The Potential and Challenges of the Use of Dynamic Software in Upper Secondary Mathematics Students’ and Teachers’ Work with Integrals in GeoGebra Based Environments

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To my daughter
Acknowledgments

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Sanela Mehanovic, Malmö, May, 2011
Preface

In the spring 2008 the Swedish government announced a possibility for in-service teachers to engage in research studies. The teachers were given a possibility to work 20% of his/her working time at the school and to be engaged in research the rest of the time during 2.5 years. At that time I had been working as a mathematics and computer science teacher in an upper secondary school in the south of Sweden. I had been interested in educational development and had, for a long time, a wish to do research. For that reason I decided to apply to this program.

In the autumn 2008 my application was approved. This thesis is the result of my work within the research program.
Abstract

An introduction of computer software into mathematics classrooms makes the didactical situation more complex compared with previous learning environments (Blomhøj, 2005). A technological tool becoming a mathematic work tool in the hands of the students is a process that has turned up unexpectedly complex (Artigue, 2002). In addition to this problem, the teachers as the users of the tool go through the same process, while, at the same time, trying to integrate the tool into their teaching activities in a meaningful way. For these reasons it seems important to contribute to the research focused on the learning and teaching conditions in environments, where computer software is newly introduced, in order to better understand impacts of the introduction of different software in mathematics classrooms.

In this study the dynamic mathematical software GeoGebra was used. GeoGebra is freely available for a number of platforms and has drawn much attention during the last years with growing user communities (www.GeoGebra.org). However, being generally available just recently, there are, comparatively, few studies on the use of GeoGebra in classroom settings.

In this thesis the introduction and integration of GeoGebra was investigated in two studies with different perspectives. In the first study students’ work with GeoGebra in their mathematical activities related to the integral concept has been researched. In the second study teachers’ utilization of the didactical potential has been investigated. The results of the two studies show that GeoGebra as a mathematical tool in the hands of the students and the teachers can have a significant role in supporting their mathematical work if exploited in a, from a didactical perspective, adequate way. A learning and teaching environment based on GeoGebra bring with it a possibility to work with mathematical concepts in a broader way compared with blackboard based classrooms. GeoGebra’s facilities makes it possible to communicate mathematics in different ways and expressing mathematical concepts in different representations in a more direct way than in non dynamical environments. Communicating mathematics in different ways and expressing mathematics knowledge through different representations is of significant importance for students, not least in relation to the new curriculum for mathematics in Sweden (The Swedish National Agency for Education, 2011), where these aspects are explicitly named as aims for students to work towards.

On the other hand, the investigations also showed that the introduction and the integration of GeoGebra was a complex process for both the students and the teachers in this research. The introduction and integration of the software in the students’ mathematical activities made the didactical situation more complex and a differentiation of students’ work with the software was observed. For some students the use of the software seemingly supported their mathematical work, and at the same time for some students the result was the opposite; the use of the software was seen as a disturbing factor in their mathematical activities. When it comes to the study of teachers’ work with GeoGebra the investigations revealed that they encountered different types of obstacles that prevented them from utilizing the full didactical potential of the software in their teaching of mathematics. Three different types of obstacles were identified:

- technical - a teacher is not able to operate the software in the intended way;
- epistemological - a teacher is not aware of the didactical potential of GeoGebra and how to exploit it in a way that supports students’ learning of integrals;
- didactical - a teacher is not aware of the complexity of technology based environments or he/she is aware of this aspect, but not comfortable with his/her competence in carrying out the process of integration of the software into his/her teaching without external help and support.

Even if it is difficult to see the software detached from the context in this research, it seems that many of the obstacles perceived by the teachers in the experimental group, as well as difficulties students perceived in their work with the software, were related to the fact that they were inexperienced with the software and, consequently, lacked in knowledge in how to exploit its features in their mathematical activities. As it seems, the teachers would encounter the same obstacles every time they try to integrate a new, to them unfamiliar, software into their teaching practice. Also many of the students would experience same difficulties if they are not adequately supported in this process. Based on this, there are reasons to believe that problems with integration of GeoGebra into mathematics classrooms identified in this research would be similar in relation to integration of other dynamic mathematic software into mathematics classrooms, or even broader, other types of software as e.g. Computer Algebra Systems (CAS), as long as the integration considers the use of an unfamiliar software.
List of papers

Paper 1

Upper Secondary Students’ Work With the Dynamic Mathematical Software GeoGebra. Mehanovic, S.

Submitted to Nordic Studies in Mathematics Education.

Paper 2


Submitted to International Journal for Technology in Mathematics Education.
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Part I

Preamble
Introduction

1.1 Background

1.1.1 International perspective

The integration of information technologies into mathematics education has been an issue for decades. Nowadays, this issue seems to gain in its relevance, as, in many countries, the use of information technologies in the mathematics classroom is included, or on its the way to be included, in a school curriculum. This gives the research related to different aspects of the integration of information technology into mathematics education an importance, from an international perspective.

On another level, each teaching situation is situated in particular educational frames with their own specificities, as well as bounded by the rules of a particular national curriculum. This study is conducted in Sweden, and, for that reason, considers specificities of the Swedish educational system.

1.1.2 National perspective

After completing compulsory education, students in Sweden are entitled to enter upper secondary school. Swedish secondary school is a free, three year long education that offers 17 different national programs. All national programs entitle students to apply for further studies at the university level (The Swedish National Agency for Education, 2011).

Mathematics is one of the compulsory subjects in upper secondary school and required for all university studies, although requirements on the level of the subject study differs for different university programs.
1.1.3 Swedish curriculum for mathematics

In the current national curriculum for upper secondary school, mathematics is described as a subject that

“aims at pupils being able to analyze, critically assess and solve problems in order to be able to independently determine their views on issues important both for themselves and society...” (The Swedish National Agency for Education, 2011)

Further, the subject aims

“at pupils experiencing delight in developing their mathematical creativity, and the ability to solve problems, as well as experience something of the beauty and logic of mathematics.” (The Swedish National Agency for Education, 2011)

In the same document, among goals to work toward, one can also read that the school in its teaching of mathematics should aim to ensure that pupils:

“... develop their knowledge of how mathematics is used in information technology, as well as how information technology can be used for solving problems in order to observe mathematical relationships, and to investigate mathematical models.” (The Swedish National Agency for Education, 2011)

In December 2010, as a part of the ongoing school reforms in Sweden, the Government made a decision to revise curriculum for compulsory and secondary education.

1.1.4 The position of information technologies in the new curriculum

The new curriculum will be officially introduced in Swedish compulsory and secondary education in September 2011 (The Swedish National Agency for Education, 2011). In contrast to the current one, by transition to the new curriculum, the use of information technology will be required in all upper secondary courses in mathematics. In the new curriculum the students are, among other qualities, expected to demonstrate “...strategies for mathematical problem solving including the use of digital media and tools.” Furthermore, for the first time in Swedish curriculum, the use of information technology is explicitly required in the assessment of students’ mathematical knowledge. This requirement is included in all upper secondary courses, and, notably, at all grading levels (with differences with respect to the quality of student’s mathematical work required for respective grade):

“In the work the student handles [...] procedures and solves ordinary tasks [...] both with and without digital tools.” (The Swedish National Agency for Education, 2011)

The increasing focus on information technology in the new curriculum can be seen in the light of developments in modern societies where the information technology, for a long time, has been considered as an important component.
1.1.5 The institutional legitimacy of information technologies

Computers and digital technologies, in general, have a broad impact on societal processes, not least when it comes to the aspects related to industry and working life. In a globalized world, digital competencies of the employees has become an important factor for companies in order to be able to deal with competing actors on the market. When it comes to the availability of the information technology, Sweden has a good starting position. Penetration of computers among young people is very large. According to recent studies more than 99% of the Swedish students in the ages 16 to 20 use computers and Internet, essentially every day (POST och telestyrelsen 2007, 2009).

Looking from a societal perspective, it is not just industry or working life that are affected by the availability and broad use of new media and information technology. The availability, together with the increasing globalization, also profoundly affects ways in which students acquire information, collaborate, and communicate (Horizonreport, 2009, 2010). School as an institution has always been depending on the development of the society:

“School, as is the case every time that it faces an evolution of scientific and/or social practices, can neither stand apart from this evolution, nor ignore the new needs it generates.” (Artigue, 2005, p.232)

Still, so far, the situation in mathematics education in Sweden has been very different and a recent report from the Swedish National Agency for Education shows that 91% of the students at the secondary level never or seldom use computers in their work with mathematics at school (Swedish National Agency for Education, 2010, p.9). One possible reason for the absence of computers in mathematics classrooms in Sweden could be a lack of legitimacy from an institutional point of view and this might cause resistance among the teachers towards the use of information technology (Artigue, 1998). In Sweden, the use of computer software has not been allowed in national tests, except for the use of symbolic calculators which is currently allowed, but not required (Swedish National Agency for Education, 2011). Without explicit requirements on the use of computers in mathematics classroom, it has been, so far, up to the teacher to decide whether he/she wants to introduce a computer software and integrate it into the teaching of mathematics. Integrating computer software into the teaching is a time demanding process (Artigue, 2002, Guin & Trouche, 2004) and the teachers might prefer to prioritize working with methods that are required in course examinations, as they

“want to be convinced that the internal efficiency of the educational system will be increased by such a change in teaching means. Computer technology are thus asked to prove that they can help teachers to face the recurrent difficulties of mathematics teaching and learning...” (Artigue, 1998, p.122)

However, if the absence of the institutional legitimacy of information technologies in mathematics classroom was one of barriers, the new curriculum might change the circumstances. Considered as an integrated part of students’ mathematical knowledge, and assessed accordingly, the use of information technology has now gained in legitimacy.
1.1.6 Information technologies enter Swedish classrooms in a large scale? Are the teachers prepared?

