i(S) = D_i \cap I(S)$ denote the set of direct interferers of user i that are active (i.e., covered) in S. Following the discussion in Section 1, user ican successfully transmit if and only if none of the users in $N_i(S) \cup D_i(S)$ is transmitting. Assuming uniform traffic and fair access (where the latter is guaranteed by the CSMA/CA protocol), the fraction of transmission time available to a user can be approximated by the reciprocal of the number of its interferers plus 1 (the user itself). This leads to the *network efficiency*:

$$e(S) = \sum_{i \in I} e(S, i) = \sum_{i \in I(S)} \frac{1}{1 + |N_i(S) \cup D_i(S)|}.$$
(1)

Using the AP location variables x_j for all $j \in J$ ($x_j = 1$ if $j \in S$ and 0 otherwise), the interferer variables y_{ih} for all $i \in I, h \in N_i$ ($y_{ih} = 1$ if $h \in N_i(S)$ and 0 otherwise), and the coverage variables z_i for all $i \in I$ ($z_i = 1$ if $i \in I(S)$ and 0 otherwise), the Maximum Efficiency Problem (MEP) can be formulated as the following 0-1 hyperbolic sum programming model:

$$\max \quad \sum_{i \in I} \frac{z_i}{1 + \sum_{h \in D_i} z_h + \sum_{h \in N_i \setminus D_i} y_{ih}}$$
(2)

(MEP) s.t.
$$\sum_{j \in J_i} x_j \ge z_i$$
 $i \in I$ (3)

$$\geqslant x_j \qquad \qquad i \in I, j \in J_i \tag{4}$$

$$y_{ih} \geqslant x_j \qquad \qquad i \in I, h \in N_i, j \in J_i \cap J_h \tag{5}$$

$$= \{0,1\} \qquad \qquad j \in J \tag{6}$$

$$i \in I, h \in N_i \tag{7}$$

$$z_i \in \{0, 1\} \qquad i \in I.$$

$$\tag{8}$$

The formulation for the Maximum Efficiency Problem with Complete Coverage (MEP-C) is obtained by substituting each variable z_i with a constant value 1, so that contraints (3) become the standard SCP constraints

 z_i

$$\sum_{j \in J_i} x_j \ge 1 \qquad i \in I.$$
(9)

In Amaldi et al. (2004, 2009) the network efficiency is approximated by neglecting direct interference. The resulting problems, which we refer to as Approximated Maximum Efficiency Problem (AMEP) and Approximated Maximum Efficiency Problem with Complete Coverage (AMEP-C), are obtained by setting $D_i = \emptyset$ for all $i \in I$ in MEP and MEP-C respectively. The models for all problem variants are summarized in Table 1. Note that constraints (4), required in MEP due to the presence of z variables in the denominator of (2), are redundant in AMEP. In all models the integrality of the x variables implies the integrality of the y and (when present) of the z ones.



Figure 2 Direct versus indirect interference.



Note. (a)-(c) The white region contains the indirect interferers $N_i(S)$ of a given TP $i \in I$, and a dashed circle encloses its direct interferers D_i . The region containing direct interferers that are not also indirect interferers is emphasized in gray, and can be large in some cases.

As illustrated in Figure 2, neglecting direct interference may lead to inaccurate results. However, the average impact on optimal solutions is often limited. Moreover, as we will see, MEP turns out to be much more difficult to solve than AMEP. Therefore such an approximation currently remains the only option for large networks. On the contrary, MEP-C is actually easier than AMEP-C, as in the former the users in D_i are considered as fixed interferers. We will report results for AMEP-C mainly for the sake of comparison with Amaldi et al. (2004, 2009).

Figure 3 Special case of indirect interference.



Note. Users i and h interfere indirectly only if at least one of them is served by AP j_2 .

In all the above models, there exists a special case where two users are considered as interferers, although interference actually depends on the choice of serving AP made by each user. Consider the scenario depicted in Figure 3, and assume that i and h are not within direct interference range. If at least one of i and h is served by AP j_2 , then the two users will be indirect interferers. But if they are served respectively by j_1 and j_3 , then interference does not occur. In this case the network efficiency overestimates the total interference. In order to address such scenarios, one needs to collect additional signal propagation informations, and to make a modeling assumption on how users select their serving APs (e.g. the one with strongest signal). On the other hand, as pointed out in Amaldi et al. (2009), the impact of these non-interfering cases on the optimal solution and on the optimal value is marginal, indicating that such situations are not frequent.

The network efficiency (1) implicitly assumes peak traffic (all users are active and attempting to access the medium). This is justified by the fact that network performance is typically an issue only under peak traffic. However, the measure can be easily generalized to account for user activity levels, if this information is available, and the ILP model and the solution algorithm presented in this paper can be directly applied. Since in reality user and traffic patterns vary over time, an aggregation of various snapshots of the active users taken at different times is often considered. Snapshot-based planning is a common approach in wireless network optimization (see e.g. Amaldi et al. 2003 and Eisenblätter et al. (2002) for third generation cellular systems design).

Although no coverage level can be a-priori guaranteed in MEP and AMEP, one can typically expect that optimal solutions cover most of the users, as the objective function contribution of an uncovered user is zero. In Section 4.4 we present computational experiments for these models, including results for the straightforward extension requiring a minimum coverage percentage.

As a final remark, all the above problems can be extended to the case in which APs can use one of k possible power levels. This is done by introducing, for each physical location, k CSs with nested covering areas. Due to the network efficiency (1), at most one of these k CSs will be selected.

2.1. Compact MILP reformulations for 0-1 hyperbolic programming

0-1 hyperbolic programming is challenging in general. The unconstrained single-ratio case is NPhard except in some special cases (Hammer and Rudeanu 1968, Hansen et al. 1991). Unconstrained multiple-ratio 0-1 hyperbolic programming, which is NP-hard even in those special cases, has been tackled with heuristics and an exact method based on decomposition (Hansen et al. 1990). Approaches to constrained single-ratio 0-1 hyperbolic problems are described in Stancu-Minasian (1997), but little is known on multiple-ratio versions.

A Mixed Integer Linear Programming (MILP) reformulation approach for multiple-ratio 0-1 hyperbolic problems is discussed in Tawarmalani et al. (2002). For our WLAN problem MEP, this reformulation technique amounts to substituting each ratio in the objective function (2) with a new continuous variable r_i . The value of r_i is defined by the bilinear constraint

$$r_i(1+\sum_{h\in D_i} z_h + \sum_{h\in N_i\setminus D_i} y_{ih}) = z_i,$$

containing the mixed (continuous-binary) bilinear terms $r_i z_h$ for $h \in D_i$ and $r_i y_{ih}$ for $h \in N_i \setminus D_i$. A standard linearization technique for such terms consists in defining upper and lower envelopes exploiting upper and lower bounds on the continuous variable r_i . As proposed in Amaldi et al. (2009), this can be improved by disjunctive arguments, separately tightening the bounds on the continuous variable r_i for each possible value of the binary variable $(z_h \text{ and } y_{ih})$. This yields a remarkable reduction in both LP-gap and computing times, as shown in Amaldi et al. (2009) for AMEP-C.

Besides applying bound tightening, in this paper we also consider model reduction by preprocessing techniques. A first simple reduction consists in merging "duplicated users", i.e., users for which e(S,i) = e(S,h) for all $S \subseteq J$, which happens if *i* and *h* are always covered by the same set of APs $(J_i = J_h)$ and have the same set of direct interferers $(D_i = D_h)$. A more substantial reduction is obtained by merging variables y_{ih} . Given two pairs of users a, b and c, d, it is easy to see that if $J_a \cap J_b = J_c \cap J_d$, then $y_{ab} = y_{cd}$. We can then replace the variables y_{ih} with a unique variable y_T for all pairs *i*, *h* for which $J_i \cap J_h = T$. The compact formulation resulting from bound tightening and model reduction will be used in Section 4.4 for comparison purposes.

2.2. Complexity issues

We now briefly discuss and extend to our other problems some complexity results for AMEP-C presented in Amaldi et al. (2009). By adapting the NP-hardness proof for AMEP-C, it can be easily verified that AMEP, MEP-C, and MEP are all NP-hard in the general case. However, WLANs are typically deployed in 2-dimensional (2D) environments, where users and AP locations are points in the plane. In this section we consider for simplicity uniform 2D Euclidean instances, where coverage and direct interference areas are disks with uniform radius ρ , as depicted in Figure 1. Complexity results for the uniform 2D Euclidean case, which is an ideal case under isotropic radio propagation, can be easily extended to the variable-radius version (APs and TPs having arbitrary coverage and interference patterns (the real case of anisotropic radio propagation) by embedding each propagation pattern into a disk.

Since positioning too many APs in a bounded region would result in unacceptable interference levels, we restrict the complexity study to uniform 2D instances in which only covers with bounded AP density are considered, that is, covers for which the number of selected CSs inside any circular region of diameter $\lambda \rho$ is bounded by a constant $C(\lambda)$, with $\lambda > 0$. Clearly, negative complexity results for such instances also hold for the general case. As shown by Amaldi et al. (2009), AMEP-C restricted to uniform 2D Euclidean instances satisfying this property remains NP-hard (see Bosio 2006 for details), but it admits a polynomial-time approximation scheme (PTAS). The NP-hardness proof directly applies to MEP-C, although there seems to be no easy way to extend it to AMEP and MEP, where no explicit coverage is required. On the other hand, the PTAS can be applied to any covering problem whose objective function can be expressed as $f(S) = \sum_{i \in I} f_i(S \cap \mathbb{B}(i, 2\rho))$, where each contribution f_i is a nonnegative function of the selected disks having center inside a ball $\mathbb{B}(i, 2\rho) \subseteq \mathbb{R}^2$ centered at *i* and with radius 2ρ . The PTAS directly holds for all our problems, relaxing the coverage requirement whenever appropriate $(f_i(S) = 0)$ if $i \notin I(S)$. For AMEP, however, we provide a more efficient PTAS in Appendix A.1.

WLAN design is also of interest in 1D environments, such as a railway platform, where users can be represented by points on the line and APs by segments. The polynomial-time algorithm given in Amaldi et al. (2009) for the 1D Euclidean version of AMEP-C can be easily extended to solve the 1D Euclidean version of problems AMEP, MEP-C, and MEP in polynomial time.

3. An enumerative ILP formulation

In this section we present a tight enumerative ILP formulation of our WLAN problems derived by Dantzig–Wolfe reformulation. As this approach is applicable to a larger class of covering problems, we derive it for the general case, using the set covering notation of groundset elements and covering subsets (TPs and CSs in WLAN terminology), and then apply it to our WLAN problems.

Consider a Groundset-Separable Set Covering Problem (GSSCP), that is an SCP whose objective function can be expressed as $f(S) = \sum_{i \in I} f_i(S \cap K_i)$, where the contribution of the *i*-th groundset element is a function $f_i : \{0,1\}^{|K_i|} \to \mathbb{R}$ of its *local solution* $S \cap K_i$. The sets $K_i \subseteq J$ are given in the input, and depend on the problem structure. For example, in MEP-C we have $f_i(S) = 1/|D_i \cup I(S \cap J_i)|$, and hence $K_i = J_i$.

