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Energy estimates and variance estimation for hyperbolic stochastic partial differential equations.

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Abstract

In this thesis the connections between the boundary conditions and the variance of the solution to a stochastic partial differential equation (PDE) are investigated. In particular a hyperbolical system of PDE's with stochastic initial and boundary data are considered. The problem is shown to be well-posed on a class of boundary conditions through the energy method. Stability is shown by using summation-by-part operators coupled with simultaneous-approximation-terms. By using the energy estimates, the relative variance of the solutions for different boundary conditions are analyzed. It is concluded that some types of boundary conditions yields a lower variance than others. This is verified by numerical computations.

Keywords: Uncertainty Quantification, Hyperbolic Partial Differential Equations, Boundary Conditions, Energy Estimates, SBP, SAT, and Variance Reduction.

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Contents

Chapter 1

Introduction

Uncertainty quantification(UQ) is the research area where one studies how uncertainties in data and parameters of a system affect the uncertainty (variance) of the solution. The system might, for example be described by a partial differential equation (PDE). In practice these uncertainties are mostly due to limited knowledge of the system at hand, measurement errors and/or noise [9].

The uncertainties for a certain situation can be found by doing simulations of the system at hand. The most common simulation method is the Monte-Carlo method where one generates a set of situations in each of which the stochastic variables are set to a fixed number [9]. One then calculates the solution in each of these cases, as if there were no uncertainties, and approximates the expected value and variance of the solution from these solutions. Numerical quadratures are also based on this principle, but here the values of the stochastic variables are set to some predetermined numbers [9]. These types of methods are called non-intrusive due to the fact that the original problem is solved a multiple number of times.

The opposite of this is obviously intrusive methods, where a different kind of problem is solved instead. One common intrusive approach is to expand the stochastic variables in an orthogonal basis and then perform a Galerkin projection [9], [11]. In this thesis we have only considered non-intrusive methods due to their simplicity.

When studying PDE's, a common method to prove well-posedness is the so called energy method [4]. Here the norm of the solution and it's derivatives are related to the norm the initial and boundary data. When dealing with uncertainties in the data we show that by making modifications of boundary conditions, the expectation of the energy estimate gives an equation for the variance of the solution. Since different types of boundary conditions yield different energy estimates it may be possible to relate the relative variances for the different boundary conditions to each other.

In this thesis we have chosen to investigate a hyperbolic system of PDE's with uncertainty in both the initial and boundary data. We will first study the energy estimates of the PDE and try to make some qualitative statements about the relation between the variances for different types of boundary conditions. These ideas will be tested by numerical computations.

Chapter 2

Main idea

2.1 Problem

Consider a system of hyperbolic PDE's where the initial- and boundary conditions depend on a stochastic variable ξ

$$\begin{aligned} u_t + Au_x &= 0 \qquad 0 \leq x \leq 1, \qquad t \geq 0 \\ u &= f(x,\xi) \qquad 0 \leq x \leq 1, \qquad t = 0 \\ \mathcal{H}_0 u &= g_0(t,\xi) \qquad x = 0, \qquad t \geq 0 \\ \mathcal{H}_1 u &= g_1(t,\xi) \qquad x = 1, \qquad t \geq 0. \end{aligned} \tag{2.1}$$

Here $A \in \mathbb{R}^{N \times N}$ is a constant matrix, $f(x,\xi) \in \mathbb{R}^N$, $g_0(t,\xi)$ and $g_1(t,\xi)$ are the data of the problem. The boundary conditions are specified by the operators H_0 and H_1 which are assumed to be linear. The number of boundary conditions required at each boundary depends on the choice of A and will be investigated in section 3.

One way to prove well-posedness for (2.1) is to use of the energy method [4], where one bounds the norm between the exact solution and a solution where perturbation has been added to the initial and boundary conditions. If it is possible to show that the norm of this difference is bounded, then the problem is well posed.

In this thesis the perturbations are chosen to be the deviation from the mean of the initial and boundary conditions. If we call the solution to the perturbated system v, it satisfies the equation

$$\begin{aligned} v_t + Av_x &= 0 && 0 \leq x \leq 1, \quad t \geq 0 \\ v &= \mathbb{E}\left[f(x,\xi)\right] && 0 \leq x \leq 1, \quad t = 0 \\ \mathcal{H}_0 v &= \mathbb{E}\left[g_0(t,\xi)\right] && x = 0, \quad t \geq 0 \\ \mathcal{H}_1 v &= \mathbb{E}\left[g_1(t,\xi)\right] && x = 1, \quad t \geq 0, \end{aligned} \tag{2.2}$$

where $v = \mathbb{E}[u]$. If we set $e = u - v = u - \mathbb{E}[u]$, it satisfies

$$\begin{aligned} e_t + Ae_x &= 0 && 0 \leq x \leq 1, \quad t \geq 0 \\ e &= \delta f(x,\xi) = f(x,\xi) - \mathbb{E}\left[f(x,\xi)\right] && 0 \leq x \leq 1, \quad t = 0 \\ \mathcal{H}_0 e &= \delta g_1(t,\xi) = g_0(t,\xi) - \mathbb{E}\left[g_0(t,\xi)\right] && x = 0, \quad t \geq 0 \\ \mathcal{H}_1 e &= \delta g_2(t,\xi) = g_1(t,\xi) - \mathbb{E}\left[g_1(t,\xi)\right] && x = 1, \quad t \geq 0. \end{aligned} \tag{2.3}$$

2.2 General remarks

For the deviation from the mean $e(x,t,\xi)$ we will use the notation

$$e = \left(e^1 \dots e^N\right)^T. \tag{2.4}$$

From here on we assume that the matrix A is symmetric and thus diagonalizable. Hence there exist a matrix S and a diagonal matrix Λ such that $A = S\Lambda S^T$. We will also factor Λ and S as

$$\Lambda = \begin{pmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{pmatrix}, \qquad S = \begin{pmatrix} S^+ & S^- \end{pmatrix}, \tag{2.5}$$

where Λ^+ is the matrix containing all positive eigenvalues and Λ^- is the matrix containing all negative eigenvalues. To simplify the forthcoming analysis we assume that all eigenvalues are non-zero and that there are m positive eigenvalues.

We will also use the characteristic variables, $w(x,t,\xi) = S^T e(x,t,\xi)$, which correspondingly are factored as

$$w(x,t,\xi) = \left(w^+, w^-\right). \tag{2.6}$$

The $N \times N$ identity matrix will be denoted by I_n.

2.3 Energy Estimate

By multiplying (2.3) by e^T and integrating over the spatial domain we obtain

$$\frac{\partial}{\partial t}\|e\|_2^2 = -e^T Ae\Big|_{x=0}^{x=1}. \tag{2.7}$$

By inserting the proper boundary conditions into (2.7), the right hand side can be bounded. In this case one have an energy estimate of the problem [4] which implies well-posedness since the norm of the solution is bounded.