In parallel with the ongoing reform of the mathematics curriculum more and more schools in Sweden, seemingly following developmental processes of the society as a whole, have decided to provide their students with computers, which in many cases mean small personal laptops. By providing their students with computers these schools implicitly oblige their teachers to use the provided technology in the students’ subject work. At the time this thesis is written there are ongoing projects that aim to computerize school education in 160 of 290 Swedish municipalities. The projects vary in their form and size: from large municipality projects to small projects conducted at a local school level. All these ongoing projects, together with the new curriculum, will most likely lead to an increased use of different information technologies in school education in Sweden. Mathematics teachers are about to encounter new, and for many of them unfamiliar, requirements in their teaching practice.

A process of integration of computer software into mathematics education is time demanding for both the teachers and the students (Goos et al. 2010). In addition, the process is also very difficult and complex (Artigue, 2002). Each time a new technology enters the mathematics classroom it means new possibilities and, at the same time, new challenges, for the students as well as for the teachers. Research suggests that, for the majority of teachers, simply providing technology is insufficient for the successful integration of technology into their teaching (Hohenwarter, Hohenwarter, Kreis & Lavicz, 2008). It has been put forward that adequate training and collegial support boost teachers’ willingness to integrate technology into their teaching and to develop successful technology-assisted teaching practices (Hohenwarter, Hohenwarter, Kreis & Lavicz, 2008). Presently, in Sweden, there are no clear national guidelines nor plans when it comes to the development of the teachers’ digital competencies with a particular reference to the need of the mathematics classroom. All such projects have been carried out at a local school level. So, how will the Swedish teachers handle this new situation? Will they be able to integrate new technologies in the teaching in a way that fully support the students’ mathematical work? Will they be able to exploit the potential of these technologies when it comes to students’ development of mathematical knowledge? Or will they experience difficulties in this process? And what will be the students’ reactions to the introduction of to them unfamiliar technologies into learning environments they are used to? And how can the study of students’ and teachers’ use of newly introduced technologies help us to organize a good technology based teaching of mathematics with the new curriculum as a fundament? Many questions could be asked in relation to this problematic. In this research some of them will be addressed, with the goal of contributing with the new knowledge about learning conditions in computer based environments. Hopefully, this new knowledge will be helpful for teachers while trying to organize computer based learning environments in a way that supports and promotes their students’ mathematical work, specially in relation to the new curriculum (and the requirements formulated in it) This is the overall aim of the thesis.

An updated map with municipalities with ongoing projects can be found at http://www2.diu.se/framlar/egen-dator/
1.2 Computer software in mathematics education

Through the years, different types of computer software have been developed for the use in mathematics education. Two of the widely used ones in mathematics classrooms across the levels are Computer Algebra Systems (CAS) and Dynamic Geometry Systems (DGS). While CAS have their focuses on manipulation of symbolic expressions, DGS concentrate on relationships between points, lines, circles, etc. During the last decade CAS have gradually included graphing capabilities in order to visualize mathematics, likewise DGS have begun to include elements of algebraic symbolization in order to be useful for a wider range of mathematical problems (Hohenwarter, 2004). An important characteristic of DGS is a dynamic changeability meaning that they allow the user to interact with the software in a way that makes it possible to get an immediate feedback on his/her work. Dynamic software link mathematic objects to each other so that every change of a property of one object leads to an immediate change of all corresponding mathematic objects. This means that, in an interaction with a dynamic software, mathematical concepts and/or simulations of real world problems can be explored in a broader way compared to non dynamical learning environments.

1.2.1 A computer software as a tool in mathematics classroom

An introduction of computer software into mathematics classroom makes the didactical situation more complex compared to a previous learning environment (Blomhøj, 2005). A technological tool becoming a mathematic work tool in the hands of the students is a process that has turned up unexpectedly complex (Artigue, 2002). In addition to this problem, the teachers as the users of the tool go through the same process, while, at the same time, trying to integrate the tool in their teaching activities in a meaningful way. For these reasons it seems important to contribute to the research focused on the learning and teaching conditions in environments where computer software is newly introduced. That type of research would lead to a better understanding of the impacts of the introduction of different software on the mathematics classroom and, consequently, on the teaching and learning of mathematics.

1.3 This study

1.3.1 The software

In this study a dynamic mathematical software GeoGebra was used. GeoGebra is freely available for a number of platforms and has drawn much attention during the last years (www.GeoGebra.org). Due to several high quality features of the program package, the GeoGebra community is growing in Sweden and in other countries, with the establishment of institutes to promote the use of the software. However, being generally available just recently, there are, comparatively, few studies on the use of GeoGebra in classroom settings.
1.3.2 The mathematical concept

As a dynamic software GeoGebra allows its users to work interactively with a wide range of mathematical concepts (Preiner, 2008). Some of the concepts are difficult for students to understand, as is the case with integrals (Orton, 1983). Integrals are an important part of calculus and their applications can be found in different scientific fields; integrals are used as a mathematical tool for explaining and analyzing different physical phenomena, such as volume, mass, work. They are also used in engineering, economy and statistics. Studies of students’ work with integrals show that this concept is difficult for students to understand when presented in the paper/pencil based teaching (Tall, 1991). On the other hand, concepts like integrals lend themselves to a computer representation as they have visual aspects that can be displayed on a computer screen along with the other representations such as algorithmic, symbolic, numerical, or natural language representation. There are studies that claim that, with the use of mathematical software as e.g. GeoGebra, the notion of integrals can be more easily adopted by students (Herceg & Herceg, 2009).

1.3.3 Aim of the study

This research was conducted at the upper secondary level in Sweden in order to investigate some aspects of the introduction of dynamic software into mathematics classroom. Considering that each activity in mathematics classrooms can be seen and explained from two perspectives, that of a student and that of a teacher, the goal of this research was to investigate the introduction of a software into mathematical activities from both perspectives. For that reason, there were two aims in the thesis, one for each perspective.

The first aim was to investigate students’ work with integral tasks in GeoGebra based learning environments. Based on results of the previous research within the field of technology enhanced mathematics education (Blomhøj, 2005; Guin & Trouche, 2002; Preiner, 2008) the hypothesis was that it would be possible to differentiate students’ use of GeoGebra in their mathematical work, and also to relate it to the findings of Guin and Trouche’s (2002) study in which they identified five students’ working methods in a technology based environment.

The second aim was partly to investigate how teachers in the experimental group exploit the didactical potential of GeoGebra in their presentation of integrals and planning of integral tasks for students, and partly to identify possible obstacles\(^2\) that might prevent the teachers from utilizing the full didactical potential of GeoGebra in this process.

Seeing the two aims in the study as the two sides of the same coin and then, putting them together, will hopefully give a better understand the potential and the limitations of GeoGebra in environments in which the software is newly introduced. This knowledge would hopefully help teachers to organize GeoGebra based environments and integrate the soft-

\(^2\) With obstacles I mean:

- difficulties that teachers encounter that could hinder them to handle the software in a, from a didactical perspective, fruitful way,

or

- difficulties that teachers perceive to hinder them in handling the software in a way that they intended to.
ware into the teaching of mathematics in a way that fruitfully supports their students' mathematical activities.

1.4 The structure of the thesis

This licentiate thesis is built up of Part I and two papers in Part II.

Part I includes 5 chapters. In chapter 1 the background to this research and the aims are described. Chapter 2 gives the theoretical perspectives and explains notions used in the thesis. This chapter also provides more information about the software and the mathematical content treated in the study. The methodology, with respect to the two aims in the thesis, is presented in chapter 3. In chapter 4 the two papers are summarized. Finally, the results, conclusions and implications of the papers are discussed in chapter 5.

In Part II the papers are presented in full versions.

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3To fit the format of this thesis the layout of the papers may have been changed compared with the submitted versions.
The didactical interplay between the software, the mathematical concept, and the user

Previous studies specialized in mathematics education show that an introduction of a new technology into mathematics education is a complex and time demanding process (Artigue, 2002; Guin & Trouche, 2002; Trouche, 2004). The process makes didactical situations between the teacher and the students more complex compared with their previous environment. It also increases differentiation in mathematics classrooms (Blomhøj, 2005) meaning that different students, even if involved in the same mathematical activity, will use the same technological tool in different ways.