By considering the subset selection variables x_j for all $j \in J$ ($x_j = 1$ if $j \in S$ and 0 otherwise), and denoting by $\{e_j^i : j \in K_i\}$ the basis of $\{0,1\}^{|K_i|}$ for each $i \in I$, the incidence vector of a local solution $S \cap K_i$ can be written as $\sum_{j \in K_i} x_j e_j^i$. Problem GSSCP can then be formulated as follows:

$$\max \quad \sum_{i \in I} f_i \Big(\sum_{j \in K_i} x_j \boldsymbol{e}_j^i \Big)$$
(GSSCP) $s.t. \quad \sum_{j \in J_i} x_j \ge 1 \qquad i \in I$

$$x_j \in \{0, 1\} \qquad j \in J_j$$

By introducing binary variables $\chi_i \in \{0,1\}^{|K_i|}$ to represent the local solutions we can write the equivalent formulation

$$\max \sum_{i \in I} f_i(\boldsymbol{\chi}_i)$$
(GSSCP') s.t. $\sum_{j \in J_i} \chi_{ij} \ge 1$ $i \in I$
(10)

$$\chi_{ij} = x_j \qquad i \in I, j \in K_i$$

$$\chi_{ij} \in \{0, 1\} \qquad i \in I, j \in K_i$$

$$x_j \in \{0, 1\} \qquad j \in J.$$

$$(11)$$

Let us assume without loss of generality that $J_i \subseteq K_i$ (missing elements can simply be included). Without constraints (11), which ensure consistency among the local-solution variables χ and the cover variables x, GSSCP' would decompose into |I| independent nonlinear subproblems, one for each $i \in I$, having as solution space the set $X_i = \{\chi_i \in \{0,1\}^{|K_i|} : \sum_{j \in J_i} \chi_{ij} \ge 1\}$ of all feasible local solution vectors of i. To exploit this structure, we apply Dantzig–Wolfe reformulation for integer programming (see e.g. Wolsey 1998) to each set X_i .

By denoting $\mathcal{P}_i = \{B \subseteq K_i : |B \cap J_i| \ge 1\}$ the collection of all local solutions of i, we can write $X_i = \{\chi_i \in \{0,1\}^{|K_i|} : \chi_i = \chi_i^B, B \in \mathcal{P}_i\}$, where $\chi_i^B = \sum_{j \in B} e_j^i$ is the incidence vector of a local solution $B \in \mathcal{P}_i$. By introducing a binary variable w_{iB} for every $i \in I$ and $B \in \mathcal{P}_i$ ($w_{iB} = 1$ if B is the local solution is selected for i, and 0 otherwise), we obtain the Dantzig–Wolfe reformulation

$$X_{i} = \left\{ \boldsymbol{\chi}_{i} \in \{0,1\}^{|K_{i}|} : \boldsymbol{\chi}_{i} = \sum_{B \in \mathcal{P}_{i}} \boldsymbol{\chi}_{i}^{B} w_{iB}, \sum_{B \in \mathcal{P}_{i}} w_{iB} = 1, \ w_{iB} \in \{0,1\}, B \in \mathcal{P}_{i} \right\}.$$
 (12)

The reformulation leads to the equations

$$\chi_{ij} = \sum_{B \in \mathcal{P}_i} \chi^B_{ij} w_{iB} = \sum_{B \in \mathcal{P}_i : j \in B} w_{iB}$$
(13)

$$f_i(\boldsymbol{\chi}_i) = f_i\left(\sum_{B \in \mathcal{P}_i} \boldsymbol{\chi}_i^B w_{iB}\right) = \sum_{B \in \mathcal{P}_i} f_i(\boldsymbol{\chi}_i^B) w_{iB} = \sum_{B \in \mathcal{P}_i} d_{iB} w_{iB}, \tag{14}$$

where d_{iB} is the coefficient resulting from the evaluation of $f_i(\boldsymbol{\chi}_i^B)$. The second equality in (14) holds because the *w* variables are binary and their sum equals one. From (13) and (14) we finally obtain the following enumerative ILP reformulation of GSSCP:

$$\max \quad \sum_{i \in I} \sum_{B \in \mathcal{P}_i} d_{iB} w_{iB} \tag{15}$$

(GSSCP-E) s.t.
$$\sum_{B \in \mathcal{P}_i} w_{iB} = 1$$
 $i \in I$ (16)

$$\sum_{\substack{B \in \mathcal{P}_i : j \in B \\ w_{iB} \in \{0, 1\}}} w_{iB} = x_j \qquad i \in I, j \in K_i$$

$$(17)$$

$$w_{iB} \in \{0, 1\} \qquad i \in I, B \in \mathcal{P}_i$$

$$x_j \in \{0, 1\} \qquad j \in J.$$

The integrality of the w variables clearly implies the integrality of the x ones. This implication also holds in the opposite direction. Assume indeed that for some feasible solution (x, w) with x binary there are two (or more) positive variables w_{iB_1}, w_{iB_2} for some $i \in I$. Then there must exist without loss of generality a $j \in B_1 \setminus B_2$, and as $x_j \ge w_{iB_1} > 0$ one gets the contradiction $1 = x_j = \sum_{B \in \mathcal{P}_i : j \in B} w_{iB} < \sum_{B \in \mathcal{P}_i : j \in B} w_{iB} < \sum_{B \in \mathcal{P}_i : j \in B} w_{iB} = 1$.

GSSCP-E can be extended to problems where coverage is not required in two possible ways. The first approach consists in including, for each $i \in I$, a variable $w_{i\emptyset}$ representing a situation in which i is not covered (this amounts to including the empty set into \mathcal{P}_i). However, as any solution in which $w_{i\emptyset} = 1$ but $x_j = 1$ for any $j \in K_i \setminus J_i$ would violate constraints (17), we also have to relax (17) into $\sum_{B \in \mathcal{P}_i : j \in B} w_{iB} \leq x_j$ and to include the additional constraint $\sum_{B \in \mathcal{P}_i : j \notin B} w_{iB} \leq 1 - x_j$ for every groundset element $i \in I$ and every $j \in K_i \setminus J_i$. The second approach consists in introducing one variable w_{iB} for each "local solution" $B \subseteq K_i \setminus J_i$ where i is not covered, which is equivalent to removing the condition $|B \cap J_i| \ge 1$ from the definition of \mathcal{P}_i .

3.1. Application to WLAN design

In the nonlinear covering problems MEP-C and AMEP-C we have $f_i(S) = 1/|D_i \cup I(S \cap J_i)|$ and $f_i(S) = 1/|I(S \cap J_i)|$ respectively, and hence $K_i = J_i$ in both problems. In this case, we can refer to a local solution for i as a *local cover*, since it contains exactly the selected APs covering i (see Figure 4 for examples of local cover). This property also holds for AMEP, where $f_i(S) = 1/|I(S \cap J_i)|$ if $i \in I(S)$ and 0 otherwise. Note that for AMEP, and more in general for problems having no coverage requirement and satisfying $K_i = J_i$, the two aforementioned extensions of GSSCP-E coincide.

Figure 4 Example of local covers.



Note. (a) TPs and CSs around a test point i. Local covers with (b) one and (c) two installed APs, whose coverage area is shown with a circle. TPs interfering with i in AMEP are indicated in black.

For MEP, in general $K_i \neq J_i$. As direct interference only occurs between covered users, to evaluate the objective function contribution of a user *i* in MEP we need to know not only the selected local cover $B \subseteq J_i$, but also which direct interference in $D_i \setminus I(B)$ are covered. Hence direct interference is a consequence of which APs in $K_i = J_i \cup_{h \in D_i} J_h$ are installed. Due to the size of K_i , neither of the two extensions of GSSCP-E is viable. However, in the local solutions of MEP we are actually not interested in *which* APs cover the direct interferences in $D_i \setminus I(B)$, but simply in *whether* they are covered. In Section 4.3 we provide an adaptation of GSSCP-E based on this observation that allows to solve MEP to optimality for instances of reasonable size.

3.2. Valid cuts

In this section we introduce a class of cuts for GSSCP-E that enforce the global consistency of the local solutions. These cuts are valid regardless of whether or not complete coverage is required. Given two groundset elements $i, h \in I$ and a nonempty collection $T \subseteq J_i \cap J_h$ of the subsets covering both i and h, the constraint

$$\sum_{B \in \mathcal{P}_i : B \cap T \neq \emptyset} w_{iB} = \sum_{B \in \mathcal{P}_h : B \cap T \neq \emptyset} w_{hB}, \tag{18}$$

states that if the local solution chosen for i includes covering subsets from T, then the local solution chosen for h must do the same. For |T| = 1 these constraints are clearly equivalent to (17), and thus only subsets $T \subseteq J_i \cap J_h$ with $|T| \ge 2$ should be considered. In particular, the constraint obtained for $T = J_i \cap J_h$ deserves some remarks. Such a constraint states that if the local solution chosen for i covers h, then the local solution chosen for h must cover i, and can be written as

$$\sum_{B\in\mathcal{P}_i:h\in I(B)} w_{iB} = \sum_{B\in\mathcal{P}_h:i\in I(B)} w_{hB}.$$
(19)

Note that, in the hyperbolic formulations presented in Section 2, this corresponds to the symmetry $y_{ih} = y_{hi}$ of the indirect interference relation. Constraints (19) are at most |I|(|I|-1)/2, and as shown in Table 2 (Section 3.3) they significantly strengthen the continuous relaxation of GSSCP-E. In the remainder we denote by GSSCP-T the tightened formulation obtained by including (19) in GSSCP-E for every pair $i \in I, h \in N_i$ with i < h.

Other similar cuts can be derived, all based on the idea that local solutions for i and h have to be coherent on $J_i \cap J_h$ or on $K_i \cap K_h$. Note that, in a column generation context, all these cuts have an impact on the objective function of the pricing problem, as they are defined in terms of the *w*-variables (see Section 4.2). Since $w_{iB} = \prod_{j \in B} x_j \prod_{j \in K_i \setminus B} (1 - x_j)$, by performing the necessary products and linearizations it can be verified that all these cuts, and in fact also constraints (17), can be obtained by the Reformulation-Linearization Technique (see e.g. Sherali and Adams 1998).

3.3. Preprocessing

The size of GSSCP-E can be reduced by applying some simple dominance rules. Consider two elements $i, h \in I$ for which *i* dominates h ($K_i \supseteq K_h$). If the local solution for *i* is $B \in \mathcal{P}_i$, the one for *h* must be $B' = B \cap K_h$. We can then remove *h*, increasing each objective function coefficient d_{iB} by $d_{hB'}$. If coverage of *h* is required, all local solutions $B \in \mathcal{P}_i$ for which $B' = \emptyset$ must be removed. At the end of the procedure, each non-dominated user *i* represents a set Δ_i of dominated users. Note that the sets Δ_i are not uniquely defined, as they depend on the order in which the users are considered. For simplicity of notation, we assume in the remainder that $i \in \Delta_i$.