Note also that

$$\begin{aligned} &\mathbb{E}\left[\|e\|_2^2\right] = \mathbb{E}\left[\int_0^1 e^T e dx\right] = \\ &\int_0^1 \mathbb{E}\left[(u - \mathbb{E}[u])^T (u - \mathbb{E}[u])\right] dx = \int_0^1 Var(u)dx, \end{aligned} \tag{2.8}$$

which means that the expected value of $\|e\|_2^2$ can be interpreted as the norm of the variance of the solution. The objective of this thesis is to investigate whether one can draw conclusions about the variance of the solution by just looking at the energy estimate.

Chapter 3

Energy estimates and well posedness

Equation (2.7) can be simplified to

$$\frac{\partial}{\partial t}\|w\|_2^2 = -w^T \Lambda w \Big|_{x=0}^{x=1} = -(w^+)^T \Lambda^+ w^+ \Big|_{x=0}^{x=1} - (w^-)^T \Lambda^- w^- \Big|_{x=0}^{x=1}. \qquad (3.1)$$

3.1 Boundary conditions

From (3.1) it is clear that for each positive eigenvalue we need a boundary condition at $x = 0$ and similarly we need at boundary condition at $x = 1$ for each negative eigenvalue. Thus we have chosen to consider boundary operators of the type

$$\begin{aligned} \mathcal{H}_0 e &= w^+(0,t,\xi) - D_0 w^-(0,t,\xi) \qquad x = 0, \\ \mathcal{H}_1 e &= w^-(1,t,\xi) - D_1 w^+(1,t,\xi) \qquad x = 1. \end{aligned} \qquad (3.2)$$

These conditions create a connection between the incoming and outgoing characteristic variables at the boundaries. If $D_0 = D_1 = 0$ we have what is called the characteristic boundary conditions. In the case when either D_0 or D_1 is non-zero we will refer to these conditions as non-characteristic boundary conditions. D_0 and D_1 are matrices of appropriate size. To show that problem (2.3) is well-posed with boundary conditions (3.2), we will use a similar approach as in [2]. By inserting (3.2) in (3.1) we get

$$\begin{aligned}
&\frac{\partial}{\partial t}\|e\|_2^2 = (\delta g_0 + D_0 w_0^-)^T \Lambda^+ (\delta g_0 + D_0 w_0^-) + (w_0^-)^T \Lambda^- w_0^- \\
&- (\delta g_1 + D_1 w_1^+)^T \Lambda^- (\delta g_1 + D_1 w_1^+) - (w_1^+)^T \Lambda^+ w_1^+ = \\
&\begin{pmatrix} w_0^- \\ \delta g_0 \end{pmatrix}^T \begin{pmatrix} \Lambda^- + D_0^T \Lambda^+ D_0 & \Lambda^+ D_0 \\ \Lambda^+ D_0 & 0 \end{pmatrix} \begin{pmatrix} w_0^- \\ \delta g_0 \end{pmatrix} + \delta g_0 (\Lambda^+) \delta g_0 \\
&- \begin{pmatrix} w_1^+ \\ g_1 \end{pmatrix}^T \begin{pmatrix} \Lambda^+ + D_1^T \Lambda^- D_1 & \Lambda^- D_1 \\ \Lambda^- D_1 & 0 \end{pmatrix} \begin{pmatrix} w_1^+ \\ \delta g_1 \end{pmatrix} - \delta g_1 (\Lambda^-) \delta g_1.
\end{aligned} \tag{3.3}$$

Here the lower case index on w denotes the value of x. Our goal is to make the first matrix negative semidefinite and the second one positive semidefinite since this would imply that the right hand side of (3.3) is bounded. Hence add and subtract the two terms $\delta g_0 G_0 \delta g_0$ and $\delta g_1 G_1 \delta g_1$, where G_0 has the same size as Λ^- and G_1 has the same size as Λ^+, to (3.3). We obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\|e\|_2^2 = \\
&\begin{pmatrix} w_0^- \\ \delta g_0 \end{pmatrix}^T \begin{pmatrix} \Lambda^- + D_0^T \Lambda^+ D_0 & \Lambda^+ D_0 \\ \Lambda^+ D_0 & G_0 \end{pmatrix} \begin{pmatrix} w_0^- \\ \delta g_0 \end{pmatrix} + \delta g_0 (\Lambda^+ - G_0) \delta g_0 \\
&- \begin{pmatrix} w_1^+ \\ \delta g_1 \end{pmatrix}^T \begin{pmatrix} \Lambda^+ + D_1^T \Lambda^- D_1 & \Lambda^- D_1 \\ \Lambda^- D_1 & G_1 \end{pmatrix} \begin{pmatrix} w_1^+ \\ \delta g_1 \end{pmatrix} - \delta g_1 (\Lambda^- - G_1) \delta g_1 = \\
&\begin{pmatrix} w_0^- \\ \delta g_0 \end{pmatrix}^T F_0 \begin{pmatrix} w_0^- \\ \delta g_0 \end{pmatrix} + \delta g_0 (\Lambda^+ - G_0) \delta g_0 - \begin{pmatrix} w_1^+ \\ \delta g_1 \end{pmatrix}^T F_1 \begin{pmatrix} w_1^+ \\ \delta g_1 \end{pmatrix} \\
&- \delta g_1 (\Lambda^- - G_1) \delta g_1.
\end{aligned} \tag{3.4}$$

The relation (3.4) show that if F_0 is negative semi-definite and F_1 is positive semi-definite, we have an energy estimate. If possible, we therefore choose G_0, G_1, D_0 and D_1 such that these criteria are satisfied. In that case (3.3) can be simplified to

$$\frac{\partial}{\partial t}\|e\|_2^2 \leq \delta g_0 (\Lambda^+ - G_0) \delta g_0 - \delta g_1 (\Lambda^- - G_1) \delta g_1, \tag{3.5}$$

which after integration over the time domain yields

$$\|e\|_2^2 \leq \|f\|_2^2 + \int_0^t \Big(\delta g_0(\Lambda^+ - G_0)\delta g_0 - \delta g_1(\Lambda^- - G_1)\delta g_1\Big)\, ds. \tag{3.6}$$

and well-posedness.

Chapter 4

Stability

In this section we perform a stability analysis of the problem considered in the previous section using a semi-discrete setting with summation-by-part (SBP) operators and simultaneous-approximation-terms (SAT) as described in [5],[1]. We assume that ξ is a fixed parameter in this section, which corresponds to using a non-intrusive method [9]. The approach will utilize the same structure and notation as in [11] .

4.1 SBP-operators

The SBP-operators are designed such that they discretely correspond to integration-by-parts. The first order derivative operator is $P^{-1}Q$, where P is positive definite and the symmetric matrix Q has the property that

$$Q + Q^T = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \vdots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}. \tag{4.1}$$

4.2 Stability

To simplify the analysis we use Kronecker products. Beside the variable e we also introduce the characteristic variable $w = (I \otimes S^T)e$ and as previously split this up as $w^+ = (I \otimes (S^+)^T)e$, $w^- = (I \otimes (S^-)^T)e$.

For the SAT formulation we will use the matrices E_0 and E_1, where

$$(E_0)_{ij} = \begin{cases} 1 & \text{i = j = 1} \\ 0 & \text{else} \end{cases}, \qquad (E_1)_{ij} = \begin{cases} 1 & \text{i = j = N} \\ 0 & \text{else} \end{cases} \tag{4.2}$$

We also use the penalty matrices Σ_0 and Σ_1 which will be determined later such that stability is achieved.