The research described in this thesis investigates students’ and teachers’ use of the dynamic system GeoGebra in their mathematical activities related to the integral concept. In order to understand how the students and the teachers in the experimental groups use GeoGebra in their mathematical work, and how the students’ use of the software can be characterized, different approaches can be taken. The instrumental approach explains these processes from a perspective of the interaction between the tool on one hand and the student or the teacher as the user of the tool on the other hand. Through this interaction a tool transforms to a mathematical instrument in the hand of the user during his/her mathematical activities.

2.1 Instrumental approach

The instrumental approach, originally introduced by Verillon and Rabardel (1995), explains the interaction between a tool and a learner as a bidirectional relation in a computerized learning environment. A learning environment itself is not neutral, but rather insists on learners’ initiative and activity. An essential starting point in the instrumental approach is the distinction between an artifact - either material or symbolic - as a given object and an instrument as a psychological construct:
The didactical interplay between the software, the mathematical concept, and the user

“(The instrument) does not exist in itself... it becomes an instrument when the subject has been able to appropriate it for himself... and has integrated it with his activity.”
(Verillon & Rabardel, 1995, p.84-85)

The instrumental approach is based on the idea of an instrument as a complex entity that combines a material or symbolic object with structures that organize the subject’s actions. While the artifact is the part of the instrument which is external to the subject, the internal part is constituted by the schemes of use. The internal part results from a subject’s personal way to use the artefact and also from appropriation of social pre-existing schemes. The process of constructing schemes, the instrumental geneses, is a two sided process. On the one side, the instrumental genesis is oriented toward the use of the artifact: the instrumentalization. On the other side, the instrumental genesis is oriented toward the task to be achieved: the instrumentation (Goos et al., 2010, p.313).

In a dynamic environment, to drag a slider can be seen as an instrument to identify geometrical properties of a mathematical object. The student must learn how to create and drag a slider, which is the process of instrumentalization. The student also must learn why to create and drag a slider, which is the process of instrumentation and this process is related to student’s conceptualization of geometrical properties. Another example is to use a drag mode1 in a dynamic environment. Classroom observations have revealed that it takes several weeks before students start to drag objects as e.g. points on their own with a mathematical intention and not only to see objects moving, and this gives an evidence of instrumental genesis and the need for the teacher to support it (Goos et al., 2010, p.313).

2.1.1 The complexity of a process of instrumental genesis

Instrumental genesis is a time demanding process, linked to the artifact’s potentialities and constraints but also to the student’s knowledge and former method of working (Trouche, 2004). This process is carried out both through student’s individual activity and through teacher organized activity (Artigue, 2002). This process has turned up unexpectedly complex (Artigue, 2002). Furthermore, recent research in mathematics education shows a differentiation of students’ behavior within the same environment: the more complex the environment, the larger the differentiation of instrumental genesis of students (Guin & Trouche, 2002).

2.1.2 A differentiation of the instrumental genesis of the students

A variety of students’ work methods in their work in symbolic calculator environments was investigated by Guin and Trouche (2002). Their investigations of students’ work in symbolic calculator environments revealed a differentiation of individual instrumental geneses of students. The investigations was generated by taking into account following elements on a lasting term:

- information sources used, which can be the previously built references,

1A drag mode is a function in a dynamic software that makes it possible to move a mathematical object in a construction by dragging it without changing the construction itself.
resort to paper/pencil, the calculator, or to the neighborhood (in particular, during practicals);

- time of tool utilization (both the global time of the calculator’s use and time spent performing each instrumented gesture);

- relationship of students to mathematics - in particular the proof method (proof can proceed from analogy, demonstration, accumulation of corroborating clues);

- their relationship to knowledge: metaknowledge that is to say knowledge which students have built on their own knowledge.

Considering the elements mentioned above Guin and Trouche propose a model with 5 typologies distinguished:

- A theoretical work method, characterized by the use of mathematical references as a systematic resource. Reasoning is essentially based on analogy and over excessive interpretation of facts with average verifying procedures of machine results.

- A rational work method, characterized by a reduced use of calculator, mainly working within traditional (paper/pencil) environment. The specificity of this behaviour is a strong student’s command process with an important role played by inferences in reasoning.

- A random work method, characterized by similar student difficulties whether in the calculator environment or in the traditional paper/pencil environment. The tasks are carried out by means of cut and paste strategies from previously memorized solutions or hastily generalized observations. Therefore, the rather weak students’ command process is revealed by trial and error procedures with very limited references to understanding tools and without verifying strategies of machine results.

- A mechanical work method, characterized by information sources more or less restricted to the calculator investigations and simple manipulations. However, reasoning is based on the accumulation of consistent machine results. Student’s command process remains rather weak, with an avoidance of mathematical references.

- A resourceful work method, characterized by an exploration of all available information sources (calculator, but also paper/pencil work and some theoretical references). Reasoning is based on the comparison and the confrontation of this information with an average degree of student’s command process. This is revealed by an investigation of a wide range of imaginative solution strategies: sometimes observations prevail, other times theoretical results predominate.

(Guin & Trouche, 2002, p. 206-207)
2.2 Dynamic software GeoGebra based teaching and learning

Computer algebra systems, such as Derive, Mathematica, Maple or MuPAD, and dynamic geometry systems, as Geometer’s Sketchpad, Cabri Geometry or GeoGebra are technological tools that have through the years been developed for teaching and learning mathematics. Studies specialized in mathematics education suggest that the use of these systems in mathematics classrooms can encourage discovery and experimentation in classrooms and that their visualization features can be effectively employed in teaching to generate conjectures (Hohenwarter, Hohenwarter, Kreis & Lavicza, 2008). Among the above, one particular system has, lately, drawn much attention - the multi-platform, open source software GeoGebra (Hohenwarter & Preiner 2007).

2.2.1 GeoGebra

GeoGebra tries to combine the ease-of-use of dynamic geometry software with the versatile possibilities of computer algebra systems. The software aims to integrate geometry, algebra, and calculus, which other packages treated separately, into a single easy-to-use package for learning and teaching mathematics from elementary through university level (Hohenwarter, Hohenwarter, Kreis & Lavicza, 2008). As the CAS-facilities of the software are rather limited, the system is considered as a dynamic software. However, according to the program home page the new release planned for August 2011 will include more symbolic features of computer algebra systems. GeoGebra has been translated to more than 40 languages, and gathers a rapidly growing worldwide user community (www.GeoGebra.org). As GeoGebra is an open-source software, the use of the software is not limited to schools or universities with site licenses, but the users can download and install GeoGebra on their private computers.

2.2.2 The affordances of GeoGebra based learning environments

GeoGebra provides the functionality of dynamic software and the user can work with points, vectors, segments, lines etc. The software also allows some capabilities of computer algebra systems in that equations and coordinates can be entered directly. When it comes to functions they can be defined algebraically and then changed dynamically. The later capabilities are characteristic of GeoGebra and the default screen provides two windows (see Figure 2.1). Each object in the left window (algebra window) corresponds to an object in the right window (geometry window) and vice versa (Hohenwarter & Jones, 2007).
2.2 Dynamic software GeoGebra based teaching and learning

Figure 2.1: GeoGebra screen showing the algebra and the geometry window with an example of Riemann's sum.

GeoGebra provides the facility to move between the algebra window and the geometry window, and it is possible for the user, on the one hand, to investigate the parameters of the equation of a curve by dragging the curve with the mouse and observing the equation change, or, on the other hand, to change the equation of the curve directly and observe the way the objects in the geometry window change (Hohenwarter & Jones, 2007).

Initially, GeoGebra was developed with the goal of letting students explore and discover mathematical concepts on their own, but GeoGebra based mathematics classroom can be organized also by taking a teacher-centered approach where the teacher plays a main role and guides the students through their mathematical work (Preiner, 2008). According to Preiner (2008), the developers of GeoGebra try to design the user interface of the software in a straightforward and clear way that supports the model of cognitive processes for learning with multimedia and reduces the cognitive load for the benefit of more successful learning. Preiner states that by visualizing and exploring mathematical concepts in multimedia environments students' understanding can be fostered in a new way compared with non-dynamical environments.

Looking from a teacher's point of view GeoGebra supports creation of static instructional materials, such as worksheets, tests, or presentations, which can be printed out and handed to the students (Preiner, 2008, p.43). In addition the software allows creation of web-based interactive instructional materials, so called dynamic worksheets and these interactive materials can be used both on local computers or via the Internet. The teachers can create interactive online learning environments, and to share their materials on the Internet (Hohenwarter, Hohenwarter, Kreis & Lavicza, 2008). The possibility of constructing interactive worksheets means that teachers can make their own GeoGebra constructions in advance and then give them to students to work with without requiring
students to construct their own GeoGebra files. By working in this way students can interact with tasks and explore mathematical concepts and problem solving strategies in GeoGebra without having a strong knowledge about how to use the software (Preiner, 2008). In that way, they can explore mathematical concepts or simulate real world problems in order to model, explore and try out different problem solving strategies (Preiner, 2008).