If we apply the above reduction to GSSCP-E and *then* introduce cuts (19), that is, for all pairs of non-dominated users, we obtain a formulation that is weaker than GSSCP-T. A stronger formulation is obtained by applying the reduction *directly* to GSSCP-T, lifting the cuts corresponding to dominated users (which have been removed) to the local solution variables of their dominants. This is in fact equivalent to applying the preprocessing to GSSCP-E, followed by introducing cuts (18) for every pair of non-dominated users $i \in I, h \in N_i, i < h$ and for every subset $T \in \mathcal{T}_{ih}$, where \mathcal{T}_{ih} is the collection of all subsets $T \subseteq J_i \cap J_h$ such that $T = J_a \cap J_b$ for some $a \in \Delta_i$ and $b \in \Delta_h$.

A significant further reduction can be obtained by considering the specific structure of our WLAN planning problems, in which there always exists an optimal solution S that is minimal with respect to inclusion (i.e., $I(S') \subset I(S)$ for all $S' \subset S$). Non-minimal local solutions can then be removed, as they cannot be part of a global minimal solution. Note that, while the previous reductions are valid in general, this reduction can be applied only if the above inclusion-minimality property holds.

3.4. Impact of valid cuts and preprocessing

In this section we present some results showing the impact of valid cuts and preprocessing on the enumerative formulation GSSCP-E. The computational experiments reported here, as well as throughout the paper, are performed on a test set of 2D instances generated with two signal propagation models, depicted in Figure 5. The first model is ideal isotropic propagation and gives rise to uniform 2D Euclidean instances (cf. Section 2.2), that can be easily reproduced and tested. The second model provides more realistic instances, corresponding to a situation of anisotropic propagation, and is obtained by "slicing" the disk into 12 sectors, each independently down-scaled by a random coefficient. For each propagation model we generated 40 instances, identified by a string "CSs-TPs-density/ID", with CSs \in {50,100}, TPs \in {300,400}, density \in {L, H} (approx. 5% and 10% respectively), and ID \in {1,...,5}. The instance density, which is the average of the ratio $|J_i|/|J|$, was controlled by changing the disk radius. More details on instance generation and on the above test set are given in Appendix A.3.

Figure 5 Signal propagation models used in the instance generator.



Note. (a) isotropic propagation. (b) anisotropic propagation.

Table 2 reports results for the solution of AMEP-C on a selection of instances with the basic enumerative formulation GSSCP-E, with formulation GSSCP-T, which coincides with GSSCP-E plus cuts (19), and with the formulation obtained after preprocessing, which we denote by GSSCP-P. For each formulation, we report the total number of variables ("Cols") and constraints ("Rows"), the percentage gap between the LP optimum and the IP optimum ("lp-gap"), and the time required to solve the LP relaxation with Cplex 8.1 ("time") on an Athlon XP 2600+ with 512MB RAM, with a time limit of 1 hour. A sign "*" indicates that the gap is zero. The table clearly shows the effectiveness of cuts (19) on the LP gap (formulation GSSCP-T) and the dramatic reduction in size, and consequently in solution time, given by the preprocessing rules (formulation GSSCP-P).

			GSSC	P-E			GSSCP-P						
instance	$_{\max}$	Cols	Rows	lp-gap	time	Cols	Rows	lp-gap	$_{\rm time}$	Cols	Rows	lp-gap	time
	$ J_i $			(%)	(sec)			(%)	(sec)			(%)	(sec)
AMEP-C, isotropic instances													
100-300-L/1	12	41,880	1,858	11.2	3.43	41,880	6,712	*	4.52	6,932	1,908	*	0.59
100-300-L/2	13	40,692	1,769	8.3	3.19	40,692	6,586	*	3.31	7,347	2,101	*	0.47
100-300-L/3	12	54,920	1,954	11.7	3.89	54,920	7,489	*	21.87	7,941	2,116	*	0.76
100-300-L/4	10	24,820	1,800	11.8	2.19	24,820	6,226	0.4	2.72	4,561	1,739	0.4	0.28
100-300-L/5	10	24,772	1,801	8.9	2.12	24,772	6,706	*	1.91	4,374	2,016	*	0.24
100-300-H/1	19	7,502,000	3,418		-	7,502,000	14,595		-	153,404	5,981	*	80.45
100-300-H/2	18	1,987,528	3,364	14.6	1027.54	1,987,528	14,890		-	99,580	5,487	*	30.16
100-300-H/3	24	63,021,492	3,380		_	63,021,492	15,547		_	129,641	5,122	*	61.84
100-300-H/4	23	29,695,064	3,524		_	29,695,064	15,450		_	160,558	5,432	*	212.48
100-300-H/5	21	12,933,680	3,566		-	12,933,680	15,727		_	272,358	7,196	*	296.15
	*	: gap equa	l to zer	0		lp	-gap :	gap bet	ween th	e integer o	optimun	1	

 Table 2
 Comparison among the different enumerative formulations.

Comparison among the basic enumerative formulation GSSCP-E, the formulation GSSCP-T obtained by introducing cuts (19), and the formulation GSSCP-P obtained by performing the preprocessing reduction.

An important remark concerns the time required to generate the formulations, which is not reported in Table 2. The time required to generate GSSCP-T and GSSCP-E is roughly the same, and depending on the formulation size it can be very high (e.g., 3.7 hours for instance 100-300-H/2). Generating GSSCP-T with a straightforward implementation of the preprocessing reductions would clearly require even more time, as one would need to first generate GSSCP-T and then apply the reduction rules. However, by performing a priori user aggregation and exploiting local solution minimality within the generation scheme (see Section 4.1 for details), GSSCP-P can be generated in significantly less time (24 seconds for the same instance 100-300-H/2).

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4. Solution approaches

Although GSSCP-P can be directly solved for small-to-medium-size WLAN instances, the worstcase exponential number of variables (the overall number of local solutions) makes large-size instances out of reach. A standard technique to deal with such large formulations is column generation. Column generation consists in solving alternately the LP relaxation of a *restricted formulation* (typically referred to as Restricted Master Linear Problem), in which only some variables are present, and a *pricing problem*, searching for a positive reduced cost variable to be added to the restricted formulation. If no such variable exists, the restricted formulation and the original one have the same LP optimum, though not necessarily the same integer optimum.

Unfortunately, the pricing problem for GSSCP-P turns out to be too hard to solve to allow for a standard column generation approach, in which the pricing problem has to be solved many times. Our approach to quickly find near-optimal solutions consists in solving a polynomial-size restricted formulation, where an appropriate selection of local solutions $Q_i \subseteq P_i$ is considered. Upper bounds on the optimum of GSSCP-P are then obtained with a single iteration of column generation.

4.1. Restricted formulation

Let \mathcal{P}_i be the collection of all local solutions for TP $i \in I$ in GSSCP-P, and for any subcollection $\mathcal{Q}_i \subseteq \mathcal{P}_i$ let $\theta(\mathcal{Q}_i) = \min_{B \in \mathcal{P}_i \setminus \mathcal{Q}_i} |B|$ be the minimum cardinality of an excluded local solution, with $\theta(\mathcal{P}_i) = \infty$. Given two parameters $d, t \in \mathbb{N}$, we define GSSCP-P(d, t) as the restricted version of GSSCP-P having local solution collections $\mathcal{Q}_i \subseteq \mathcal{P}_i$ defined as follows. In a first phase, each set \mathcal{Q}_i is initialized by including all the local solutions $B \in \mathcal{P}_i$ with $|B| \leq d$. If their overall number is larger than t the procedure is stopped, otherwise in a second phase a TP i with smallest \mathcal{Q}_i over all TPs for which $\mathcal{Q}_i \subset \mathcal{P}_i$ is selected, and a few (see below) local solutions $B \in \mathcal{P}_i \setminus \mathcal{Q}_i$ with $|B| = \theta(\mathcal{Q}_i)$ are generated and included into \mathcal{Q}_i . This is repeated until the limit t is reached or until $\mathcal{Q}_i = \mathcal{P}_i$ for all $i \in I$. In the latter case we say that the formulation GSSCP-P(d, t) is complete, as it coincides with GSSCP-P.

This procedure can be efficiently implemented by exploiting local solution minimality within the generation scheme. For each TP *i* we maintain a list \mathcal{L}_i of minimal local solutions that have to be expanded successively. The list is initialized by $\mathcal{L}_i = (\emptyset)$. When a local solution has to be added to \mathcal{Q}_i in any phase of the above procedure, we remove the local solution *B* at the head of the list and generate a new local solution $B_j = B \cup \{j\}$ for each AP $j \in K_i$ such that j < k for every $k \in B$ (so as to avoid duplicates). All the minimal local solution is generated this way are added both to \mathcal{P}_i and to the tail of \mathcal{L}_i . If no minimal local solution is generated in considering *B*, a new element is taken from the head of \mathcal{L}_i . If at some point \mathcal{L}_i becomes empty, then $\mathcal{Q}_i = \mathcal{P}_i$.

If there exists an optimal cover S for GSSCP such that $|S \cap K_i| < \theta(Q_i)$ for all TPs $i \in I$, then S is feasible and optimal also for GSSCP-P(d, t). Otherwise, the solution provided by GSSCP-P(d, t) is likely to be near optimal for our WLAN problems, as their objective functions penalize coverage overlaps. Note that, if the parameters d, t are too small, GSSCP-P(d, t) can be infeasible.