The semi-discrete formulation of (2.7) with characteristic boundary conditions becomes

$$\begin{aligned} &(I \otimes I)w_t + (P^{-1}Q \otimes \Lambda)w = \\ &(P^{-1} \otimes I)(E_0 \otimes \Sigma_0)(w^+ - D_0 w^- - g_0) \\ &+ (P^{-1} \otimes I)(E_1 \otimes \Sigma_1)(w^- - D_1 w^+ - g_1). \end{aligned} \tag{4.3}$$

We now multiply (4.3) with $w^T(P \otimes I)$

$$\begin{aligned} &w^T(P \otimes I)e_t + w^T(Q \otimes \Lambda)e = \\ &w^T(E_0 \otimes \Sigma_0)(w^+ - D_0 w^- - g_0) + w^T(E_1 \otimes \Sigma_1)(w^- - D_1 w^+ - g_1). \end{aligned} \tag{4.4}$$

Next add the transpose of the whole equation (4.4) to itself,

$$\begin{aligned} &\frac{\partial}{\partial t}\|w\|^2_{(P\otimes I)} + w^T(Q + Q^T \otimes \Lambda)w = \\ &2w^T(E_0 \otimes \Sigma_0)(w^+ - D_0 w^- - g_0) + 2w^T(E_1 \otimes \Sigma_1)(w^- - D_1 w^+ - g_1). \end{aligned} \tag{4.5}$$

By now using the SBP property (4.1) we get

$$\begin{aligned} &\frac{\partial}{\partial t}\|w\|^2_{(P\otimes I)} + w_1^T A w_1 - w_0^T \Lambda w_0 = \\ &2w_0^T \Sigma_0 (w_0^+ - g_0) + 2w_1^T \Sigma_1 (w_1^- - D_1 w^+ - g_1). \end{aligned} \tag{4.6}$$

Let the penalty matrices be $\Sigma_0 = -\begin{pmatrix} \Lambda^+ & 0 \\ 0 & 0 \end{pmatrix}$ and $\Sigma_1 = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda^- \end{pmatrix}$ respectively so that

$$\frac{\partial}{\partial t}\|w\|^2_{(P\otimes I)} + (w_1^+)^T\Lambda^+w_1^+ + (w_1^-)^T\Lambda^-w_1^- - (w_0^+)^T\Lambda^+w_0^+ - (w_0^-)^T\Lambda^-w_0^- = \\ -2(w_0^+)^T\Lambda^+(w_0^+ - D_0w^- - g_0) + 2(w_1^-)^T\Lambda^-(w_1^- - D_1w^+ - g_1). \tag{4.7}$$

By rearranging the terms we arrive at

$$\frac{\partial}{\partial t}\|w\|^2_{(P\otimes I)} = -\begin{pmatrix} w_0^+ \\ w_0^- \\ g_0 \end{pmatrix}^T \begin{pmatrix} \Lambda^+ & -2\Lambda^+D_0 & -2\Lambda^+ \\ 0 & -\Lambda^- & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_0^+ \\ w_0^- \\ g_0 \end{pmatrix} \\ -\begin{pmatrix} w_1^- \\ w_1^+ \\ g_1 \end{pmatrix}^T \begin{pmatrix} -\Lambda^- & 2\Lambda^-D_1 & 2\Lambda^- \\ 0 & \Lambda^+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_1^- \\ w_1^+ \\ g_1 \end{pmatrix} \tag{4.8}$$

Hence we proven that a condition for stability is that both the matrices in (4.8) are positive semi-definite. In appendix B we have derived an explicit condition on D_0 and D_1 for which stability is achieved.

Chapter 5

Numerical solution

In this section we discuss how the stochastic nature of the problem is considered in practice by numerical computations.

5.1 Specific for stochastic PDE's

There are several ways to solve PDE's with stochastic parameters. The goal here is to calculate integrals of the type

$$\int_{-\infty}^{\infty} f(e(x,t,s))p_{\xi}(s)ds, \tag{5.1}$$

where $f(\cdot)$ is a function that specifies some statistic quantity of interest and $p_{\xi}(\cdot)$ is the probability density function (pdf) for ξ. For example the choice $f(x) = x$ would give us the expected value. Here will use what is called a non-intrusive method [9]. Non-intrusive methods basically solves the original problem many times. Each of these times the stochastic variable ξ is set to a fix value ξ_i. Assuming we have taken K samples the approximation looks like

$$\int_{-\infty}^{\infty} f(e(x,t,\xi))p_{\xi}(\xi)d\xi = \sum_{i=1}^{K} f(e(x,t,\xi_i))w_i + R_K, \tag{5.2}$$

where w_i is the weight for each value of ξ_i and R_K is the error term. The weights have to satisfy the condition that

$$\sum_{i=1}^{K} w_i = 1, \tag{5.3}$$

since we are approximating a pdf.

The most classical non-intrusive method is the Monte-Carlo method [9]. The Monte-Carl method simply generates the points ξ_i from the distribution of ξ by a random number generator and all points are then equally weighted, i.e. $w_i = 1/K$. The main advantage of the Monte-Carlo method is that a new distribution of ξ only requires a minor change in the algorithm (one has to sample from the new distribution instead). Hence the Monte-Carlo is suitable when dealing with complicated distributions. In our case we consider normal distributions and then more effective methods exist.

We will use a Gauss-Hermite quadrature, where it is necessary that $\xi \sim \mathcal{N}(0,1)$. The points and weights are determined by using Hermite polynomials (see appendix D for a short introduction to these). If we let H_n denote the n^{th} Hermite polynomial and $\{h_i\}_{i=1}^{n}$ all its roots, then we choose

$$\xi_i = \sqrt{2}h_i, \qquad w_i = \frac{2^{K-1}K!}{K^2(H_{K-1}(h_i))^2}. \tag{5.4}$$

If these are used in (5.2) then there exist a finite $z \in \mathbb{R}$ such that the error R_K is

$$R_K = \frac{K!\sqrt{\pi}}{2^K(2K)!}\frac{d^{2K}}{ds^{2K}}\left(f(e(x,s,z))\right). \tag{5.5}$$

The expected value and variance of the solution are approximated as

$$\mathbb{E}\left[u(x,t,\xi)\right] \approx \sum_{i=1}^{K} u(x,t,y_i)w_i, \tag{5.6}$$

$$Var(u(,t,\xi)) \approx \sum_{i=1}^{K}\left(u(x,t,y_i) - \sum_{j=1}^{K} u(x,t,y_j)w_j\right)^2 w_i. \tag{5.7}$$

Chapter 6

A specific case

Here we consider the case when $N = 2$ and

$$A = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}, \tag{6.1}$$

where $\alpha > 0$ will be varied. The characteristic equation for this matrix is

$$|A - \lambda I| = \begin{pmatrix} 1-\lambda & \alpha \\ \alpha & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - \alpha^2 = 0, \tag{6.2}$$

which has the solution $\lambda = 1 \pm \alpha$ and corresponding eigenvectors $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$. Since $\alpha > 0$ we have three different situations

- $0 < \alpha < 1 :\ \Lambda^+ = \begin{pmatrix} 1-\alpha & 0 \\ 0 & 1+\alpha \end{pmatrix}$, $\Lambda^- = 0$, $S^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $S^- = 0$
- $\alpha = 1 :\ \Lambda^+ = 1 + \alpha$, $\Lambda^- = 0$, $S^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $S^- = 0$
- $\alpha > 1 :\ \Lambda^+ = 1+\alpha$, $\Lambda^- = 1-\alpha$, $S^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $S^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

We will not consider the case $\alpha = 1$. When $0 < \alpha < 1$ we will only have positive eigenvalues and thus the characteristic and non-characteristic boundary conditions will coincide. Hence we will only investigate the case of $\alpha > 1$.