2.3 GeoGebra and integrals

2.3.1 Previous investigations of GeoGebra based learning environments

In a GeoGebra based mathematics classroom different types of mathematical concept can be explored. Previous research reports from the integration of the software in the teaching of a wide range of concepts, such as calculus and geometry (Fahlberg-Stojanovska & Stojanovski, 2009; Hohenwarter, Hohenwarter, Kreis, Lavicza, 2008; Herceg & Herceg, 2009; Little, 2008), statistics (Hewson, 2009), and economics (Siebel, 2011).

However, looking particularly on the integral concept, although there are examples working material on the software homepage, in which the concept is treated, there are few scientific reports from investigations of the GeoGebra based learning or teaching of this important concept in calculus. Herceg and Herceg (2009) reported on a study that investigated university students’ theoretical, visual and practical knowledge of the definite integral. In the quantitative study conducted in two university classes, the computer software GeoGebra and Mathematica were used for visualization of the concept. The results of the study revealed that the use of the software had a positive impact on the students’ knowledge and skills. In another study Attorps et al. (2011) reported on a teaching experiment regarding the definite integral concept in university mathematics teaching. Also this study was quantitative and conducted at an university. The collected data consisted of engineering students’ answers to pre and post tests. The experiment revealed that by using the GeoGebra it is possible to create learning opportunities of the definite integral concept that support the students’ learning. It was also observed that the use of the GeoGebra software during the lecture increased the students’ possibilities to experience the concept of the definite integral as a real number.

2.3.2 Positioning this research into the scientific field

The few studies that investigate GeoGebra based learning and teaching of the integral concept, focus on the students and their learning (Herceg & Herceg, 2009; Attorps, Björk & Radic, 2011). To obtain a more complete view the aspects of the teacher must be included. The situation is described with Figure 2.2.
Although this research does not focus on the interplay between the students and the teachers, it does consider GeoGebra based environments from both the perspective of the learning- as well as the perspective of the teaching of integrals. Rather than focusing on one of the perspectives the research investigates the problematique of GeoGebra based learning and teaching of integrals as a whole, and brings some of its aspects to the light.

### 2.3.3 Upper secondary students’ understanding of integrals

In the upper secondary education, the integral is usually defined in the following way:

Let $f(x)$ be a continuous function in a closed interval $[a, b]$ divided in subintervals with equal length $\triangle x$. Then, for $n$ subintervals we have the area $S \sim \sum_{i=1}^{n} f(x_i) \triangle x_i$. If we let $n \to \infty$ then $\triangle x_i \to 0$ and it can be shown that $\sum_{i=1}^{n} f(x_i) \triangle x_i$ approaches a limit called the integral of $f$ from $a$ to $b$, which is denoted $\int_{a}^{b} f(x)dx$.

This definition, based on a Riemann sum, is difficult for students to understand. Students’ difficulties with integrals are not a new behavior in the mathematics classroom. Integrals have, for a long time been difficult for students to deal with (Orton, 1983; Tall, 1991;
Machin & Rivero, 2003). In the early eighties Orton observed that students had difficulties while solving tasks related to the understanding of integration as limit of sums (1983). In this study students were able to apply, with some facility, the basic techniques of integration, but further probing indicated fundamental misunderstanding about the underlying concepts. Students interpreted the integral as a procedure which transforms an input into some output. Also, in the same study, it was found that students’ technical ability could be quite strong, despite showing minimal conceptual understanding. Apart from showing strong procedural skills the students were found to demonstrate a great reluctance to using geometric interpretations to complete an algebraic process, and when possible, were more inclined to move to an algebraic context (Orton, 1983). Another study from Orton (1980) revealed that students had problems with the integral if \( f(x) \) is negative or \( b \) is less than \( a \) (1980). More recent studies specialized in mathematics education show that this concept is still difficult for students’ to grasp; they are not able to write meaningfully about the definition of a definite integral nor can they without difficulties interpret problems calculating areas and definite integrals in wider contexts (Rasslan and Tall, 2002). The students also tend to identify the definite integral with an area (Attorps et al. 2011).

The integral concept is one of the concepts that have clear visual aspects and can be presented in different representations. Working with different representations of the integral can be important for students’ learning of this mathematical concept.

### 2.3.4 The importance of semiotic representations for students’ understanding of integrals

According to Duval (2006), in the field of mathematics, in contrast to phenomena of physics, astronomy, biology, etc., objects are not accessible by perception or by instruments such as microscopes, telescopes, and other measurement apparatus. The only way to access objects and deal with their properties is by using signs and semiotic representations. Further, he claims that

> “mathematical activity needs to have different semiotic representation systems that can be freely used according to the task to be carried out... Some processes are easier in one semiotic system than in another one... But in many cases it is not only one representation system that is implicitly or explicitly used but at least two.” (Duval, 2006, p.108)

Duval defines a register of representation as a representation system that permit a transformation of representations. (E.g. in a case of natural language there can be a big gap between this and other semiotic representations (Duval, 2006, p. 111)). According to him, involving simultaneous mobilization of at least two registers of representation, or the possibility of changing at any moment from one register to another, in a mathematical activity allows students to reach a conceptual comprehension (Duval, 2006, p.126).

> “Even if a single representation register is enough from a mathematical point of view, from a cognitive point of view mathematical activity involves the simultaneous mobilization of at least two registers of representation, or the possibility of changing at any moment from one register to another. In other words, conceptual comprehension in
2.3 GeoGebra and integrals

"mathematics involves a two-register synergy, and sometimes a three-register synergy."
(Duval, 2006, p.126)

Furthermore, not involving the simultaneous mobilization of at least two registers of representation in a mathematical activity

“is the reason why what is mathematically simple and occurs at the initial stage of mathematical knowledge construction can be cognitively complex and requires a development of a specific awareness about this coordination of registers.” (Duval, 2006, p.126-127)

Among other things, dynamic software offer learners the possibility to simultaneously access different semiotic representation registers of the integral and work with the concept in an investigative way. In the next section the didactic potential of GeoGebra with respect to integrals is described.

2.3.5 The didactical potential of GeoGebra in supporting upper secondary students’ mathematical work with integrals

The integral is usually introduced to upper secondary students in terms of a Riemann sum, a definition many of them have difficult to understand. Here, by the use of dynamic systems as GeoGebra, they can visualize and explore integrals in a new way compared with non dynamic learning environments (Herceg & Herceg, 2009; Tall, 1991). As an example, by using the in advance constructed file shown in Figure 1, the teacher can let the students work with integrals in an investigative way by exploring different aspects of its upper and lower sums. The students can easily change the number of rectangles for the respective sum by dragging the slider \( n \) in the geometry window, or change the length of the interval by moving points that defines it along the \( x \)-axis. While making changes in the geometry window they can follow changes in algebraic expressions in the algebra window and vice versa. This can be seen from a perspective of an immediate access to different representation registers, as formulated by Duval (2006) and, thus has a potential to support students’ mathematical understanding of the concept. Also, getting an immediate feedback on their work in GeoGebra students can learn integrals (through a discovery learning) by experimenting with a domain, and inferring rules from the results of these experiments (Hohenwarter & Preiner, 2007). By interacting with the file shown in Figure 1 students also design their own experiments in the domain and infer the rules of the domain themselves. That can help them to understand the mathematical concept in a better way: constructive activities are expected to help students to understand the domain at a higher level than when the necessary information is just presented by a teacher or an expository learning environment (van Joolingen, 1999, p. 386).
This research was conducted at the upper secondary level in Sweden in order to investigate some aspects of the introduction of dynamic software into mathematics classroom. Considering that each mathematical activity can be seen and explained from two perspectives, that of a student and that of a teacher, it seemed important that an investigation of the introduction of the software involves both of these. For this reason there were two aims in this study including the both perspectives. By putting together the two aims the goal was to get a better understanding of the potential and limitations of GeoGebra in environments in which the software has been newly introduced. In this study the decision was made to separate investigations of the students’ and the teachers’ work with GeoGebra in two smaller studies. With this structure, the respective study would fully focus on one perspective (that of a student or that of a teacher) at a time by taking into account specificities of each perspective.

As both of the studies considered a mathematical work with GeoGebra it seemed appropriate to base the investigations on a method that focuses on the user’s (the student’s as well as the teacher’s) interaction with the software. This means to analyze the interaction between this software as a tool on the one side and the students and the teachers as the users of the tool on the other side. For this reason an instrumental approach approach has been taken in both studies.

### 3.1 The two studies

The investigations started with a study that focused on students’ work with GeoGebra. After this study had been conducted a second study followed, in which the teachers’ utilization of the didactical potential of the software was investigated. Below follows a description of the two studies, including the aims, the research questions, and the methods used in respective study.
3.2 The first study

3.2.1 Aim, hypothesis and research questions

The first study was conducted at the upper secondary level in Sweden. The aim of this study was to investigate students’ work with integral tasks in GeoGebra based learning environments. Based on results of previous research within the field of technology enhanced mathematics education (Blomhøj, 2005; Guin & Trouche, 2002; Preiner, 2008) the hypothesis was that it would be possible to differentiate students’ use of GeoGebra in their mathematical work. The research questions that have guided this investigation were:

- How do the students in the experimental classes work with GeoGebra while solving integral tasks?
- How can the students’ work with GeoGebra be related to their perceptions of the role of GeoGebra in the process of learning of integrals?