4.2. The Pricing problem

Let $\boldsymbol{\pi} = \{\pi_i \in \mathbb{R}, i \in I\}, \boldsymbol{\gamma} = \{\gamma_{ij}, i \in I, j \in K_i\}$, and $\boldsymbol{\lambda} = \{\lambda_{ih}^T \in \mathbb{R}, i \in I, h \in N_i, i < h, T \in \mathcal{T}_{ih}\}$ be the dual variables associated respectively with constraints (16), (17) and (18) of GSSCP-P, as defined in Section 3.4. The pricing problem for GSSCP-P decomposes into |I| problems, one for each $i \in I$. Introducing variables $y_{ih}^T (y_{ih}^T = 1 \text{ if } i \text{ and } h \text{ are both covered by some AP } j \in T$, and 0 otherwise), the pricing problem GSSCP-PP_i for a given TP *i* reads

$$\xi_i(\boldsymbol{\pi}, \boldsymbol{\gamma}, \boldsymbol{\lambda}) = \max \quad \sum_{h \in \Delta_i} f_h\left(\sum_{j \in J_h} \chi_{ij} \boldsymbol{e}_j\right) - \sum_{j \in K_i} \gamma_{ij} \chi_{ij} - \sum_{h \in N_i} \sum_{T \in \mathcal{T}_{ih}} (\lambda_{hi}^T - \lambda_{ih}^T) y_{ih}^T - \pi_i(20)$$

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$$(\text{GSSCP-PP}_i(\boldsymbol{\pi}, \boldsymbol{\gamma}, \boldsymbol{\lambda})) \quad s.t. \quad \sum_{i \in I_i} \chi_{ij} \ge 1 \qquad \qquad h \in \Delta_i$$
(21)

$$\begin{aligned}
y_{ih}^T &\geq \chi_{ij} & h \in N_i, T \in \mathcal{T}_{ih}, j \in T \\
y_{ih}^T &\leq \sum \chi_{ij} & h \in N_i, T \in \mathcal{T}_{ih}
\end{aligned}$$
(22)

$$\begin{array}{l} \substack{j \in T \\ \chi_{ij} \in \{0, 1\} \\ y_{ih}^T \in \{0, 1\} \end{array} \qquad \qquad j \in K_i \\ h \in N_i, T \in \mathcal{T}_{ih}, \end{array}$$

where $\lambda_{ih}^T = 0$ if $i \ge h$. Local solution minimality can be enforced with a straightforward modification, which is omitted for the sake of simplicity. Let $(\boldsymbol{\pi}^*, \boldsymbol{\gamma}^*, \boldsymbol{\lambda}^*)$ be an optimal dual solution of the LP relaxation of the restricted formulation GSSCP-P(d, t), with objective function value z^* . It can be verified (see e.g. Lübbecke and Desrosiers 2005) that $z^* + \sum_{i \in I} \xi_i(\boldsymbol{\pi}^*, \boldsymbol{\gamma}^*, \boldsymbol{\lambda}^*)$ is an upper bound on the LP optimum of GSSCP-P, and hence on the integer optimum of GSSCP. Clearly, we need to solve GSSCP-PP_i only for users for which $\mathcal{Q}_i \ne \mathcal{P}_i$.

As proved in Appendix A.2, GSSCP-PP_i is NP-hard even when $\Delta_i = \emptyset$, which also implies that the pricing problem for GSSCP-T is NP-hard as well. Although it inherits the nonlinearity of GSSCP, problem GSSCP-PP_i is smaller and hence easier to solve, as Δ_i typically contains few elements. As is turns out, solving GSSCP-PP_i to integer optimality with the MILP reformulation described in Section 2.1 becomes practicable, although not efficient enough to allow for a standard column generation approach, in which pricing has to be routinely performed.

4.3. Two-level enumerative ILP formulation and heuristics for MEP

As mentioned in Section 3, in MEP the objective function contribution of a user *i* depends not only on its local cover (the subset $S \cap J_i$ of installed APs that cover the user), but also on the local cover of its direct interferers, and we have $K_i = J_i \cup_{h \in D_i} J_h$. Due to the size of K_i , even the polynomial-size restricted formulation GSSCP-P(d,t) becomes too large to be directly solved. Therefore, we propose a two-level enumerative formulation, where instead of enumerating local solutions $B \subseteq K_i$ we enumerate local covers $B \subseteq J_i$ and associated subsets of direct interferers.

Given a local cover $B \subseteq J_i$, let $U_i(B) = D_i \setminus I(B)$ denote the set of direct interferers that are not covered by B. Let $\mathcal{P}_i = \{B \subseteq J_i : |B| \ge 1\}$ denote here the set of all possible local covers for i, and $\mathcal{U}_i(B) = \{U \subseteq U_i(B)\}$ be the collection of all subsets of $U_i(B)$. By introducing a binary variable w_{iBU} for every $B \in \mathcal{P}_i$ and $U \in \mathcal{U}_i(B)$ ($w_{iBU} = 1$ if B is the local cover selected for i and U are the active direct interferers not covered by B, and 0 otherwise), MEP can be formulated as follows:

$$\max \sum_{i \in I} \sum_{B \in \mathcal{P}_i} \sum_{U \in \mathcal{U}_i(B)} f_i(B, U) w_{iBU}$$

(MEP-E) s.t.
$$\sum_{B \in \mathcal{P}_i} \sum_{U \in \mathcal{U}_i(B)} w_{iBU} + w_{i\emptyset} = 1 \qquad i \in I$$
(24)

$$\sum_{B \in \mathcal{P}_i: j \in B} \sum_{U \in \mathcal{U}_i(B)} w_{iBU} = x_j \qquad i \in I, j \in J_i$$
(25)

$$\sum_{\substack{B \in \mathcal{P}_i: h \notin I(B) \\ w_{iBU} \in \{0,1\}}} \sum_{\substack{U \in \mathcal{U}_i(B): h \notin U \\ i \in I, m \in \mathcal{P}_i, u \in \mathcal{U}_i(B)}} w_{iBU} \leqslant w_{h\emptyset} \qquad i \in I, h \in D_i$$
(26)

$$\begin{array}{ll} w_{iBU} \in \{0,1\} & i \in I, B \in \mathcal{P}_i, U \in \mathcal{U}_i(B) \\ w_{i\emptyset} \in \{0,1\} & i \in I \\ x_j \in \{0,1\} & j \in J. \end{array}$$

Constraints (24) and (25) have the same meaning as (16) and (17) respectively. Constraints (26) state that if a direct interferer $h \in D_i$ is covered $(w_{h\emptyset} = 0)$, then it is not possible to select a local

cover $B \in \mathcal{P}_i$ and a set $U \in \mathcal{U}_i(B)$ for which h is neither covered by B nor included in U. Note that the inequality relation accounts for the case when none of i and h is covered.

The valid cuts (19) can be easily adapted to MEP-E and extended to consider direct interferers:

$$\sum_{B \in \mathcal{P}_i: h \in I(B)} \sum_{U \in \mathcal{U}_i(B)} w_{iBU} = \sum_{B \in \mathcal{P}_h: i \in I(B)} \sum_{U \in \mathcal{U}_h(B)} w_{hBU} \quad i \in I, h \in N_i$$
(27)

$$\sum_{B \in \mathcal{P}_i} \sum_{U \in \mathcal{U}_i(B): h \in U} w_{iBU} = \sum_{B \in \mathcal{P}_h} \sum_{U \in \mathcal{U}_h(B): i \in U} w_{hBU} \qquad i \in I, h \in D_i.$$
(28)

Since computational experiments show that constraints (28) increase the formulation size without providing noticeable benefits, only constraints (27) are used.

As with GSSCP-P(d, t), we consider a restricted polynomial-size formulation MEP-E(d, t). First, note that not all the subsets in $\mathcal{U}_i(B)$ need to be considered. Let $H_i(B) = \bigcup_{h \in U_i(B)} J_h \setminus J_i$ be the set of APs that cover some direct interferer in $U_i(B)$ but not *i* itself. Knowing which of these APs are installed allows to determine which direct interferers in $U_i(B)$ are covered. Each subset $E \subseteq H_i(B)$ of APs induces (not uniquely) a subset $U \subseteq U_i(B)$ of interferers, but typically many $U \subseteq U_i(B)$ do not correspond to any $E \subseteq H_i(B)$. Moreover, we need to consider only subsets *E* for which $E \cup B$ is minimal. The generation procedure for MEP-E(d, t) exploits minimality and increasing cardinality in a two-level enumeration scheme, enumerating on the first level the local covers $B \subseteq J_i$ and on the second level the corresponding extensions $E \subseteq H_i(B)$, which are then mapped (removing duplicates) into interfering sets. Without entering into the details, all pairs (B, E) for which $B \cup E$ is minimal and $|B \cup E| \leq d$ are considered, and (if room is available) additional pairs are generated by nondecreasing cardinality until the overall limit *t* is reached.

As shown in Section 4.4, MEP-E(d, t) allows to tackle small-size instances, but is not viable for large ones. Therefore, we now discuss which MEP variants provide optimal solutions that are good heuristic solutions for MEP. The optimum of MEP-C is clearly a lower bound to the optimum of MEP. Another lower bound is obtained by evaluating optimal solutions of AMEP with the objective function of MEP. The same holds also for AMEP-C, but the resulting bound is dominated by the optimum of MEP-C. Consider then the following problem:

$$\max \sum_{i \in I} \frac{z_i}{1 + |D_i \setminus G_i| + \sum_{h \in D_i \cap G_i} z_h + \sum_{h \in N_i \setminus D_i} y_{ih}}$$
(29)
(MEP-G) s.t. (3) - (8),

where $G_i = \{h \in I : J_h \subseteq J_i\}$ is the set of TPs covered only by the APs in J_i . It is easy to see that in MEP-G we have $K_i = J_i$, as in (29) the TPs in $D_i \setminus G_i$ are considered as fixed direct interferers, while knowing the selected local cover $B \subseteq J_i$ allows to decide which TPs in $D_i \cap G_i$ are active. It is also easy to see that MEP-G dominates MEP-C. Indeed, MEP-G is a relaxation of MEP-C, as the latter is obtained by simply setting $z_i = 1$, and when evaluated with the objective function of MEP the value of an optimal solution of MEP-G may increase, while for MEP-C it does not change. As no theoretical dominance relation exists between AMEP and MEP-G, both their optimal solutions will be considered as heuristic solutions for MEP.

4.4. Computational results

In this section we present computational results for the approaches discussed in the previous sections on the WLAN design instances presented in Section 3.4. In Table 3 we provide results for problems AMEP, AMEP-C, and MEP-C. Results for MEP are separately reported in Table 4. All the formulations are solved with Cplex 8.1 on an Athlon XP 2600+ with 512MB RAM.