One way to proceed would be to derive the conditions of D_0 and D_1 for which the matrices of (3.4) and (4.8) are positive/negative semi-definite. Instead we have chosen to consider an alternative derivation of stability and well-posedness in appendix A-B. Here it is proven that a sufficient condition on D_0 and D_1 for well-posedness and stability is

$$|D_0||D_1| \leq 1. \tag{6.3}$$

Chapter 7

Analysis of the energy estimates

In this section we consider the energy estimates for the different boundary conditions and compare these. We also analyze specific types of data.

7.1 The estimates

First consider the general case of non-characteristic boundary conditions (i.e. either D_0 or or D_1 are non-zero). A first estimate are obtained by inserting the boundary conditions (3.2) into the general estimate (3.1) and then taking the expectation

$$\begin{aligned}&\frac{\partial}{\partial t}\mathbb{E}\left[\|w\|_2^2\right] + (\lambda^+ + \lambda^- D_1^2)\mathbb{E}\left[(w_1^+)^2\right] - (\lambda^- + D_0^2\lambda^+)\mathbb{E}\left[(w_0^-)^2\right] = \\ &2\lambda^+ D_0\mathbb{E}\left[w_0^-\delta g_0\right] - 2\lambda^- D_1\mathbb{E}\left[w_1^+\delta g_1\right] + \lambda^+\mathbb{E}\left[\delta g_0^2\right] - \lambda^-\mathbb{E}\left[\delta g_1^2\right].\end{aligned} \tag{7.1}$$

By then extracting δg_0 and δg_1 from the boundary conditions and inserting these into the mixed terms above we get

$$\begin{aligned}&\frac{\partial}{\partial t}\mathbb{E}\left[\|w\|_2^2\right] + (\lambda^+ - \lambda^- D_1^2)\mathbb{E}\left[(w_1^+)^2\right] - (\lambda^- - D_0^2\lambda^+)\mathbb{E}\left[(w_0^-)^2\right] = \\ &2\lambda^+ D_0\mathbb{E}\left[w_0^- w_0^+\right] - 2\lambda^- D_1\mathbb{E}\left[w_1^+ w_1^-\right] + \lambda^+\mathbb{E}\left[\delta g_0^2\right] - \lambda^-\mathbb{E}\left[\delta g_1^2\right].\end{aligned} \tag{7.2}$$

Now take the average of these two to get

$$\frac{\partial}{\partial t}\mathbb{E}\left[\|w\|_2^2\right] + \lambda^+\mathbb{E}\left[(w_1^+)^2\right] - \lambda^-\mathbb{E}\left[(w_0^-)^2\right] =$$
$$\lambda^+ D_0\mathbb{E}\left[w_0^-(w_0^+ + \delta g_0)\right] - \lambda^- D_1\mathbb{E}\left[w_1^+(w_1^- + \delta g_1)\right] + \lambda^+\mathbb{E}\left[\delta g_0^2\right] - \lambda^-\mathbb{E}\left[\delta g_1^2\right]. \tag{7.3}$$

For the characteristic boundary conditions we set $D_0 = D_1 = 0$ in any of the estimates (7.1)-(7.3) to get

$$\frac{\partial}{\partial t}\mathbb{E}\left[\|w\|_2^2\right] + \lambda^+\mathbb{E}\left[(w_1^+)^2\right] - \lambda^-\mathbb{E}\left[(w_0^-)^2\right] = \lambda^+\mathbb{E}\left[\delta g_0^2\right] - \lambda^-\mathbb{E}\left[\delta g_1^2\right]. \tag{7.4}$$

Now we will compare (7.4) to (7.1)-(7.3) for some different types of data. In particular we will consider the influence on $\frac{\partial}{\partial t}\mathbb{E}\left[\|w\|_2^2\right]$, i.e. the decay of the norm of the variance.

7.2 Zero variance on the boundary

If $\delta g_0 = \delta g_1 = 0$ we see from (7.1) compared to (7.4) that we should, regardless of the choices of D_0 and D_1, get a higher variance for the non-characteristic boundary conditions. The reason for this is that the norm of the variance decays by $\lambda^-\mathbb{E}\left[(w_0^-)^2\right] - \lambda^+\mathbb{E}\left[(w_1^+)^2\right]$ for the characteristic boundary conditions, whereas it decays by $(\lambda^- + D_0^2\lambda^+)\mathbb{E}\left[(w_0^-)^2\right] - (\lambda^+ + D_1^2\lambda^-)\mathbb{E}\left[(w_1^+)^2\right]$ in the non-characteristic case. These are thus similar up to the constants multiplying the boundary terms, which are larger in the characteristic case.

Initially we know that variance on the boundary are equal in both cases and thus we see that the decay for the characteristic boundary conditions are larger. A reasonable assumption is that the decay of variance are somewhat equally spread out through space. This mean that the norm of the variance decays exponentially in both cases, but with a larger speed in the characteristic case.

7.3 Small or decaying variance on the boundary

Now consider the case when the variance of the data boundary data is either considered small relative to the solution or is decaying. In this case we will compare (7.4) to (7.3). The difference between these two is that the estimate for the non-characteristic conditions has the extra terms $\lambda^+ D_0 \mathbb{E}\left[w_0^-(w_0^+ + \delta g_0)\right]$ and $-\lambda^- D_1 \mathbb{E}\left[w_1^+(w_1^- + \delta g_1)\right]$. We thus have to establish how these effect the decay of the norm.

For short times, the terms in the non-characteristic estimate will be well approximated by the initial data. We thus use the approximation

$$\begin{aligned} \mathbb{E}\left[w_0^-(w_0^+ + \delta g_0)\right] &\approx 2\mathbb{E}\left[\delta f^+(0,\xi)\delta f^-(0,\xi)\right] - D_0 \mathbb{E}\left[\left(\delta f^-(0,\xi)\right)^2\right] \\ \mathbb{E}\left[w_1^+(w_1^- + \delta g_1)\right] &\approx 2\mathbb{E}\left[\delta f^+(1,\xi)\delta f^-(1,\xi)\right] - D_1 \mathbb{E}\left[\left(\delta f^+(1,\xi)\right)^2\right]. \end{aligned} \tag{7.5}$$

By calculating these we can determine whether $\lambda^+ D_0 \mathbb{E}\left[w_0^-(w_0^+ + \delta g_0)\right] - \lambda^- D_1 \mathbb{E}\left[w_1^+(w_1^- + \delta g_1)\right]$ is positive or negative. If it is positive it means that that the decay of the norm of the variance becomes smaller in the non-characteristic case.