3.2.2 Methods for data collection

Two classes coming from two different upper secondary schools participated in the first study. During the classroom experiments the students were working with a material, specially constructed for the purpose of the study (see Appendix A). The material, consisting of integral tasks, was constructed in accordance to existing course literature and national course curricula. The tasks were a combination of paper/pencil- and GeoGebra based problems.

During classroom experiments, the students’ work with integral tasks have been observed through regular classroom visits.

After classroom experiments individual interviews with students were conducted. In the interviews, students were asked to answer questions related to their perception of the role of GeoGebra in their work with integral tasks. They were also asked to reproduce one of their solutions to GeoGebra based integral tasks while explaining how they were thinking.

The collected data consisted of the students’ interviews response, information gathered from observation of the students’ classroom work with GeoGebra, and the students’ solutions to the integral tasks, both their paper/pencil work and GeoGebra files. Related to the collected GeoGebra files, construction protocols of students’ solutions of integral tasks had an important role.

3.2.3 A construction protocol

A construction protocol is a feature of GeoGebra that makes it possible to display the series of steps taken while solving a task with the software. By entering the Construction Protocol window, each construction in GeoGebra can be displayed step by step. In this study, construction protocols were used to give more information about steps students

\[1\] With ‘students’ perception’ I mean what students express to be the role of GeoGebra in their learning of integrals, e.g. affordances that the software contributes (or expects to contribute) with in their learning process and/or constraints and limitations of the software in the same process.
took in solving GeoGebra related integral tasks. Here, it is important to note that objects constructed in GeoGebra and then deleted from the construction are not shown as steps in the protocol. This means that it is possible for the student to define two functions, manipulate them in GeoGebra and then to change his solution by deleting one of the functions, and the Construction protocol will still show a step-by-step process only including steps with the remaining function. Although in an incomplete way, the construction protocol gives access to part of students’ mathematical thinking.

3.2.4 Methods for data analysis

After collecting students’ solutions to the integral tasks their paper/pencil work and GeoGebra files were pre analyzed. The pre analysis started with the look at Construction protocols of students’ tasks in order to reconstruct and understand their GeoGebra based solutions. Based on the pre analysis of the work collected from students, on information gathered through observations of their classroom work, and also on the information from the teacher about students’ individual work with mathematics, a group of students was selected for complementary interviews.

Research question 1: In order to answer the first research question, the students’ use of GeoGebra was analyzed by taking an instrumental approach. Guin and Trouche’s differentiation of the instrumental genesis of the students has been be used as a starting point in the analysis of their work with the integral tasks in the GeoGebra based environment in the study.

Research question 2: Turning to the second research question, students’ perception of GeoGebra in relation to their work with integrals was investigated through an analysis of their interview response.

3.3 The second study

3.3.1 Aim and research questions

The second study was conducted with two aims. The first aim was to investigate how teachers exploit the didactical potential of GeoGebra in their presentation of integrals and planning of integral tasks for students. The second aim was to identify possible obstacles that might prevent teachers from utilizing the full didactical potential of GeoGebra in this process.

The study has been guided by the following research questions:

- What do teachers in the experimental group express as the didactical potential of GeoGebra in the teaching of integrals?

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2With obstacles I mean:
- difficulties that teachers encounter that could hinder them to handle the software in a, from a didactical perspective, fruitful way, or
- difficulties that teachers perceive to hinder them in handling the software in a way that they intended to.
- To what degree and in which ways do they utilize the theoretically stated potential of the software in their presentation of integrals and preparing of integral tasks for students?

- What types of obstacles prevent the teachers in the experimental group from utilizing the full didactical potential of GeoGebra?

### 3.3.2 Methods for data collection

In order to answer the research questions a decision was made to ask teachers of varied experience of the use of GeoGebra in school mathematics to make an introduction of integrals in GeoGebra.

It was planned to relate teachers’ introductions of integrals to their utilization of GeoGebra’s didactical potential in supporting an investigative way of working, immediate feedback and access to different representation registers. A method was needed that would focus on teachers’ mathematical activities in the GeoGebra screen while they were explaining the integral concept with the use of GeoGebra. For that reason the teachers were asked to make their GeoGebra based introductions of integrals outside the classroom and at the same time record the introduction with a screen recording program. This arrangement allows the teachers to fully concentrate on their presentation of integrals, without any disruptions from students which would be the case in a normal classroom situation.

Except for recorded GeoGebra presentations, the teachers were asked to prepare integral tasks for students related to their presentations. The prepared tasks were used as complementary data in the investigations of teachers’ utilization of the didactical potential of GeoGebra.

After collecting teachers presentations and integrals tasks that they prepared for students, individual interviews were conducted with all participating teachers. In the interviews the teachers were asked to elaborate on their perception of a didactical potential of GeoGebra in teaching of integrals. They were also asked to elaborate on obstacles they encountered while using GeoGebra that hindered them in handling the software in the way they intended to.

Data collected in the second study consisted of the teachers’ interview response, integral tasks they planned for students and their screen recorded presentations of integrals in GeoGebra.

### 3.3.3 Screen recorded presentations

A screen recording software captures every activity on the computer screen including the entire desktop, windows, menus, video with sound, and sound from external sources e.g. the user’s voice. By the use of screen recording software teachers can visualize a mathematical concept for students and at the same time record both their activities on the screen and their explanations. Recorded presentations can be distributed to students or uploaded on the Internet. This opens for new ways of communication mathematics to students, as they are able to access presentations with teachers’ explanations even outside the school.
e.g. in case of absence. Also, this way of presenting mathematics is very different from
the statical way used in course books. From a research perspective the recordings can
give information about what aspects of GeoGebra have been utilized in the teachers intro-
duction of the integral concept.

In this study the teachers were asked to produce their introduction of integrals in Geo-
Gebra while recording it with an optional screen recorder program (Screen-o.matic, Cam-
Studio, ScreenToaster e.t.c.).

### 3.3.4 Methods for data analysis

Research question 1: In order to investigate how teachers in the experimental group uti-
lize the didactical potential of GeoGebra, each teacher’s recorded presentation of integrals
and integral tasks that he/she prepared for students have been analyzed. In recorded pre-
sentations I was looking for: indications of a teacher relating different representations
registers to each other, and teacher’s mathematical activities that would indicate that he or
she exploits a possibility of the immediate feedback and/or explorative way of working.
In integral tasks I was looking for the same elements as described above, but here investi-
gating if the teacher plans to provide students with tasks that allow them to explore and
investigate the integral concept by interacting with GeoGebra and in which way.

Research question 2: In order to investigate what teachers in the experimental group ex-
press as GeoGebra’s didactical potential in teaching of integrals, individual interviews
were conducted with the teachers. In the interviews, the teachers were asked to elaborate
on their perception of the didactical potential of GeoGebra with respect to teaching of
integrals. In analysis of the teachers’ interviews I was looking for what the teachers ex-
press to be affordances that GeoGebra did contribute and/or could contribute in their
teaching of integrals.

Research question 3: The third research question was related to possible obstacles in
teachers’ utilization of GeoGebra. To answer this question, teachers’ presentations, in-
tegral tasks, and their interview response were analyzed. Their exploitation of GeoGebra’s
potential was related to the didactical potential of the software.

### 3.4 Putting the two studies together

Aims, research questions and methods for data collection in both studies are shown in
*Figure 3.1*, see below.
3.5 Ethical considerations

In this section the ethical considerations are described.

The first study, described in Paper 1, investigated students’ work with and perception of the dynamic software GeoGebra. Research data were gathered from a collection of students’ solutions to mathematical tasks, classroom observations of their mathematical work and from interviews with the students. The students that participated in the study were all over 15 years of age, which means that their parents’ permissions were not needed. Instead, it is the students themselves that had to give their consent. Before signing the approval, the students were informed about the aim of the research and the activities they were supposed to take part in. They were informed that the study would follow ethical principles formulated by the Swedish Research Council (SRC, 2010) with the demands on information, approval, confidentiality and how the research material will be used. The students were guaranteed confidentiality meaning that data collected within the study would not be tracked to any specific individual, except for the purpose of this re-
search and that nobody outside of the investigation team will have access to the collected information.

The second study, described in Paper 2, investigated teachers’ utilization of the didactical potential of GeoGebra. Research data were gathered from a collection of teachers’ screen recorded presentations of integrals, integral tasks produced by the teacher, and from interviews with the teachers. The teachers took part in the study on a voluntarily basis. Otherwise, the same demands were applied on information, approval, confidentiality and how the research material will be used as in the first study, in accordance to principles formulated by Swedish Research Council.
Summary of papers

The two studies in this research are described in two papers, one for each study. In this chapter a summary of the papers is provided. The two papers are enclosed in the second part of the thesis.

4.1 Paper1


Submitted to Nordic Studies in Mathematics Education.

The aim of the study conducted in two upper secondary classes in Sweden was to investigate how students work with and perceive GeoGebra in their learning of the integral concept. The analysis was based on data gathered from classroom experiments (student’ solutions to integral tasks and observations of their classroom work), and from the interviews with the students. The study had a starting point in a hypothesis of a possible differentiation of students’ mathematical work in GeoGebra.