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Tabl	e 3	Com	putati	onal r	esults fo	or AM	EP, A
		MILP		GS	SCP-P(d,t)	
instance	sol	gap	time	sol	gap	time	cov
	o.f.	(%)	(sec)	o.f.	(%)	(sec)	(%)
	A	MEP,	isotrop	ic insta	ances		
100-300-L/1	19.237	6.17	-	19.563	* X	0.8	95.0
100-300-L/2	20.664	7.97	-	20.969	$(0.26) \times$	1.9	95.7
100-300-L/3	19.115	8.56	_	19.457	* X	1.0	92.3
100-300-L/4	21 628	0.84 5.76	_	21 750	(0.37) X	21.8	92.0
100-400-L/1	17.498	8.63	-	17.969	* X	3.7	92.5
100-400-L/2	17.577	9.06	_	18.129	$(0.20) \times$	8.8	93.2
100-400-L/3	18.091	8.18	-	18.209	$(0.08) \times$	7.3	92.0
100-400-L/4	17.401	8.93	-	18.106	* X	4.2	94.5
100-400-L/5	18.922	6.28	-	19.001	* X	1.9	96.5 01.0
100-300-H/2	8.029	20.0 36.1	_	11.328	* ^	19.4	91.0 95.3
100-300-H/3	6.423	47.1	_	10.893	$(0.13) \times$	159.4	95.0
100-300-H/4	7.939	30.5	_	10.591	××	32.4	92.3
100-300-H/5	8.572	29.6	-	10.846	* ×	220.8	89.0
100-400-H/1	8.174	32.2	-	10.752	* ×	86.2	96.0
100-400-H/2	8.899	26.2	-	11.067	* X	46.2	92.8
100-400-H/4	4.000	25.1	_	10.005	0.15	1431.0	00.0 94.0
100-400-H/5	9.676	23.1	_	11.513	* X	58.8	94.8
avg		20.80			0.01		93.3
std		15.76			0.03		2.1
	AN	IEP-C.	isotro	pic ins	tances		
100-300-1/1	18 099	0.42		18 099	* *	0.7	
100-300-L/2	20.499	*	1505.2	20.499	* ^	0.6	
100-300-L/3	18.143	3.48	_	18.143	* X	0.9	
100-300-L/4	18.487	*	1952.1	18.487	$(0.38) \times$	4.5	
100-300-L/5	20.708	*	1461.0	20.708	* X	0.3	
100-400-L/1	16.076	6.56	-	16.368	* X	4.2	
100-400-L/2	16.257	7.04	_	10.422	* X	2.7	
100-400-L/3	15.515	10.0	_	16.217	* ^ * X	4.6	
100-400-L/5	18.029	5.23	_	18.255	* X	1.6	
100-300-H/1	7.702	33.1	-	10.092	* X	86.3	
100-300-H/2	10.358	14.5	-	10.696	* X	34.2	
100-300-H/3	9.676	17.7	-	10.476	* ×	66.7	
100-300-H/4	8.685	21.2	_	9.758	* X	219.4	
100-400-H/1	8.163	29.0	_	9.679	$(0.14) \times$	470.4	
100-400-H/2	8.317	27.8	_	10.341	* X	68.6	
100-400-H/3			_	9.018	0.12	1326.4	
100-400-H/4	8.723	20.0	-	9.793	* X	197.6	
100-400-H/5	10.040	16.5	-	10.424	$(0.06) \times$	475.8	
avg		12.83			0.01		
500		10.00			0.00		
	M	EP-C,	isotrop	oic inst	ances		
100-300-L/1	14.748	*	639.3	14.748	* ×	0.6	
100-300-L/2	15.704	*	1680.7	15.704	* X	1.3	
100-300-L/3	14.272	1.45	134.6	14.557	* X * Y	0.8	
100-300-L/5	15.303	*	422.0	15.303	* X	0.2	
100-400-L/1	13.023	1.71	-	13.087	* X	3.0	
100-400-L/2	13.137	3.21	-	13.226	* ×	1.8	
100-400-L/3	14.187	0.76	-	14.223	* ×	0.8	
100-400-L/4	12.643	4.47	-	14.758	* ×	5.6 1 F	
100-300-H/1	7.402	2.08 11.4		7.921	* ×	1.0 35.2	
100-300-H/2	7.742	7.95	_	7.973	* ×	26.9	
100-300-Н/З	7.636	8.03	_	7.864	$(0.14) \times$	161.1	
100-300-Н/4	7.535	10.1	-	7.803	* ×	107.3	
100-300-H/5	7.576	9.25	-	7.872	* ×	89.6	
100-400-H/1	6.922	16.5	_	7.662	* X	116.3	
100-400-H/3	6.517	19.0	_	7.393	* ×	406.5	
100-400-H/4	6.788	17.0	_	7.600	$(0.07) \times$	568.9	
100-400-H/5	7.584	12.5	_	8.073	* X	121.2	
avg		7.10			0.00		
std		6.45			0.00		

	INEF	-0.01	1 geon	Terric	∠⊔ inst	ances.				
		MILP		GSSCP-P(d, t)						
instance	sol	gap	time	sol	gap	time	CO			
	o.f.	(%)	(sec)	o.f.	(%)	(sec)	(%			
	AM	EP, a	nisotro	pic ins	tances					
100-300-L/1	19.614	5.44	-	19.729	$(0.34) \times$	20.6	9			
100-300-L/2	18.890	12.4	-	19.364	$(0.85) \times$	239.6	9			
100-300-L/3	18.587	8.33	-	19.026	* ×	0.8	9			
100-300-L/4	19.207	13.8	-	20.124	* ×	3.5	9			
100-300-L/5	18.035	12.6	-	18.815	$(0.38) \times$	140.8	8			
100-400-L/1	18.673	9.86	-	19.153	$(0.03) \times$	13.5	9			
100-400-L/2	18.942	10.4	-	19.652	* ×	3.2	9			
100-400-L/3	16.029	10.5	_	18.929	* X	4.5	9			
100-400-L/4	10.938	13.1	_	19 544	* X	28.5	9			
100-400-L/5	6743	14.2		10.944	(0.05) X	92.0	9			
100-300-H/2	9 251	28.6	_	11.621	0.08	510^{-2}	8			
100-300-H/3	7 769	39.4	_	11.664	0.03	303.4	9			
100-300-H/4	3.811	69.5	_	10.764	$(0.12) \times$	473.9	9			
100-300-H/5	4.710	63.1	_	10.958	(0.12) X	89.1	9			
100-400-H/1	7.459	41.4	_	11.454	0.10	545.5	9			
100-400-Н/2	6.846	47.0	_			_				
100-400-Н/З			_	10.690	0.86	1584.9	9			
100-400-H/4			_	10.983	1.61	1976.6	9			
100-400-H/5	5.000	61.2	-	11.259	0.46	1097.9	9			
avg		28.03			0.18		9			
std		20.96			0.41					
	AME	EP-C.	anisotr	opic in	stances					
100-300-L/1	17.655	3.49	_	17.706	* X	1.6				
100-300-L/2	18.166	6.57	_	18.312	$(0.23) \times$	61.7				
100-300-L/3	17.777	*	2544.8	17.777	(0.20) X * X	0.4				
100-300-L/4	18.647	4.04	_	18.710	$(0.72) \times$	63.3				
100-300-L/5	17.371	6.79	_	17.605	* X	3.4				
100-400-L/1	16.980	9.65	-	17.163	* X	12.0				
100-400-L/2	17.878	7.40	_	18.001	* ×	3.4				
100-400-L/3	16.687	6.73	_	17.012	* ×	4.4				
100-400-L/4	16.487	7.78	-	16.674	* ×	14.9				
100-400-L/5	16.251	10.5	-	16.676	$(0.48) \times$	218.4				
100-300-H/1			-			-				
100-300-H/2			-	10.621	0.10	2663.7				
100-300-Н/З	8.754	28.2	-	10.292	0.23	845.9				
100-300-H/4	8.337	31.0	-	10.012	$(0.47) \times$	1568.2				
100-300-H/5	8.504	29.7	-	9.962	* X	384.1				
100-400-H/1			_	10.303	0.36	1221.4				
100-400-H/2			_	9.910		_				
100-400-H/3			_	9 792	0.68	1528.0				
100-400-11/4			_	0.152	0.08	1020.0				
avg		11.67	_		0.09	-				
std		10.16			0.18					
	ME.	P-C, a	nisotro	pic ins	tances	1.0				
100-300-L/1	14.000	* 2.40	1814.5	14.000	× X	1.3				
100-300-L/2	13.424	2.49	106.8	13.405	(0.08) ×	0.9				
100-300-I/4	13 685	1 20	150.0	13 710	* ~	4.2				
100-300-I /5	13 689	0.98	_	13 689	* ~	3.4				
100-400-I./1	13.615	1.81		13.645	* ×	5.1				
100-400-L/2	14.298	1.92	_	14.326	* ×	2.3				
100-400-T./3	13.587	0.89	_	13.607	* ×	3.8				
100-400-L/4	13.133	2.73	_	13.204	* X	7.9				
100-400-L/5	13.333	2.58	_	13.427	$(0.21) \times$	48.6				
100-300-H/1	7.376	11.2	-	7.742	0.10	856.5				
	1		_	7.987	0.01	360.8				
100-300-H/2		16 5	_	7.855	*	226.0				
100-300-H/2 100-300-H/3	6.956	10.5			(0.00)	×				
100-300-H/2 100-300-H/3 100-300-H/4	$6.956 \\ 7.320$	10.5 13.0	-	7.768	$(0.03) \times$	599.1				
100-300-H/2 100-300-H/3 100-300-H/4 100-300-H/5	$6.956 \\ 7.320 \\ 7.564$	13.0 8.49		7.768 7.732	(0.03) × * ×	599.1 111.7				
100-300-H/2 100-300-H/3 100-300-H/4 100-300-H/5 100-400-H/1	$ \begin{array}{r} 6.956 \\ 7.320 \\ 7.564 \\ 6.970 \\ \end{array} $	10.5 13.0 8.49 18.0		7.768 7.732 7.943	$(0.03) \times \\ * \times \\ 0.08$	599.1 111.7 541.8				
100-300-H/2 100-300-H/3 100-300-H/4 100-300-H/5 100-400-H/1 100-400-H/2	6.956 7.320 7.564 6.970	10.5 13.0 8.49 18.0		7.768 7.732 7.943 7.613	$(0.03) \times \\ * \times \\ 0.08 \\ 0.35$	599.1 111.7 541.8 2178.1				
100-300-H/2 100-300-H/3 100-300-H/4 100-300-H/5 100-400-H/1 100-400-H/2 100-400-H/3	$ \begin{array}{r} 6.956 \\ 7.320 \\ 7.564 \\ 6.970 \\ \end{array} $	10.3 13.0 8.49 18.0		7.768 7.732 7.943 7.613	$(0.03) \times \\ * \times \\ 0.08 \\ 0.35$	599.1 111.7 541.8 2178.1 -				
100-300-H/2 100-300-H/3 100-300-H/4 100-300-H/5 100-400-H/1 100-400-H/2 100-400-H/3 100-400-H/4	6.956 7.320 7.564 6.970	10.5 13.0 8.49 18.0		7.768 7.732 7.943 7.613 7.608	$(0.03) \times \\ * \times \\ 0.08 \\ 0.35 \\ 0.41$	599.1 111.7 541.8 2178.1 - 1288.1				

gap : gap between the best lower and upper bounds found $\times~$: complete formulation, with nonzero LP gap in parenthesis

* : gap equal to zero- : time limit exceeded

0.12

22.06.497.08

avg std

16

Table 3 presents results for the largest instances obtained with the standard MILP linearization described in Section 2.1, in which bound tightening and model reduction have been applied, and the restricted Dantzig-Wolfe reformulation GSSCP-P(d, t), with d = 4 and t = 350000. Formulation GSSCP-P(d, t) is solved first to LP optimality (to obtain a dual optimal solution) and then to IP optimality. Next, the pricing problem is solved, and the upper bound is evaluated. For each formulation we report in the column "gap" the percentage gap between the best lower and upper bounds found within the time limit of 1 hour, and in the column "time" the overall solution time (see Table A.2 in Appendix A.3 for the time required by each step). We indicate with a sign "×" that GSSCP-P(d, t) is complete (equivalent to GSSCP-P). In this case the actual gap provided by the algorithm is zero, and we report in parenthesis the LP gap (if larger than zero). Column "cov" reports the percentage of TPs covered in an optimal solution of AMEP. Table entries "avg" and "std" show the average value and standard deviation of the gap and, for MEP-U, of the coverage percentage.