The effect of each of these terms on the variance is solely determined by the corresponding correlation at the initial time. The reason for this is that the correlation determines the sign of term which in turns describes whether the term makes the variance decay or increase. For example consider a case when $D_0 > 0$. Then a positive correlation between $\delta f^-(0,\xi)$ and $\delta f^+(0,\xi)$ implies that the corresponding term $\lambda^+ D_0 \mathbb{E}\left[\delta f^-(0,\xi)\delta f^+(0,\xi)\right]$ makes the norm of the variance increase.

For long times (by the assumption in this section), the variance from the data is either small or decaying. The variance of the solution will thus be much larger then the variance on the boundary and hence we can approximate the terms in the non-characteristic estimate as

$$\begin{aligned} &\lambda^+ D_0 \mathbb{E}\left[w_0^-(w_0^+ + \delta g_0)\right] - \lambda^- D_1 \mathbb{E}\left[w_1^+(w_1^- + \delta g_1)\right] \\ &\approx \lambda^+ D_0 \mathbb{E}\left[w_0^- w_0^+\right] - \lambda^- D_1 \mathbb{E}\left[w_1^+ w_1^-\right]. \end{aligned} \tag{7.6}$$

The small variance of the boundary data also allows us to infer from the boundary conditions that

$$w_0^+ = D_0 w_0^- + \delta g_0 \approx D_0 w_0^-, \qquad w_1^- = D_1 w_1^+ + \delta g_1 \approx D_1 w_1^+. \tag{7.7}$$

As seen in (7.7) the magnitude of the correlation between w_0^- and w_0^+ becomes close to one with the same sign as D_0 and similar for w_1^- and w_1^+. By combining (7.6) and (7.7) we get that

$$\begin{aligned} &\lambda^+ D_0 \mathbb{E}\left[w_0^-(w_0^+ + \delta g_0)\right] - \lambda^- D_1 \mathbb{E}\left[w_1^+(w_1^- + \delta g_1)\right] \\ &\approx \lambda^+ D_0^2 \mathbb{E}\left[(w_0^-)^2\right] - \lambda^- D_1^2 \mathbb{E}\left[(w_1^+)^2\right]. \end{aligned} \tag{7.8}$$

As seen above we get the same situation as in the case when $\delta g_0 = \delta g_1 = 0$. Hence the characteristic boundary conditions will produce a smaller variance than the non-characteristic for long times.

7.4 Large variance on the boundary

If the variance of the boundary data is large, no general conclusion can be drawn since terms of type $\mathbb{E}\left[w_i^\pm g_i\right]$ will dominate.

7.5 Noise process

In this thesis we have so far only considered the case when ξ is a single stochastic variable. However in practice it is more common, when ξ represent some noise, to assume that it is a continuous-time stochastic process. The most general stochastic process to model noise is the white noise process [10], W_t. White noise has the properties [10]

1. $t_1 \neq t_2 \Rightarrow W_{t_1}$ and W_{t_2} are independent.

2. W_t is stationary.
3. $\mathbb{E}[W_t] = 0, \forall t$.

If we set $\xi = W_t$, the terms w and $g(t, \xi_t)$ becomes almost independent since the we know from property 1 above that W_t is independent of the past. An implication is thus that the terms that differs (7.1) from (7.4) on the right hand side are approximately

$$\begin{aligned} 2\lambda^+ D_0 \mathbb{E}\left[w_0^- \delta g_0\right] &\approx 2\lambda^+ D_0 \mathbb{E}\left[w_0^-\right] \mathbb{E}\left[\delta g_0\right] = 0, \\ -2\lambda^- D_1 \mathbb{E}\left[w_1^+ \delta g_1\right] &\approx -2\lambda^- D_1 \mathbb{E}\left[w_1^+\right] \mathbb{E}\left[\delta g_1\right] = 0. \end{aligned} \tag{7.9}$$

This means that now have exactly the same result as in the case when $\delta g_i = 0$, in which case the characteristic boundary conditions always gave a lower variance.

Chapter 8

Computer Simulations

The hypothesis that a sharper energy estimate implies a smaller variance will be tested in this section. We will investigate different types of data in the initial and the boundary conditions. Notice that the data needs to be chosen under the following conditions

$$\begin{cases} \delta g_0(0,\xi) = \mathcal{H}_0 \delta f(0,\xi) & t = 0, x = 0, \\ \delta g_1(0,\xi) = \mathcal{H}_1 \delta f(1,\xi) & t = 0, x = 1, \\ \mathbb{E}\left[\delta f(x,\xi)\right] = 0 & t = 0, \forall x, \\ \mathbb{E}\left[\delta g_i(t,\xi)\right] = 0 & \forall t, i = 0, 1. \end{cases} \tag{8.1}$$

The first two conditions state that the initial and boundary data must be compatible. The last two conditions state that the problem is defined such that the expected value is zero. In this section we will for simplicity assume that δf specify the initial condition for w (not for e as defined in section 2.1).

We will consider the four different combinations of D_0 and D_1

- $D_0 < 0$ and $D_1 < 0$
- $D_0 > 0$ and $D_1 < 0$
- $D_0 < 0$ and $D_1 > 0$
- $D_0 > 0$ and $D_1 > 0$.

For all cases, we will simulate the whole interval $t \in [0, 1]$

8.1 Method of comparison

To compare the variance between the different types of boundary conditions, we will use the Riemann sum approximation of the norm of the variance, i.e.

$$\begin{aligned} \mathbb{E}\left[\|e\|_2^2\right] &= \mathbb{E}\left[\|e^1\|_2^2 + \|e^2\|_2^2\right] = \sum_{i=1}^{2} \int_0^1 Var(u^i)dx \\ &\approx \Delta x \sum_{i=1}^{2}\sum_{j=1}^{n} Var(u_j^i) = \mathbb{E}_{\Delta x}\left[\|e\|_2^2\right]. \end{aligned} \tag{8.2}$$

This can be seen as a measure of the total amount of uncertainty that is currently in the system, which can easily be observed over time.

To get a single number to determine the implication of the variance on the system we will integrate (8.2) over time, i.e.

$$\int_0^T \mathbb{E}\left[\|e\|_2^2\right] dt = \int_0^T \sum_{i=1}^{2} \int_0^1 Var(u^i)dx \approx \Delta t \sum_{l=0}^{N\Delta t} \left(\mathbb{E}_{\Delta x}\left[\|e\|_2^2\right]\right). \tag{8.3}$$

8.2 Choices

We will only consider the case of $\alpha = 2$ in (6.1). The means that we will have one positive and one negative eigenvalue. The constants D_0 and D_1 in the boundary conditions will in our case obviously be scalars. We know from (6.3) that they need to satisfy $|D_0|D_1| \leq 1$ and thus we have chosen the values to be $D_0 = \pm 0.5$ and $D_1 \pm 1.5$.