4.1.1 Results

The study revealed a differentiation in students’ work with GeoGebra in their solving of integral tasks and thus confirmed the initial hypothesis. Four different students’ work methods were identified. Table 4.1 shows the relation between the four types of students’ work with GeoGebra identified in this study and Guin and Trouche’s differentiation of the instrumental genesis of the students.
Summary of papers

Four work methods in this study | Work methods in Guin and Trouche’s model
---|---
Work method 1 | Theoretical work method
| Resourceful work method
Work method 2 | Rational work method
| Mechanical work method
Work method 3 | Random work method
Work method 4

*Table 4.1*: Four work methods identified in this study in relation to Guin and Trouche’s work methods (2002).

The analysis revealed that, compared with students with weak work methods, students with strong work methods were better in exploiting the potential of GeoGebra. They were also better in realizing the constraints of this software while using it as an instrument in their work with integrals. These students were able to link their theoretical knowledge of the integral concept and their knowledge of how to use GeoGebra in order to explore the integral concept. On the other side, lacking both a mathematical knowledge of the integral concept and a technical knowledge of how to use features of GeoGebra in their work with integrals, students with the weakest work methods demonstrated great difficulties while exploring the integral concept in GeoGebra. Their weakness in both the mathematical part and the technical part made it difficult for these students to distinguish between the knowledge of how to operate GeoGebra and how to use this program for doing and learning mathematics.

Linking students’ perception of the role of GeoGebra in their learning of the integral concept to students’ work methods in this study it was found that similar characteristics of the software have been appreciated across the work methods, as long as students fulfilled their mathematical work, or, at least, a part of it, in GeoGebra. Students with the weakest work method, who did not work with GeoGebra and did not know how to use the program in their work with integral tasks in this study, did not see any advantages with using it in their mathematical work. Furthermore, these students found it even more confusing to work in GeoGebra environment, compared with their previous environment based on paper/pencil and/or a calculator.

Students with weak work methods were also more concerned about the time aspect than students with strong work methods. They were more critical about the time devoted to instruct students how to use GeoGebra in their mathematical work in this study.

On the other hand, all of the students, regardless of characterization of their work method, found it very important to master paper/pencil based solving techniques. As they expressed, solving tasks by hand was an important element of the course exam and the national tests in mathematics, and, for that reason, this solving technique was considered as more important than solving the tasks with the use of GeoGebra.
4.2 Paper2


Upper secondary teachers were asked to make an introduction of integrals and prepare integral tasks for students at the upper secondary level. The teachers were asked to construct this material by using the dynamic geometry software GeoGebra. Through an analysis of the introductions and tasks produced by the teachers and their interview response, the teachers’ perception and utilization of the didactical potential of GeoGebra was investigated.

4.2.1 Results

The results of the study have shown that the teachers in the experimental group encountered obstacles that prevented them from utilizing the full potential of the software in their teaching of integrals. With respect to the nature of the obstacles three different types have been distinguished: epistemological-, technical- and didactical obstacles.

Epistemological obstacles

An immediate access to different representation registers of a mathematical concept can support students’ understanding of mathematics (Duval, 2006). GeoGebra supports parallel work with different representation (Preiner, 2008, Hohenwarter, 2004) and furthermore, by interacting with the software it is possible for students to explore mathematical concepts in a broader way compared with non dynamic learning environments (Preiner, 2008). The teacher can e.g. let the students explore the properties of the integral or to investigate the connection between the derivative and the primitive of the function \( f \). However, not all of the teachers in the experimental group expressed awareness that GeoGebra could be exploited in a broader sense than as a tool for visualization only. Some of them do not link different semiotic representations of integrals in their presentations nor give a possibility to students to explore the integral concept in the prepared tasks. Some of the teachers in the experimental group seemed to have a rather limited perspective of the teaching of integrals. What seems important to these teachers is to teach the students to be able to do exercises and tasks in the course book and in the final exam later. They do not seem to realize the didactical potential of GeoGebra when it comes to giving a broader idea of the mathematical concept of integrals to the students (see Teacher 1 - 4.1.3 Analysis, Paper 2).

Technical obstacles

The second type of obstacles was of a technical nature. Although being able to make a recorded presentation of integrals on their own, some teachers experienced difficulties in using the software while e.g. looking for functions in GeoGebra that they intended to use
in the presentation of integrals and/or in integral tasks. In their planning of integral tasks for students some teachers decided not to construct their own tasks feeling uncomfortable with their ability to operate the technical aspects of GeoGebra and to use functions and features of the software in a way they intended to. Here, it is important to recall that, for some of the teachers in the experimental group this study was the first contact with a dynamical geometry software, and some of them had a very limited experience of GeoGebra. Still, all of the teachers were experienced in the use of some mathematical software and were able to make a presentation of integrals in GeoGebra on their own, without any external help. On the other hand, despite their ability to handle technical aspects of non dynamical software, or in some cases even some (although limited) experience of the use of GeoGebra, these teachers experienced significant technical obstacles in their exploitation of the didactical potential of this dynamical software which hindered them from using it in the intended way.

Didactical obstacles

The third type of obstacles observed in this study were didactical obstacles. Integration of technological tools into mathematics education makes the didactical situation more complex and increases the differentiation in students’ work with mathematics. The complex process of instrumental genesis of students requires an active role of the teacher and puts great demands on his/her way of organizing the classroom and orchestrating students’ mathematical activities. Some of the teachers in the experimental group were not aware of the complexity of the process and thus were not able to deal with encountered difficulties in a successful way (see Teacher 1 - 4.1.3 Analysis, Paper 2). On the other hand, some of the teachers were aware of different aspects of the instrumental genesis of students and the need of teacher to support this process, but did not seem to feel comfortable in their own competence in carrying out the process on their own. These teachers felt a need for external help and support in order to integrate the software into their students’ mathematical activities in a, from a didactical perspective, fruitful way (see 4.3 Other teachers, Paper 2).
Conclusions and perspectives

5.1 The first study, Paper 1

The results of this study can be seen, on one hand, in the light of the didactical potential of GeoGebra in supporting students’ learning of mathematical concepts, and on the other hand, in the light of the complexity of introduction of software like GeoGebra into mathematics classrooms.

From the first perspective, students in the study were generally positive to the use of GeoGebra in their work with integrals. They related their positive views to a variation in teaching methods in general, but also to the use of GeoGebra in their mathematical work in particular. GeoGebra, as they experienced, gave them an opportunity to see and to learn mathematics in a new way, and the possibility of a visualization and a simultaneous work with different representations of the integral concept in GeoGebra was appreciated.

From the second perspective, the integration of GeoGebra into students’ mathematical work made a didactical situation in classes that participated in our study more complex and differentiation in students’ work with the software while solving integral tasks was observed. The differentiation was broad; from students who were able to explore the integral concept in GeoGebra in a new way, by exploiting the potential of GeoGebra, to students who lacking in both mathematical knowledge and knowledge of how to use GeoGebra and not being able to distinguish between these two types of knowledge, experienced that learning mathematics with GeoGebra was even more difficult compared with their previous learning environment. Here, it seems important to consider the institutional legitimacy of the software in the classrooms that participated in this study. Some students in the experimental group were pragmatic in their way of prioritizing problem solving methods in their mathematical work and prefer solving methods allowed for use in course exams. As GeoGebra was not included into course examination some students decided to not put much efforts in learning how to use this software in their mathematical work.
Conclusions and perspectives

The results of the first study suggest that we, in order to organize good GeoGebra based learning environments, have to consider a number of aspects. Firstly, we have to make students aware of the didactical potential of the software, meaning that the software can help them to explore mathematical concepts, and to express these in different ways. They also need to be aware of how the work with the software can support their understanding of mathematics. Secondly, students’ process of learning of how to use the software as a mathematical tool has to be supported in an adequate way, according to each student’s individual needs. Last, the legitimacy of the software in mathematics classroom, specially its position in the course exams, has to be considered, as it absence can demotivate students from using the software in their mathematical activities.

5.2 The second study, Paper 2

The results of this study should be seen from the perspective of the complexity of the introduction of software like GeoGebra into mathematics classrooms. All teachers that participated in this study considered GeoGebra as a software with a significant didactical potential in teaching of integrals, but, their utilization of the software’s didactical potential, varied from teacher to teacher. Some of the teachers in the experimental group did not seem to perceive nor to exploit GeoGebra’s potential in a wide meaning. Instead, it was seen as a visualization tool only. The results showed that an introduction of computer software like GeoGebra into mathematics education can be difficult for teachers with no experience of the use of such a software, but even other teachers in the experimental group expressed and/or experienced some type of obstacles (didactical, epistemological and/or technical) that prevented them from utilizing the full potential of the software in the teaching of integrals.

The results of the second study suggests that the teachers, in order to be able to organize good GeoGebra based teaching environments, need an adequate support in the process of integration of the software into their teaching practice. There is a need of supporting and developing their competencies with respect to different aspects of the use of the software as a mathematical tool: technical competencies in how to operate the software, competencies in how a specific mathematical concept can be treated in GeoGebra in a, from the perspective of students learning of the concept, optimal way, as well as competencies in how to organize students’ mathematical activities in GeoGebra based classrooms in a way that fully supports students’ individual needs.