Table 3 shows a remarkable improvement of GSSCP-P(d, t) over the MILP formulation. Optimality can be proved with the MILP formulation only for some of the smallest instances for problems MEP-C and AMEP-C, and for no instance for AMEP. Whenever the MILP formulation can be solved to integer optimality within the time limit, GSSCP-P(d, t) turns out to be complete, and is solved in a few seconds. Many instances for which the MILP formulation cannot be solved to optimality can be solved with GSSCP-P(d, t). For 93 instances the formulation is complete (and hence optimal), and for two more instances the percentage gap is zero. When none of the two formulation is solved to optimality, the percentage gaps obtained with GSSCP-P(d, t) are by far smaller than those provided by the MILP formulation. This is even more evident for problem AMEP, for which the MILP formulation is much less effective due to the additional variables and constraints required to take into account user coverage.

The applicability of our approach to WLAN design is clearly demonstrated by the size of instances for which GSSCP-P(d,t) provides optimal or near-optimal solutions. Its limits can be seen on the largest anisotropic instance class 100-400-H, where for many instances the formulation GSSCP-P(d,t) is too large and Cplex stops due to an excessive memory usage. It is worth noting that for these instances most of the computing time is spent on solving the LP relaxation (see Table A.2 in Appendix A.3). The LP relaxation is solved by dual simplex, with the default solver parameter settings. We have also tried the other LP solvers in Cplex (primal simplex and barrier), but for these instances dual simplex turns out to be the best option.

Table 4 reports computational results for problem MEP, comparing the standard MILP linearization (see Section 2.1), the enumerative formulation MEP-E(d, t) with d = 4 and t = 200000, and the optimal solutions of AMEP and MEP-G. The latter are re-evaluated with the objective function of MEP. As MEP turns out to be much harder to solve than the other WLAN problems, the results are reported only for isotropic instances with 50 APs. For each formulation we report the objective function value of the best solution found within the time limit of 1 hour and the computing time. A sign " \times " indicates that MEP-E(d, t) is complete (equivalent to MEP-E).

MEP-E(d, t) clearly outperforms the MILP linearization both in terms of solution quality and computing time. Only one instance can be solved within the time limit with the MILP formulation, and the lower bound provided for the other becomes weak as instance density or size increases. On the contrary, MEP-E(d, t) can be solved to integer optimality for all but two instances: 050-300-H/5, due to the time limit, and 050-400-H/5, because the formulation is not complete.

As for the heuristic solutions, there is no empirical dominance relation between AMEP and MEP-G when they are used to approximate MEP. Each of them provides better solutions than the other for roughly half of the instances. The best solution among AMEP and MEP-G provides an average gap of 1.75% with respect to the best known solution, with standard deviation 1.10. The time needed to obtain these results, however, is far lower than that required to solve MEP-E(d, t), and these approaches remain the only choice for larger instances.

					1				
	MI	LP	MEI	P-E		AME	Ρ	MEP	-G
instance	sol	time	sol	time	1	MEP	$_{\rm time}$	MEP	time
	o.f.	(sec)	o.f.	(sec)		o.f.	(sec)	o.f.	(sec)
		М	nsta	nces					
050-300-L/1	14.245	2028.0	14.245 imes	4.0		14.047	0.0	$14.061 \dagger$	0.1
050-300-L/2	12.489	-	12.575 imes	22.7		$12.297 \dagger$	0.1	12.249	0.1
050-300-L/3	14.273	-	14.273 imes	48.9		$14.150 \dagger$	0.0	13.938	0.0
050-300-L/4	14.786	-	$14.786 \times$	2.5		$14.537 \dagger$	0.0	14.454	0.1
050-300-L/5	13.469	-	13.476 imes	54.8		$12.756 \dagger$	0.0	12.669	0.0
050-400-L/1	15.766	-	15.835 imes	13.2		15.700	0.0	$15.742 \dagger$	0.0
050-400-L/2	12.713	_	$12.941 \times$	81.5		12.303	0.2	$12.518 \dagger$	0.3
050-400-L/3	15.558	_	15.592 imes	53.9		15.281	0.0	$15.428 \dagger$	0.0
050-400-L/4	16.435	_	16.477 imes	11.5		16.177†	0.0	16.095	0.0
050-400-L/5	14.251	-	14.385 imes	31.0		$14.247 \dagger$	0.0	14.142	0.0
050-300-H/1	3.747	-	7.289 imes	835.2		$7.204 \dagger$	0.7	7.153	1.3
050-300-H/2	3.091	_	7.319 imes	1266.6		$7.167 \dagger$	0.8	$7.167 \dagger$	0.8
050-300-H/3	4.640	-	$7.606 \times$	300.1		7.476^{+}	2.5	7.362	1.4
050-300-H/4	6.591	-	$7.275 \times$	485.7		$7.177 \dagger$	1.4	7.061	2.9
050-300-H/5	4.000	_	7.743 imes	_		$7.657 \dagger$	0.2	7.579	0.3
050-400-H/1	4.709	-	8.319 imes	612.9		8.143^{+}	0.8	8.050	1.1
050-400-H/2	4.151	-	8.285 imes	2004.2		8.242	0.9	8.268†	0.8
050-400-H/3		_	8.285 imes	3196.6		8.118†	0.6	8.073	0.8
050-400-H/4	5.224	_	$8.456 \times$	874.2		8.191	0.9	8.205†	0.9
050-400-H/5		_	8.191	994.8		8.091 †	2.1	8.013	2.3

Table 4 Computational results for MEP on geometric 2D instances with 50 APs.

-: time limit exceeded ×: complete formulation †: best solution among AMEP and MEP-G

Table 5 Results for the inclusion of a partial coverage requirement in AMEP.

	AMEP			eta=92%			eta=94%			eta=96%			$\beta = 98\%$			AMEP-C	
instance	sol	$_{\rm time}$	cov	sol	$_{\rm time}$	cov	sol	$_{\rm time}$	cov	sol	$_{\rm time}$	cov	sol	time	cov	sol	time
	o.f.	(sec)	(%)	o.f.	(sec)	(%)	o.f.	(sec)	(%)	o.f.	(sec)	(%)	o.f.	(sec)	(%)	o.f.	(sec)
	isotropic instances																
100-300-L/1	19.563	0.8	95.0							19.553	0.9	96.0	19.293	2.4	98.0	18.099	0.7
100-300-L/2	20.969	1.9	95.7							20.946	4.1	96.3	20.855	3.9	98.0	20.499	0.6
100-300-L/3	19.457	1.0	92.3				19.370	3.6	94.7	19.300	1.4	96.0	18.979	2.0	98.0	18.143	0.9
100-300-L/4	19.411	21.8	92.0				19.338	23.7	94.3	19.273	28.6	96.0	19.120	2.2	98.0	18.487	4.5
100-300-L/5	21.750	0.4	93.7				21.731	0.9	94.3	21.622	1.7	96.3	21.372	16.8	98.0	20.708	0.3
100-300-H/1	10.696	101.4	91.0	10.689	148.9	92.0	10.626	882.9	95.0	10.606	209.3	96.0	10.386	2761.8	98.0	10.092	86.3
100-300-H/2	11.328	19.4	95.3							11.281	152.5	98.0	11.281	32.2	98.0	10.696	34.2
100-300-H/3	10.893	159.4	95.0							10.880	234.0	96.0	10.817	256.3	98.0	10.476	66.7
100-300-H/4	10.591	32.4	92.3				10.560	46.0	94.0	10.396	427.6	96.0	10.245	192.3	98.0	9.758	219.4
100-300-H/5	10.846	220.8	89.0	10.842	308.3	94.0	10.842	234.1	94.0	10.747	639.5	96.0	10.630	459.6	98.0	10.211	308.8

Results for the inclusion of a partial coverage requirement in AMEP, for various values of the coverage parameter β . Values of β smaller than the coverage percentage obtained with AMEP are not considered.

In Table 5 we report computational results for the variant of AMEP in which a minimal coverage percentage β is required. This formulation is obtained by including in GSSCP-P(d, t) the constraint $\sum_{i \in I} \sum_{B \in \mathcal{P}_i} c_{iB} w_{iB} \ge \beta |I|$, where $c_{iB} = |I(B) \cap \Delta_i|$ accounts for the dominated users. According to Table 5, the time required to solve this problem is always higher than that required to solve AMEP, although sometimes lower than that required to solve AMEP-C. The highest computing times are observed for values of β close to the average of the coverage given by AMEP and the complete coverage. The problem variant in which coverage is required for a specific subset of TPs can be simply obtained by removing some local covers from AMEP. As doing so does not have any significant impact on solution time, we do not report results of this case.

5. Concluding remarks

In this paper we investigated the problem of designing WLANs with maximum network efficiency. We considered 0-1 hyperbolic set covering formulations accounting for relevant protocol and planning features such as direct interference and partial coverage of the service area. To find a trade-off between model accuracy and computational tractability, we also considered variants in which the network efficiency is approximated by neglecting direct interference.

We proposed a Dantzig–Wolfe reformulation-based solution approach that is applicable to covering problems whose objective function is a sum of nonlinear contributions over the groundset elements. The approach is extremely efficient for solving the problem version in which complete coverage is required as well as the variants with approximated network efficiency. By applying instance reduction rules and strengthening the formulation with appropriate valid cuts, the approach yields optimal solutions for most of the considered instances, and very small percentage gaps for the remaining ones. The design problem MEP, that accounts for direct interference and allows for partial coverage, is the most challenging one. Since the above reformulation is not viable for MEP, due to the large number of covering sets that have to be considered in evaluating each objective function contribution, we proposed a two-level enumerative ILP reformulation that provides promising results for instances of up to moderate size. For large size instances, the approximated problem versions yield for MEP good quality heuristic solutions, and remain the best option.

The network efficiency and the solution approach presented here can be easily extended to account for the data rate experienced by the users, which in WLANs depends on the signal quality of the AP to which each user communicates (rate adaptation). The most common AP selection rule used in practice is the best-signal AP, but any selection rule based on local information can be considered. Note that the reduction based on minimality described in Section 3.3 cannot be applied to this case, as the inclusion-minimality property is no longer guaranteed.

Future work includes the investigation of extensions to multiple-frequency WLAN design and the application of our approach to other nonlinear problems that admit a GSSCP formulation.

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Complexity proofs and additional information

We provide here the complexity proofs for the PTAS of AMEP and the NP-hardness of the pricing problem $GSSCP-PP_i$, and give additional information on the instance generation procedure.

A.1. Polynomial-time approximation scheme for AMEP

Let us recall that, in a uniform 2D Euclidean instance, coverage areas are disks with uniform radius ρ . Given such an instance, we say that a cover has *bounded AP density* if the number of installed APs inside any circular region of diameter $\lambda \rho$ is bounded by a constant $C(\lambda)$.

PROPOSITION A.1. When restricted to uniform 2D Euclidean instances in which only covers with bounded AP density are considered, AMEP admits a PTAS.