8.3 Periodic and decaying variance

For this case we choose the initial data to be

$$\delta f^+(x,\xi) = 3\xi^3, \qquad \delta f^-(x,\xi) = 3\xi. \tag{8.4}$$

The boundary data for the characteristic boundary conditions are chosen to be

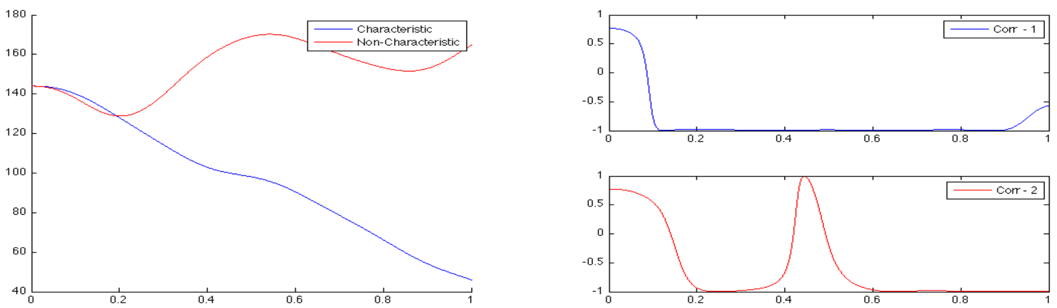
$$\begin{aligned} \delta g_0(t,\xi) &= 3e^{-0.5t}\sin(2\pi t)\xi^3, \\ \delta g_1(t,\xi) &= 3e^{-0.5t}\sin(4\pi t)\xi. \end{aligned} \tag{8.5}$$

To make the data for the non-characteristic boundary conditions satisfy (8.1) and be consistent with the data for the characteristic boundary conditions leads to

$$\begin{aligned} \delta g_0(t,\xi) &= 3e^{-0.5t}\sin(2\pi t)\xi^3 - 3D_0 e^{-0.5t}\sin(4\pi t)\xi, \\ \delta g_1(t,\xi) &= 3e^{-0.5t}\sin(4\pi t)\xi - 3D_1 e^{-0.5t}\sin(2\pi t)\xi^3, \end{aligned} \tag{8.6}$$

for the non-characteristic boundary conditions.

The figures below show the norm for a specific non-characteristic boundary condition compared to the characteristic boundary condition. We also plot the correlation at the boundaries between the characteristic variables for each type of non-characteristic condition. Note that the scales in the correlation plots are different.



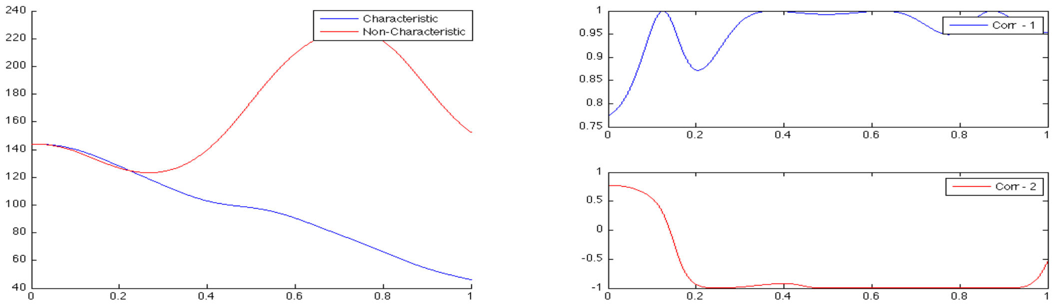
(a) The norm of the variances as a function of time.

(b) The correlation between the characteristic variables at the left (upper figure) and right (lower figure) boundary for the non-characteristic boundary condition as a function of time.

Figure 8.1: Characteristic boundary conditions compared to non-characteristic with $D_0 = -0.5$ and $D_1 = -1.5$

	Char.	$D_0 < 0$ $D_1 < 0$	$D_0 > 0$ $D_1 < 0$	$D_0 < 0$ $D_1 > 0$	$D_0 > 0$ $D_1 > 0$
$\int_0^T \mathbb{E}\left[\|e\|_2^2\right] dt$	97.00	152.22	169.29	189.11	170.53

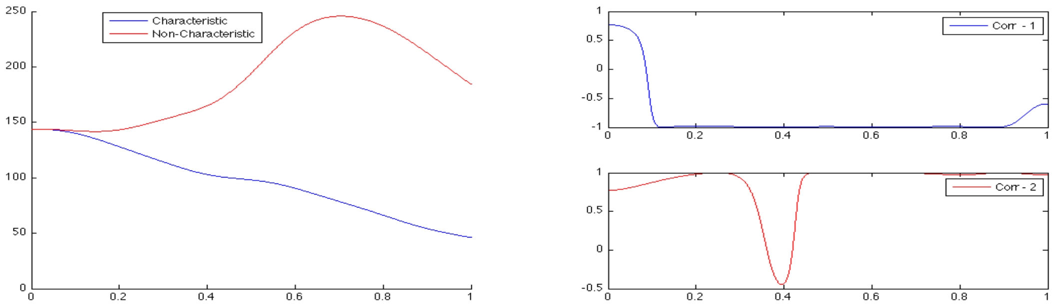
Table 8.1: The total norm of the variance in the case of periodic and decaying boundary data variance.



(a) The norm of the variances as a function of time.

(b) The correlation between the characteristic variables at the left (upper figure) and right (lower figure) boundary for the non-characteristic boundary condition as a function of time.

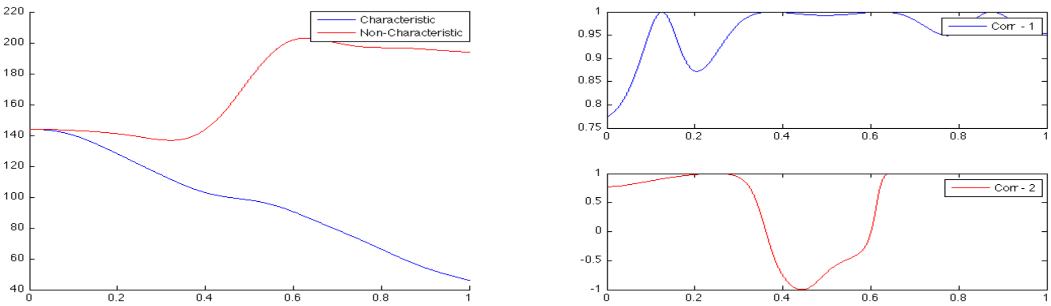
Figure 8.2: Characteristic boundary conditions compared to non-characteristic with $D_0 = 0.5$ and $D_1 = -1.5$



(a) The norm of the variances as a function of time.

(b) The correlation between the characteristic variables at the left (upper figure) and right (lower figure) boundary for the non-characteristic boundary condition as a function of time.

Figure 8.3: Characteristic boundary conditions compared to non-characteristic with $D_0 = -0.5$ and $D_1 = 1.5$



(a) The norm of the variances as a function of time.

(b) The correlation between the characteristic variables at the left (upper figure) and right (lower figure) boundary for the non-characteristic boundary condition as a function of time.