The development of teachers competencies, discussed above, is of a particular importance in environments where computer technology expects to largely enter mathematics classrooms, as the case is in Sweden.

5.3 The new curriculum

From a national perspective, in Sweden, the findings of this study should also be seen in the light of the new curriculum for mathematics, that will be officially introduced in September 2011. The new curriculum requires information technologies to be used in
all upper secondary courses in mathematics, and students are, among other qualities, expected to demonstrate strategies for mathematical problem solving including the use of digital media and tools. Further, considered as an integrated part of students’ mathematical knowledge, and assessed accordingly, the use of information technology has now gained in legitimacy.

Considering the situation in Sweden, it seems very important to give teachers a possibility to develop their competencies related to the integration of information technologies into the mathematics classroom. The teachers need to possess these competencies in order to be able to organize good technology based learning environments that supports students’ learning of mathematics in a, from a didactical perspective, fruitful way.

5.4 Some reflections about the generalizability of the findings

The features of GeoGebra exploited by the students and the teachers in this research are not characteristic for GeoGebra only, but also for other dynamic software and for some of the features also for other types of mathematical software that integrates graphical, numerical and computer algebra systems. In general, such types of mathematical software provide new ways for students to create and interactive with mathematical objects and, in this sense, they can be characterized as advanced mathematical software in educational contexts. Therefore, the conclusions drawn from this study can be expected to be valid also in a relation to the use of other types of advanced mathematical software in learning and teaching of mathematics.

5.5 Future research

Information technologies enter the mathematics classroom in different formats. One particular technology has lately gained a lot of attention; a technology that is more and more present in the mathematics classroom, although, so far, not exploited in students’ mathematical activities in a wider meaning, except for local school projects. The question is about the last generation of mobile phones, so called smart phones. In addition to computer software, it would be interesting to investigate mobile phone applications for following reason. Software developed for smart phones promise, in many cases, the same functionality like software installed on computers do. However, from a human-computer-interaction point of view, there are reasons to investigate if the user’s work with an application on a mobile phone supports his/her learning of mathematics in a same way as an interaction with an application installed on a computer does, and, if not, what would be the difference. This type of research is important as development of different types of educational applications for smart phones is fastly growing. As, in industrialized countries, students’ access to smart phones is very high, this technology is expected to position itself as a legitimate actor in mathematics classrooms in a very near future. It is important that the teachers are armed with the competencies needed for a, from a didactical perspective, successful integration of smart phones into their students’ mathematical activities.


Appendix A

Integral Tasks
Integralövningar med GeoGebra

Detta övningsmaterial är avsett att användas inom ramen för forskningsprojektet Teknologiska hjälpmedel i matematikundervisningen. Materialet är framtaget i syfte att undersöka hur den matematiska förståelse hos eleverna utvecklas i en lärandemiljö baserad på det dynamiska programmet GeoGebra. Till materialet hör ett antal GeoGebra filer. För att få tillgång till dessa filer maila till uppgiftskonstruktörerna:
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eller
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Uppgift 1
Bestäm en primitiv funktion till \( f(x) = x^5 \).

Uppgift 2
Bestäm den primitiva funktionen \( F(x) \) till \( f(x) = 2x \), om det gäller att \( F(1) = 2 \).

Uppgift 3
Bestäm samtliga primitiva funktioner till \( f(x) = 5 \sin(5x) \).

Uppgift 4
Integralen \( \int_{0}^{2} x(x - 3) \, dx \) har värdet \( -\frac{10}{3} \). Visa hur man kommer fram till detta resultat med hjälp av primitiv funktion.

Uppgift 5
Bestäm det positiva talet \( a \) så att \( \int_{1}^{a} \frac{1}{x} \, dx = 2 \).

Läs in filen Uppgift5.ggb i GeoGebra.

**Uppgift 6**

Kurvorna $f(x) = e^{0.2x}$ och $g(x) = x^2$ innesluter tillsammans med $y$-axeln ett område i första kvadranten. Inför lämpliga beteckningar och teckna integralen för områdets area.

Läs in filen *Uppgift 6.ggb* i GeoGebra.
Bestäm områdets area med minst tre värdesiffror. Beskriv och förklara ditt tillvägagångssätt.

**Uppgift 7**

Teckna ett integraluttryck för arean av det område som begränsas av kurvorna \( f(x) = 3x^2 \) och \( g(x) = 16 - x^2 \) samt beräkna denna area för hand. Motivera dina räkningar.

Läs in filen *Uppgift7.ggb* i GeoGebra.

![Diagram](image)

Ändra i funktionerna genom att ställa dig på en funktion och dra. Vad händer med funktionsuttrycket som visas under *Fria objekt* i algebrafönstret när du flyttar en kurva? Varför?

a) Kan du justera kurvorna så att det område som begränsas av dem är lika med 100 areaenheter? Hur många sådana lösningar finns det? Skriv ner din motivering.

b) Vilken är den minsta area du kan få fram? Skriv ner ditt svar.

**Uppgift 8**

Låt \( g(x) = \int_0^x \frac{1}{1 + t^2} \, dt \).
a) Använd figuren ovan och beskriv vad $g(3)$ betyder.

b) Läs in filen Uppgift8.ggb i GeoGebra. Bestäm med hjälp av GeoGebra ett närmevärde till $g(3)$. 

c) Kan du i ord beskriva den geometriska betydelsen av funktionen $g(x)$?

**Uppgift 9**

a) Ändra antalet rektanglar $n$ genom att ställa sig på glidaren och dra. Hur påverkas differensen mellan över- och undersumman? Skriv ner dina slutsatser.

b) Stämmer dina slutsatser i a) även för andra intervall $[a,b]$? Kontrollera detta genom att ändra både intervallgränser och antalet rektanglar. Skriv ner dina slutsatser.

c) Kan man göra differensen mellan över- och undersumman godtyckligt liten (hur liten som helst)? Hur gör man det? Skriv ner din förklaring så noga som möjligt.

Uppgift 10

Gå hit www.georgiostheodoridis.se/archives/MDIntegralerV5.html. Följande figur visas


b) Kryssa i rutor Visa undersumman och Visa integralen. Ändra värdet på variablerna $A$, $B$,... Jämför med dina observationer i 1. och skriv ner dina slutsatser.


Uppgift 11


Gå till Visa och kontrollera att Algebrafönster är markerat.

Rita kurvan $f(x) = 4e^{-0.5x}$ genom att mata in funktionen $f(x)$ i Input-fältet (se figur...
nedan).

Tänk på att du måste använda multiplikationstecken, att du måste ha parentes kring $-0.5x$ samt att man använder decimalpunkt istället för decimalkomma. Observera att $e$ skrivs genom att välja 'e' i rullningslista till höger om inmatningsfältet.

Klicka på Enter. Nu visas $f(x)$ i GeoGebra fönstret.

Rita kurvan $g(x) = 2x - x^2$. 

Ställ dig på funktionen $f(x)$, högerklicka och välj *Egenskaper*. Klicka sedan på *Färg*. Välj en valfri färg och klicka på *Stäng*.

Gör om proceduren för $g(x)$.

Justera koordinatplanet genom att klicka på knappen med pilar och sedan ställa dig på $x$-axeln och dra.
Appendix A: Integral tasks

Infoga rutnät genom att klicka på Visa och välja Rutnät.

**a)** Uppskatta det området i figuren som begränsas av $f(x) = 4e^{-0.5x}$, $g(x) = 2x - x^2$ samt $y$-axeln och linjen $x = 2$. Använd rutnätet och uppskatta arean genom att grovt räkna antal rutor. Skriv ner ditt svar.

**b)** Den exakta arean kan bestämmas genom att beräkna arean under $f(x)$ i intervall $[0, 2]$ och sedan subtrahera arean under $g(x)$ för samma intervall:

$$
\int_0^2 f(x) \, dx - \int_0^2 g(x) \, dx,
$$

vilket även kan skrivas som

$$
\int_0^2 (f(x) - g(x)) \, dx.
$$


**c)** Nu ska vi kontrollera svaret med hjälp av GeoGebra. I rullningslista Kommando (se figur nedan) väljer du Integral.

Nu visas Integral [ ] i Input-fältet.

Kompletera inmatningsuppgifter på följande sätt:

Integral [f, g, 0, 2 ]
Tryck på Enter.
Jämför detta resultat med resultatet i b).

**Uppgift 12**

Vad händer med integralvärdet om man, i *Uppgift 11*, byter plats på \( f(x) \) och \( g(x) \) och skriver:

\[
\text{Integral} \; [g, \; f, \; 0, \; 2]
\]
i *Input*-fältet? Hur påverkas resultatet?

**Uppgift 13**

I *Uppgift 11* fick du bestämma värden av integralen

\[
\int_{0}^{2} (f(x) - g(x)) \, dx, \quad (3)
\]

där \( f(x) = 4e^{-0.5x} \) och \( g(x) = 2x - x^2 \).