Proof. We prove Proposition A.1 for any objective function f(S) that can be written as a $f(S) = \sum_{i \in I} f_i(S \cap \mathbb{B}(i, \rho))$, where each contribution only depends on the selected CSs contained in a ball of radius ρ centered in *i*. We also require the f_i to be monotonic with respect to subset inclusion, i.e., $f_i(A) \leq f_i(B)$ for all $B \subseteq A, B \neq \emptyset$. Note that we do not impose non-negativity. It is easy to verify that the objective function of AMEP satisfies the above properties. Indeed, in AMEP we have $f_i(S) = 1/|I(S \cap J_i)|$ if $i \in I(S)$ and 0 otherwise, which is clearly monotonic, and the distance between *i* and each CS $j \in J_i$ is by definition smaller than ρ .

The result is based on the *shifting lemma* (see e.g. Hochbaum and Maass 1985). Given an integer $l \ge 3$ and a uniform 2D Euclidean instance, the algorithm partitions its bounding box (the smallest rectangle containing all TPs and CSs) with a grid of mesh size $l\rho$, where ρ is the disk radius. Shifting horizontally and vertically the grid by multiples of ρ , l^2 different partitions P_k are obtained, with $k \in K = \{0, \ldots, l^2 - 1\}$ (see Figure A.1(a) for an example). For each partition P_k , the local solutions found for each $l\rho \times l\rho$ block of the partition are merged to obtain a global partition solution ALG_k . The solution ALG returned by the algorithm is the best among all the partition solutions.

The reduced instance for a single $l\rho \times l\rho$ block (see Figure A.1(b)) is obtained by removing all users and CSs that lie outside the block, as well as all the CSs contained in the border of width ρ inside the block. Due to the bounded AP density property, an optimal solution for this reduced instance can be found by enumeration in polynomial time in |I| and |J|, and exponential time in $C(\lambda)$ for some $\lambda = \lambda(l)$. Due to the border removal, the objective function value of the global solution ALG_k is simply the sum of the objective function values of the local solutions of the blocks of partition P_k .

Let OPT be an optimal solution, and OPT_k be the solution obtained by removing from OPT all the CSs that belong to some border in the partition P_k . It is easy to see that $f(OPT_k) \leq f(ALG_k) \leq f(ALG) \leq f(OPT)$ for every $k \in K$, so that we can bound f(OPT) - f(ALG) by bounding the loss $f(OPT) - f(OPT_k)$.

Given an arbitrary solution $S \subseteq J$, we evenly distribute the objective function contribution of each user over its covering APs by defining the coefficients $w(S,j) = \sum_{i \in I_j} f_i(S)/|J_i \cap S|$, so that

$$\sum_{j \in S} w(S,j) = \sum_{j \in S} \sum_{i \in I_j} \frac{f_i(S)}{|J_i \cap S|} = \sum_{i \in I(S)} \sum_{j \in J_i \cap S} \frac{f_i(S)}{|J_i \cap S|} = \sum_{i \in I(S)} f_i(S) = f(S).$$
(A.1)

We can now bound the loss obtained by removing some CSs from a given solution.

CLAIM A.1. For any arbitrary $S \subseteq J$ and $R \subseteq S$, $f(S) - f(R) \leqslant \sum_{j \in S \setminus R} w(S, j)$.



Note. (a) The bounding box of the instance and two possible partitions, with l = 4, where users are points and CSs are squares. (b) The reduced instance for one $l\rho \times l\rho$ block with the coverage disks for the remaining CSs.

Applying (A.1), proving Claim A.1 amounts to prove that $\sum_{j \in R} w(S, j) \leq \sum_{j \in R} w(R, j)$, which follows from the monotonicity of the f_i . By summing the loss over all the partitions and applying Claim A.1 we obtain

$$\sum_{k \in K} \left[f(OPT) - f(OPT_k) \right] \leqslant \sum_{k \in K} \sum_{j \in OPT \setminus OPT_k} w(OPT, j) = \sum_{j \in OPT} \sum_{k \in K: j \notin OPT_k} w(OPT, j)$$

Since each AP $j \in OPT$ is removed exactly in 4l - 4 partitions (those where it falls inside one of the 4l - 4 border squares), we have that

$$\sum_{k \in K} \left[f(OPT) - f(OPT_k) \right] \leqslant \sum_{j \in OPT} (4l - 4)w(OPT, j) = (4l - 4)f(OPT).$$

The average loss is greater than f(OPT) - f(ALG), and hence is at most $\frac{4l-4}{l^2}f(OPT)$, so that

$$f(ALG) \geqslant \left(1 - \frac{4l - 4}{l^2}\right) f(OPT) = \left(1 - \frac{2}{l}\right)^2 f(OPT). \quad \Box$$

It is worth pointing out that, while the PTAS given in Amaldi et al. (2009) uses a grid with mesh size 2ρ , the above PTAS for AMEP is based on a grid with mesh size ρ . For any given approximation guarantee (value of the parameter l), the local instances solved by enumeration in the second PTAS are thus four times smaller. Under the reasonable assumption that the bounding constant $C(\lambda)$ grows linearly in the area of the circular region considered, the required computing time is the 4-th root of that needed with the first PTAS. In spite of this improvement, the geometrical enumerative search remains unpractical for large values of the parameter l.

A.2. Complexity of the pricing problem $GSSCP-PP_i$

PROPOSITION A.2. Consider a given $TP \ i \in I$. If the objective function contribution f_i is bounded, then the pricing problem $GSSCP-PP_i(\pi, \gamma, \lambda)$ is NP-hard, even if $\Delta_i = \emptyset$ (i.e., $TP \ i$ does not dominate any other TP).

Proof. Let us first remark that if f_i is well defined (i.e., $f_i(B)$ is finite for each local solution $B \in \mathcal{P}_i$) then it is bounded, as \mathcal{P}_i is finite. In problems AMEP-C, MEP-C, and AMEP we have indeed $0 \leq f_i(B) \leq 1$ for every $B \in \mathcal{P}_i$. We prove that GSSCP-PP_i is NP-hard by polynomial-time

reduction from the minimum cardinality Set Covering Problem (SCP). Given a covering instance, defined by a groundset I and a collection $\mathcal{J} = \{I_j\}_{j \in J}$ of its subsets, in SCP one looks for a cover of minimum cardinality. Let us recall that a cover is a subset $S \subseteq J$ for which $\bigcup_{i \in S} I_i = I$.

Let us consider a covering instance (I, \mathcal{J}) for which $I \notin \mathcal{J}$, as otherwise the problem is trivial, and for which there is no $h, k \in I$ for which $\mathcal{J}_h \subseteq J_k$, as otherwise h can be safely removed. Note that this implies that there is no h for which $h \in I_j$ for every $j \in J$. Given the covering instance (I, \mathcal{J}) , we define an instance (I, \mathcal{J}) of GSSCP (which is also a covering instance) as follows. We define the groundset as $I = I \cup \{i\}$, where i is a new element with index greater than any $h \in I$. The collection \mathcal{J} is then defined by creating, for each $j \in J$, a subset $I_j \subseteq I$ with $I_j = I_j \cup \{i\}$, and introducing in the collection the subset I, which we assumed was not contained in \mathcal{J} .

Element *i* clearly does not dominate any other element, as the covering set $I \in \mathcal{J}$ contains every element but *i*. On the other side, no element $h \in \tilde{I}$ can dominate *i*, otherwise it would be covered by all the subsets in $\tilde{\mathcal{J}}$. Therefore, there is no dominance relation in the instance, so that no preprocessing can be performed. As a consequence, *i* will belong to the formulation GSSCP-P, and we can consider the pricing problem GSSCP-PP_{*i*}. It is also easy to see that $N_i = \tilde{I}, J_i = \tilde{J}$, and that for every $h \in N_i$ the collection \mathcal{T}_{ih} contains as unique element the intersection $\{J_i \cap J_h\}$.

We do not assume $K_i = J_i$, and we make no assumptions on f_i , apart from its boundedness. Let therefore l, u be lower and upper bounds on f_i , and let $\epsilon > 0$ be a small positive constant. We define the objective function coefficients of (20) (see Section 4.2 in the article) as follows. We set $\pi_i = l$, $\gamma_{ij} = (u - l + \epsilon)$ for every $j \in J_i$, and $\pi_i = 0$ for all $j \in K_i \setminus J_i$ (if any). Then we set $\lambda_{hi} = -(u - l + \epsilon)(|J_i| + 1)$ for every $h \in N_i$. Note that $\lambda_{ih} = 0$ for all $h \in N_i$, because i > hby construction. Any solution with $y_{ih} = 1$ for every $h \in N_i$ has objective function value at least $l + (u - l + \epsilon)(|J_i| + 1)|N_i| - (u - l + \epsilon)|J_i| - l$, while any solution with $y_{ih} = 0$ for at least one h has objective function at most

$$u + (u - l + \epsilon)(|J_i| + 1)(|N_i| - 1) - l = (u - l + \epsilon)(|J_i| + 1)|N_i| - (u - l + \epsilon)|J_i| - \epsilon.$$

Thus, any optimal solution $(\boldsymbol{\chi}_i^*, \boldsymbol{y}_i^*)$ of GSSCP-PP_i must be such that $y_{ih}^* = 1$ for every $h \in N_i$. As a consequence, due to constraints (21) we have that $S^* = \{j \in J_i : \chi_{ij}^* = 1\}$ is a cover for $N_i = \tilde{I}$. Let $g(S) = f_i(S) - |S|(u - l + \epsilon) - l$ denote the objective function (20) for an arbitrary cover S of \tilde{I} (neglecting the contribution of the y variables, that is identical for all covers). Exploiting upper and lower bounds on f_i we have

$$l-|S|(u-l+\epsilon)-l\leqslant u-|S|(u-l+\epsilon)-l<-(|S|-1)(u-l+\epsilon).$$

This implies that it is possible to determine whether a cover S_1 has a smaller cardinality than another cover S_2 by checking the values $g(S_1)$ and $g(S_2)$. Therefore, any optimal solution $(\boldsymbol{\chi}_i^*, \boldsymbol{y}_i^*)$ of GSSCP-PP_i provides a minimum cardinality cover S^* for \tilde{I} . \Box

A.3. Instance generator

The input parameters for the generation of an instance are the number of TPs and APs, the signal propagation model (isotropic or anisotropic), the radius, and the minimum distance among APs. The instances are generated in a (discretized) squared region of unit size as follows. First the APs are randomly placed, respecting the minimum distance. Then the TPs are randomly placed. To prevent trivial reductions, TPs are placed only in regions that are covered by at least two APs. In a last step, the APs that do not cover any TP are removed and randomly placed so as to cover at least one TP (still respecting the minimum distance among APs).