Figure 8.4: Characteristic boundary conditions compared to non-characteristic with $D_0 = 0.5$ and $D_1 = 1.5$

It is clear from Table 8.1 that the characteristic boundary condition gives the lowest variance. However as seen in the figures initially some of the non-characteristic boundary yield a lower variance. This was as explained in section 7.3 and it is due to initial correlations between the data.

From the figures the initial correlation between the solutions at boundaries is observed to be 0.77. The correlation is the same at both boundaries due to the fact that the initial uncertainty is constant in space. From 7.3 and more specifically the analysis below (7.5) we can conclude that the case of $D_0 < 0$ and $D_1 < 0$ should give the highest variance and the case of $D_0 > 0$ and $D_1 > 0$ the lowest among the non-characteristic cases. From table 8.1 we see that the case of $D_0 < 0$ and $D_1 < 0$ gives the lowest and the that case of $D_0 > 0$ and $D_1 > 0$ only gives the second largest variance.

This can by explained by figure 8.4b where we see that correlation at the right boundary changes sign for a long time in the middle of the interval, compared to the other cases where this does not happens. This is most certainly due to the fact that the boundary data (8.6) has different forms for the different cases and in this particular case it becomes large at the particular time.

For long times we see that the characteristic boundary conditions yield a lower variance just as predicted.

8.4 Perfect boundary knowledge

Here we choose the initial data such that the values on the boundary are equal to zero. We have

$$\delta f^{+}(x,\xi) = 2\sin(2\pi x)\xi^3, \qquad \delta f^{-}(x,\xi) = -3\sin(2\pi x)\xi. \qquad (8.7)$$

This allows us to set the boundary functions to

$$\delta g_0(t,\xi) = 0, \qquad \delta g_1(t,\xi) = 0. \qquad (8.8)$$

Table 8.2 shows the total norm of the variance for the different cases. In figure 8.5 we have plotted the norm of the variance for the characteristic boundary conditions and the case of $D_0 < 0$ and $D_1 < 0$. The plots of norm of the variance for other cases looks almost exactly the same as in the case$D_0 < 0$ and $D_1 < 0$ and are thus omitted.

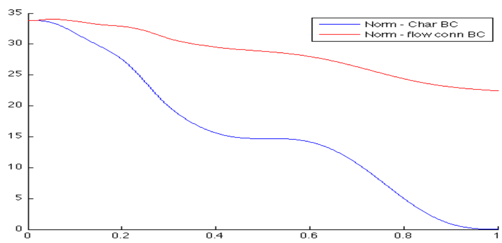


Figure 8.5: The norm of the variance as a function of time for characteristic boundary conditions compared to non-characteristic boundary conditions with $D_0 = -0.5$ and $D_1 = -1.5$.

	Char.	$D_0 < 0$ $D_1 < 0$	$D_0 > 0$ $D_1 < 0$	$D_0 < 0$ $D_1 > 0$	$D_0 > 0$ $D_1 > 0$
$\int_0^T \mathbb{E}\left[\|e\|_2^2\right] dt$	15.75	28.61	28.61	28.61	28.62

Table 8.2: The total norm of the variance in the case of a perfect boundary knowledge of the boundary data.

In this case, the characteristic boundary conditions gets a significantly lower variance compared to all the different non-characteristic boundary conditions. We also see that the variance for the non-characteristic boundary conditions are independent of D_0 and D_1. These result are exactly those expected from the analysis in section 7.1, since there we concluded that the characteristic boundary conditions will give a lower variance. We also see that the decay for the characteristic case is significantly larger.

Chapter 9

Conclusions

By studying the energy estimates for different boundary conditions to a hyperbolic PDE, we first showed that these have a connection to the variance of the solution. The idea that the relative size of the variance can be inferred from the energy estimate was first derived by analysis and intuition. This was then verified by numerical computations.

Depending on the situation for the initial and boundary data, we show that in all cases, the characteristic boundary conditions gives a smaller variance then non-characteristic conditions.

Depending on the structure of the data, one might be able to a priori choose the boundary conditions to minimize the variance due to correlation between the data and the solution. This may also involve a tradeoff between minimizing the variance for short or long times.

We also discussed how the same idea as exploited above can be extended to the more common practical case of a noise process. Here we believe the characteristic boundary conditions would be even more superior, in terms of low variance, than in the case of a single stochastic variable.

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Appendix A

Well-posedness

Here we use a similar approach as in [2]. The trick is to show that a positive linear combination of the component wise norm of w is bounded, which in turn also implies that $\|w\|_2^2$ itself is bounded. We will use the positive definite diagonal matrix B and also C defined as

$$B = \begin{pmatrix} |D_1|(\Lambda^+)^{-1} & 0 \\ 0 & -|D_0|(\Lambda^-)^{-1} \end{pmatrix}, C = \begin{pmatrix} |D_1|I_m & 0 \\ 0 & -|D_0|I_{N-m} \end{pmatrix}. \quad (A.1)$$

The matrix C will also be used in this approach. As the matrix norm $|\cdot|$ we use

$$|G| = \rho(G^T G) \quad (A.2)$$

where G is a matrix and $\rho(\cdot)$ is the spectral radius of a matrix [6]. The reason why this the linear combination defined by B is used is that it removes all the dependence of Λ from which D_0 and D_1 for which the problem is well-posed. Our starting point is to consider (2.3) in the characteristic variables w and multiply it by B to get

$$Bw_t = Cw_x. \quad (A.3)$$

By then multiplying (A.3) by w^T and integrate over x we get

$$\frac{\partial}{\partial t}\|B^{1/2}w\|_2^2 = -w^T Cw\Big|_{x=0}^{x=1}. \quad (A.4)$$

Just as in [2] we set $g^0 = g^1 = 0$ to simplify the analysis, without any loss of generality. By inserting (3.2) in (A.4) we therefore get

$$\frac{\partial}{\partial t}\|B^{1/2}w^-\|_2^2 = (w_1^+)^T(|D_0|D_1^T D_1 - |D_1|I)w_1^+ + (w_0^-)^T(|D_1|D_0^T D_0 - |D_0|I)w_0^-. \tag{A.5}$$

If both the constant matrices on the left hand side in the estimate (A.5) are negative semi-definite, the equation may be reduce to

$$\frac{\partial}{\partial t}\|B^{1/2}w\|_2^2 \leq 0. \tag{A.6}$$

In that case it follows that the problem is well-posed. The conditions on D_0 and D_1 are thus

$$\begin{aligned} |D_0|D_1^T D_1 - |D_1|I \leq 0 \\ |D_1|D_0^T D_0 - |D_0|I \leq 0. \end{aligned} \tag{A.7}$$

As in [2] we observe that the matrix products can be approximated as $D_i^T D_i I \leq |D_i|^2 I$. When this is applied to the conditions (A.7), both of these becomes implies that

$$|D_0||D_1| \leq 1. \tag{A.8}$$

To summarize; we have shown that (2.3) is well-posed under the boundary conditions (3.2) if the matrices D_0 and D_1 satisfies the condition (A.8).