Integralens värde är \( \frac{-8}{e} + \frac{20}{3} \) eller 3,724 efter en avrundning till tre decimaler.
I denna uppgift skall vi undersöka hur värden av denna integral påverkas vid vertikal förskjutning av det område som begränsas av kurvorna $f(x)$ och $g(x)$ längs $y$-axeln. För att göra detta definierar vi $f(x)$ och $g(x)$ på något annorlunda sätt (jämfört med 
Uppgift 11) genom att använda oss av en glidare.


Gå till Visa och kontrollera att Algebrafönstret är markerat.

Nu ska du infoga en glidare $a$ genom att klicka på

och sedan vänsterklicka någonstans i GeoGebra-fönstret, se bilden nedan.

Låt glidaren $a$ anta värden mellan 0 och 10 och klicka på Verkställ.

För att ändra värden på glidaren från 1 till 0 gör du följande:

- klicka på pilen i övre vänstra hörn i GeoGebra-fönstret,
- ställ dig nu på glidaren och dra tills $a = 0$.

Rita nu kurvan $f(x) = 4e^{-0.5x} - a$ genom att mata in den i Input-fältet (se figur nedan).
Rita kurvan \( g(x) = 2x - x^2 - a \) på motsvarande sätt.

Nu ska du bestämma värdet av integralen för område som begränsas av kurvorna \( f(x) = 4e^{-0.5x} - a \) och \( g(x) = 2x - x^2 - a \) i intervallet [0, 2] och detta gör du genom att skriva

\[
\text{Integral} \left[ f, g, 0, 2 \right]
\]

i Input-fältet.

Har du gjort rätt har du fått följande bild, se nedan:

a) Betrakta nu integralvärdet för area av det område som begränsas av kurvorna \( f(x) \) och \( g(x) \) i intervall [0, 2]. Hur påverkas detta värde vid förskjutning av \( f(x) \) och \( g(x) \) längs \( y \)-axeln? Använd glidare \( a \) för att ta reda på svaret.

Förklara skriftligt det du observerar så noga som möjligt.
Appendix A: Integral tasks

**Uppgift 14**

Bestäm areaen som begränsas av kurvan \( f(x) = x^2 \), x-axeln samt linjerna \( x = 2 \) och \( x = -2 \) för hand. Kontrollera din lösning med hjälp av GeoGebra.

**Uppgift 15**

Rita kurvan \( f(x) = 5 - x^2 \) och linjen \( g(x) = 5 - x \).

a) Beräkna för hand var linjen skär kurvan.

b) Ställ upp en integral för beräkning av areaen av det område som begränsas av linjen och kurvan.

c) Hur stor är denna area? Beräkna för hand.

d) Kontrollera din lösning med hjälp av GeoGebra.

**Uppgift 16**

Inför och rita funktionen \( f(x) = x \sin(x) \) i GeoGebra. Inför nu en glidare \( a \) som går mellan 0 och 7.

a) Beräkna integralen \( \int_{0}^{a} f(x) \, dx \) genom att skriva

\[
\text{Integral} [f, 0, a]
\]

i inmattningsfältet. Dra i glidaren. Du kommer att se att integralen kan anta både positiva och negativa värden. Förklara varför det är så och skriv ner ditt svar.

b) Bestäm det värdet på \( a \) för vilket integralvärdet blir så litet som möjligt. Förklara hur du tänker för att få fram detta.

**Uppgift 17**

Kurvorna \( f(x) = \sqrt{2x + k} \) och \( g(x) = x \) begränsar tillsammans med x-axeln ett område. Bestäm värdet på \( k \).

Kontrollera ditt svar med hjälp av GeoGebra.
Uppgift 18
Beräkna för hand arean av det område som begränsas av $x$-axeln och kurvan $f(x) = x^2 - 3x$ i intervall $[-1, 4]$.
Kontrollera ditt svar med hjälp av GeoGebra.

Uppgift 19
Betrakta kurvan $f(x) = \sin(x)$.

a) Ställ upp en integral för beräkning av arean av det område som begränsas av kurvan $f(x)$ och $x$-axeln i intervall $[0, 2\pi]$.

b) Beräkna integralen för hand.

c) Kontrollera ditt svar med hjälp av GeoGebra.

Uppgift 20
Lös följande uppgift med hjälp av GeoGebra.

Kurvan $f(x) = 3x^2 - x^3$ begränsar tillsammans med $x$-axeln ett ändligt område. Beräkna förhållandet mellan areorna av de båda delar, i vilka kurvan $g(x) = 0.5x^3$ delar området.

Uppgift 21
Kurvan $f(x) = ax^2(6 - x)$ där $a > 0$ begränsar tillsammans med $x$-axeln ett område med arean 18 areaenheter. Bestäm $a$ för hand.

Kontrollera ditt svar med hjälp av GeoGebra.

Tips: Inför en glidare $a$ som går mellan 0 och 1.

Integralkalkylens huvudsats
Läs in filen Uppgift23.ggb i GeoGebra.
Figuren visar grafen till \( f(x) = ax^3 + bx^2 + cx + d \)

Area under grafen till \( f(x) \) i intervall \([a, x]\) kan beskrivas med areamätningsfunktionen

\[
A(x) = \int_a^x f(t) \, dt \text{ där } x > a.
\]

Vi kan konstatera att \( A(x) \) är en funktion ty till varje värde på \( x \) får vi ett funktionsvärde som är givet av integralen. Tidigare har du sett att detta värde kan beräknas som gränsvärdet av Riemansummor. Nu ska vi undersöka andra egenskaper denna funktion har, men innan vi gör det ska vi repetera lite om derivatan från C-kursen:

Derivatan av funktionen \( g(x) \) definieras som gränsvärdet

\[
g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h}.
\]

Geometriskt kan \( g'(x) \) tolkas som riktningskoefficienten för tangenten till kurvan \( g(x) \) i punkten \((x, g(x))\). Som du redan vet används derivata ofta vid beräkningar av förändringar, t.ex. förändringar av populationsstorlek, temperaturförändringar m.m. Vi tittar nu på areamätningsfunktionen \( A(x) \). Area som begränsas av \( f(x) \) och \( x \)-axeln visas för två intervall (se figuren nedan):

- area1 - area under grafen till \( f(x) \) i intervall \([-1, x]\),
- area2 - area under grafen till \( f(x) \) i intervall \([-1, x + h]\).
Betrakta nu $\lim_{h \to 0} \frac{A(x + h) - A(x)}{h}$. Detta gränsvärde är derivatan av areamättningsfunktionen $A(x)$ och visar förändring av areastorlek då intervallet ändras från $[-1, x]$ till $[-1, x + h]$. Med hjälp av GeoGebra ska vi nu undersöka hur detta gränsvärde påverkas då $h$ går mot noll:

a) Ställ dig på punkten $Q$ i ditt GeoGebra-fönster och dra mot punkten $P$. Kan du se något samband mellan $\lim_{h \to 0} \frac{A(x + h) - A(x)}{h}$ och $f(x)$? Om ja, hur ser sambandet ut?

Skriv ner ditt svar.

Sats 1: Om $f$ är kontinuerlig i $[a, b]$ och

$$A(x) = \int_a^x f(t) \, dt$$

så gäller att $A'(x) = f(x)$ d.v.s. $A(x)$ är primitiv till $f(x)$.

Denna sats leder sedan till insättningsformeln:

Sats 2: Om $f$ är kontinuerlig i $[a, b]$ och om $F$ är en godtycklig primitiv funktion till $f$ i $[a, b]$ så är

$$\int_a^b f(t) \, dt = F(b) - F(a).$$

Bevis: Låt $F$ vara en godtycklig primitiv funktion till $f$ och låt

$$A(x) = \int_a^x f(t) \, dt$$

Då är $A'(x) = F'(x) = f(x)$ och $A(x) = F(x) + C$.

Vidare gäller att $A(a) = \int_a^a f(t) \, dt = 0$. 
b) Komplettera beviset.

**Uppgift 24, Np vt05**

Figuren visar en parabel och en rektangel i ett koordinatsystem. Det skuggade området är begränsat av parabeln och $x$-axeln. Arean av det skuggade området kallas i fortsättningen parabelarean.

Två av rektangelns hörn sammanfaller med kurvans skärningspunkter med $x$-axeln. En av rektangelsidorna tangerar kurvans maximipunkt.

I den här uppgiften ska du undersöka förhållandet mellan parabelarean och rektangelarean. Låt parabelns ekvation vara $f(x) = b - ax^2$, där $a$ och $b$ är positiva tal.

a) Börja med att sätta $b = 9$ och $a = 1$ och rita grafen till funktionen $f(x) = 9 - x^2$. Bestäm därefter förhållandet mellan parabelarean och rektangelarean för hand.

b) Välj själv andra exempel med hjälp av GeoGebra och försök formulera en slutsats utifrån dina valda exempel.

c) Undersök om din slutsats även gäller i det allmänna fallet med parabeln $f(x) = b - ax^2$ och kontrollera din lösning med hjälp av GeoGebra.

Skriv ner dina svar.
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