A TP is covered by an AP if the former falls within the coverage pattern of the latter. Similarly, two TPs are direct interferers if they fall within the respective patterns of each other. As in practice the patterns could be expected to be similar for close stations, the instances generated

			1	able A	•1	Test set descripti	on.				
instance	instance density max (%) $ J_i $		instance	density	max	instanco	density	max	instance	density	max
mstance			Instance	(%)	$ J_i $	Instance	(%)	$ J_i $	Instance	(%)	$ J_i $
	is	otropi	c instances				an	isotrop	oic instances		
050-300-L/1	6.02	7	050-300-H/1	11.23	11	050-300-L/1	5.20	5	050-300-H/1	10.07	10
050-300-L/2	6.66	9	050-300-H/2	11.43	14	050-300-L/2	5.22	7	050-300-H/2	10.20	11
050-300-L/3	6.29	6	050-300-H/3	11.57	12	050-300-L/3	5.06	5	050-300-H/3	9.07	10
050-300-L/4	5.91	7	050-300-H/4	11.67	13	050-300-L/4	4.85	6	050-300-H/4	9.97	10
050-300-L/5	6.37	6	050-300-H/5	9.55	11	050-300-L/5	5.49	7	050-300-H/5	10.51	13
050-400-L/1	5.43	6	050-400-H/1	10.14	10	050-400-L/1	5.11	6	050-400-H/1	11.07	13
050-400-L/2	6.49	9	050-400-H/2	9.73	11	050-400-L/2	5.19	6	050-400-H/2	10.93	10
050-400-L/3	5.57	7	050-400-H/3	9.70	11	050-400-L/3	5.23	5	050-400-H/3	10.38	13
050-400-L/4	5.16	6	050-400-H/4	10.01	10	050-400-L/4	5.37	7	050-400-H/4	9.82	13
050-400-L/5	5.78	6	050-400-H/5	10.57	12	050-400-L/5	5.45	6	050-400-H/5	10.62	12
100-300-L/1	5.19	12	100-300-H/1	10.39	19	100-300-L/1	5.35	11	100-300-H/1	10.28	22
100-300-L/2	4.90	13	100-300-H/2	10.21	18	100-300-L/2	5.21	12	100-300-H/2	10.08	18
100-300-L/3	5.51	12	100-300-H/3	10.27	24	100-300-L/3	5.13	12	100-300-H/3	10.40	23
100-300-L/4	5.00	10	100-300-H/4	10.75	23	100-300-L/4	5.37	12	100-300-H/4	9.87	17
100-300-L/5	5.00	10	100-300-H/5	10.89	21	100-300-L/5	5.49	12	100-300-H/5	9.60	18
100-400-L/1	5.80	17	100-400-H/1	10.55	22	100-400-L/1	5.34	12	100-400-H/1	9.90	21
100-400-L/2	5.68	13	100-400-H/2	10.36	18	100-400-L/2	5.04	11	100-400-H/2	10.04	20
100-400-L/3	5.54	11	100-400-H/3	11.22	21	100-400-L/3	5.35	14	100-400-H/3	10.60	22
100-400-L/4	6.19	13	100-400-H/4	10.59	21	100-400-L/4	5.82	13	100-400-H/4	10.22	19
100-400-L/5	5.24	11	100-400-H/5	10.04	20	100-400-L/5	5.84	16	100-400-H/5	10.18	22

Table A.1Test set description.

This table shows the density and the maximum coverage cardinality for each instance.

in this way have less structure, and are harder to solve. To prevent instance decomposition, each instance has to pass a connectivity test to ensure that the adjacency graph derived from the AP-TP cover relation is connected. Table A.1 reports, for each instance, the density and the maximum cardinality max $|J_i|$ of the covering sets. Table A.2 reports details on the computational results for GSSCP-P(d, t), with d = 4 and t = 350000, applied to AMEP-C, MEP-C and MEP.

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Τα	ible A.	4 (Joinpu	tation	lai res	suits ioi	GDDC	1 - 1 (u, u)							
		A	MEP				AN	MEP-C				Μ	EP-C		
instance	gap	time	t_{LP}	t_{IP}	t_{PR}	gap	time	t_{LP}	t_{IP}	t_{PR}	gap	time	t_{LP}	t_{IP}	t_{PR}
	(%)	(sec)	(sec)	(sec)	(sec)	(%)	(sec)	(sec)	(sec)	(sec)	(%)	(sec)	(sec)	(sec)	(sec)
	I												L		
						isotro	opic ins	tances							
100-300-L/1	* ×	0.8	0.6	0.2		* X	0.7	0.6	0.2		* X	0.6	0.4	0.1	
100-300-L/2	$(0.26) \times$	1.9	1.0	0.9		* ×	0.6	0.5	0.2		* ×	1.3	0.7	0.6	
100-300-L/3	* ×	1.0	0.8	0.2		* ×	0.9	0.8	0.2		* ×	0.8	0.6	0.2	
100-300-L/4	$(0.37) \times$	21.8	0.5	21.3		$(0.38) \times$	4.5	0.3	4.2		* X	0.2	0.2	0.1	
100-300-L/5	* X	0.4	0.3	0.1		* ×	0.3	0.2	0.1		* ×	0.3	0.2	0.1	
100-400-L/1	* X	3.7	2.9	0.9		* ×	4.2	3.5	0.7		* ×	3.0	2.3	0.7	
100-400-L/2	$(0.20) \times$	8.8	2.3	6.5		*×	2.7	2.4	0.3		* X	1.8	1.5	0.3	
100-400-L/3	$(0.08) \times$	7.3	1.7	5.6		*×	1.0	0.8	0.2		* ×	0.8	0.6	0.2	
100-400-L/4	* ×	4.2	3.5	0.7		*×	4.6	4.0	0.6		* ×	5.6	4.9	0.6	
100-400-L/5	*×	1.9	1.0	0.9		*×	1.6	1.4	0.2		* ×	1.5	1.3	0.2	
100-300-H/1	* X	101.4	94.8	6.5		* ×	86.3	80.4	5.9		* ×	35.2	29.4	5.8	
100-300-H/2	* ×	19.4	15.1	4.3		*×	34.2	30.2	4.0		* ×	26.9	22.8	4.0	
100-300-H/3	$(0.13) \times$	159.4	47.9	111.4		*×	66.7	61.8	4.9		$(0.14) \times$	161.1	38.6	122.5	
100-300-H/4	*×	32.4	24.9	7.5		*×	219.4	212.5	6.9		* X	107.3	100.4	6.9	
100-300-H/5	*×	220.8	207.6	13.2		*×	308.8	296.1	12.6		* ×	89.6	76.9	12.7	
100-400-H/1	* X	86.2	75.4	10.8		$(0.14) \times$	470.4	202.8	267.5		* X	116.3	106.3	10.0	
100-400-H/2	*×	46.2	39.8	6.4		*×	68.6	62.7	5.9		* ×	37.0	31.0	6.0	
100-400-H/3	0.13	1431.0	552.2	22.2	856.6	0.12	1326.4	864.3	22.5	439.6	*	406.5	152.1	21.9	232.5
100-400-H/4	*×	128.2	112.2	16.1		*×	197.6	182.4	15.2		$(0.07) \times$	568.9	211.4	357.5	
100-400-H/5	*×	58.8	53.1	5.7		$(0.06) \times$	475.8	210.0	265.8		*×	121.2	116.0	5.2	
						anisoti	ropic in	stances	3						
100-300-L/1	$(0.34) \times$	20.6	1.8	18.7		*×	1.6	1.4	0.2		* ×	1.3	1.0	0.3	
100-300-L/2	$(0.85) \times$	239.6	5.8	233.8		$(0.23) \times$	61.7	2.7	59.0		$(0.08) \times$	8.9	5.0	3.9	
100-300-L/3	* ×	0.8	0.6	0.2		* ×	0.4	0.3	0.1		* ×	0.4	0.2	0.2	
100-300-L/4	* ×	3.5	3.1	0.4		$(0.72) \times$	63.3	2.3	61.0		* ×	4.2	2.5	1.8	
100-300-L/5	$(0.38) \times$	140.8	4.1	136.7		* ×	3.4	2.8	0.6		* ×	3.4	2.8	0.6	
100-400-L/1	$(0.03) \times$	13.5	7.7	5.8		* ×	12.0	11.3	0.7		* ×	5.1	4.4	0.7	
100-400-L/2	* ×	3.2	2.7	0.5		*×	3.4	3.0	0.4		* ×	2.3	1.9	0.4	
100-400-L/3	* ×	4.5	3.6	0.9		*×	4.4	3.7	0.7		* ×	3.8	3.1	0.7	
100-400-L/4	* ×	28.5	26.7	1.9		* ×	14.9	13.4	1.5		* ×	7.9	6.4	1.5	
100-400-L/5	$(0.05) \times$	92.6	23.6	68.9		$(0.48) \times$	218.4	13.8	204.5		$(0.21) \times$	48.6	4.9	43.6	
100-300-H/1		-	1835.8	-	-		-	-	-	-	0.10	856.5	718.7	17.9	120.0
100-300-H/2	0.08	510.2	406.7	15.4	88.1	0.10	2663.7	2566.8	16.1	80.7	0.01	360.8	309.9	15.4	35.5
100-300-H/3	0.03	303.4	144.8	19.4	139.2	0.23	845.9	663.0	19.7	163.1	*	226.0	131.5	19.1	75.4
100-300-H/4	$(0.12) \times$	473.9	221.3	252.6		$(0.47) \times$	1568.2	558.8	1009.4		$(0.03) \times$	599.1	179.6	419.5	
100-300-H/5	* ×	89.1	79.8	9.4		* ×	384.1	375.0	9.1		* ×	111.7	102.8	8.9	
100-400-H/1	0.10	545.5	345.4	22.2	177.9	0.36	1221.4	1061.3	22.6	137.5	0.08	541.8	442.1	22.6	77.1
100-400-H/2		_	-	_	-		-	3541.0	29.3	-	0.35	2178.1	1760.6	28.3	389.1
100-400-H/3	0.86	1584.9	730.8	30.4	823.7		-	-	_	-		_	-	_	-
100-400-H/4	1.61	1976.6	1412.1	28.5	536.0	0.68	1528.0	1042.5	26.0	459.6	0.41	1288.1	1014.1	26.3	247.7
100-400-H/5	0.46	1097.9	732.8	26.4	338.7		_	2726.2	_	_	0.10	837.4	635.9	25.7	175.9

Table A.2 Computational results for GSSCP-P(d, t) on AMEP-C, MEP-C and MEP.

gap : gap between the best lower and upper bounds found \times : complete formulation, with nonzero LP gap in parenthesis

* : gap equal to zero
 - : time limit exceeded

Computational results for GSSCP-P(d, t) on AMEP-C, MEP-C and MEP for the largest isotropic and anisotropic instances (100 APs). For each problem, we report the percentage gap between the best lower and upper bounds found within the time limit, the overall computing time, and the time required by each step of the algorithm (LP relaxation, solution of the IP, and pricing problem). A sign "×" indicates that the formulation is complete (equivalent to GSSCP), and hence that the solution found is optimal for GSSCP independently on the upper bound. In this case no solution time is shown for the pricing, as solving the pricing is not necessary.