Appendix B

Stability

The semi-discrete formulation of (A.3) with the boundary conditions (3.2) becomes

$$
\begin{aligned}
&(I \otimes B)w_t + (P^{-1}Q \otimes C)w = \\
&(P^{-1} \otimes I)(E_0 \otimes \Sigma_0)(w^+ - D_0 w^-) + (P^{-1} \otimes I)(E_1 \otimes \Sigma_1)(w^- - D_1 w^+).
\end{aligned} \tag{B.1}
$$

We now multiply (B.1) with $w^T(P \otimes I)$

$$
\begin{aligned}
&w^T(P \otimes B)w_t + w^T(Q \otimes C)w = \\
&w^T(E_0 \otimes \Sigma_0)(w^+ - D_0 w^-) + w^T(E_1 \otimes \Sigma_1)(w^- - D_1 w^+).
\end{aligned} \tag{B.2}
$$

Next add the transpose of the whole equation (B.2) to itself,

$$
\begin{aligned}
&\frac{\partial}{\partial t}\|w\|^2_{(P \otimes B)} + w^T(Q + Q^T \otimes C)w = \\
&2w^T(E_0 \otimes \Sigma_0)(w^+ - D_0 w^-) + 2w^T(E_1 \otimes \Sigma_1)(w^- - D_1 w^+).
\end{aligned} \tag{B.3}
$$

By now using the SBP property (4.1) we get

$$
\begin{aligned}
&\frac{\partial}{\partial t}\|w\|^2_{(P \otimes B)} + w_1^T C w_1 - w_0^T C w_0 = \\
&2w_0^T \Sigma_0 (w_0^+ - D_0 w_0^-) + 2w_1^T \Sigma_1 (w_1^- - D_1 w_1^+).
\end{aligned} \tag{B.4}
$$

Let the penalty matrices be $\Sigma_0 = -\tau|D_1|\begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}$ and $\Sigma_1 = -\tau|D_0|\begin{pmatrix} 0 & 0 \\ 0 & I_{N-m} \end{pmatrix}$ respectively, where τ is a scalar penalty parameter, so that

$$\frac{\partial}{\partial t}\|w\|^2_{(P\otimes B)} + |D_1|(w_1^+)^T w_1^+ - |D_0|(w_1^-)^T w_1^- - |D_1|(w_0^+)^T w_0^+ + |D_0|(w_0^-)^T w_0^- = \\ -2\tau|D_1|(w_0^+)^T(w_0^+ - D_0 w_0^-) - 2\tau|D_0|(w_1^-)^T(w_1^- - D_1 w_1^+). \tag{B.5}$$

To ease the notation we denote the discrete vector norm by $|w| = \sqrt{w^T w}$. Now use the same approximation as in [2] to approximate the cross terms on the right hand side in (B.5) as

$$(w_0^+)^T D_0 w_0^- \leq |w_0^+||D_0||w_0^-|, \qquad (w_1^-)^T D_1 w_1^+ \leq |w_1^-||D_1||w_1^+|. \tag{B.6}$$

When these approximations are applied to the corresponding terms in (B.5) we get

$$\frac{\partial}{\partial t}\|w\|^2_{(P\otimes B)} + |D_1||w_1^+|^2 + |D_0||w_1^-|^2(2\tau - 1) + |D_1||w_0^+|^2(2\tau - 1) \\ + |D_0||w_0^-|^2 - 2\tau|D_1||w_0^+||D_0||w_0^-| - 2\tau|D_0||w_1^-||D_1||w_1^+| \leq 0. \tag{B.7}$$

By completion of squares on the terms in (B.7) it becomes equivalent to

$$\frac{\partial}{\partial t}\|w\|^2_{(P\otimes B)} + |D_1|\left(|w_1^+|^2 - |D_0||w_1^-|\right)^2 + |w_1^-|^2\left(|D_0|(2\tau - 1) - \tau^2|D_0|^2|D_1|\right) \\ + |D_0|\left(|w_0^-| - |D_1||w_0^+|\right)^2 + |w_0^+|^2\left(|D_1|(2\tau - 1) - \tau^2|D_0||D_1^2|\right) \leq 0. \tag{B.8}$$

The condition for stability is that the norm of w is bounded. From equation (B.8) we see that if the constants multiplying the boundary terms are positive, the derivate of the norm is negative and hence bounded. Thus we get the following conditions on the constants

$$|D_0|(2\tau - 1) - \tau^2|D_0|^2|D_1| \geq 0 \\ |D_1|(2\tau - 1) - \tau^2|D_0||D_1|^2 \geq 0. \tag{B.9}$$

More explicit we see that both of the conditions in (B.9) are equivalent to $2\tau - 1 \geq \tau^2|D_0||D_1|$. Since we know from the stability analysis that $|D_0||D_1| \leq 1$, it is possible to factorize the previous condition as

$$\left(\tau - \frac{1-\sqrt{1-|D_0||D_1|}}{|D_0||D_1|}\right)\left(\tau - \frac{1+\sqrt{|D_0||D_1|}}{|D_0||D_1|}\right) \leq 0. \tag{B.10}$$

Out of (B.10) we see that the problem is stable when τ satisfies

$$\frac{1-\sqrt{1-|D_0||D_1|}}{|D_0||D_1|} \leq \tau \leq \frac{1+\sqrt{1-|D_0||D_1|}}{|D_0||D_1|}. \tag{B.11}$$

Appendix C

Probability theory

As this thesis investigate the behavior of function under subject to uncertainty, we here clarify some basic facts of probability theory.

C.1 Probability spaces, stochastic variables and processes

A probability space is defined as a measurable space $(\Omega, \mathcal{F}, P)$, where Ω is the sample space, $\mathcal{F}$ is a corresponding sigma-algebra and P is a probability measure [3]. A probability measure, P, satisfies all the same properties as a usual measure beside the normalization property, $\int_\Omega dP = 1$

Given a probability space $(\Omega, \mathcal{F}, P)$, a stochastic variable $X(\omega)$ is a measurable function from Ω to $\mathbb{R}$, i.e. $\forall c \in \mathbb{R}$ it holds that $\{\omega : X(\omega) \leq c\} \in \mathcal{F}$ [3]. For practical applications one often considers stochastic process, which simply is a collection of random variables. So for example one could for example define a process $X(\omega, t)$ for all $t \geq 0$, which would simply mean the collection $\{X(\omega, t), t \geq 0\}$ commonly written as $\{X_t\}$.

C.2 Expectation and moments of a stochastic variable

The expectation of a random variable, $X(\omega)$ is defined as

$$\mathbb{E}\left[X(\omega)\right] = \int_{\Omega} X(\omega) dP(\omega) \tag{C.1}$$

Similarly the k^{th} moment of a stochastic variable, $X(\omega)$, is defined as

$$\mathbb{E}\left[X(\omega)^k\right] = \int_{\Omega} X(\omega)^k dP(\omega) \tag{C.2}$$

The most common higher moment in practice is the second centered moment, called the variance. To center a moment means to consider the variable minus the mean instead of the variable itself. Hence the variance of $X(\omega)$ is calculated as

$$Var\left[X(\omega)\right] = \int_{\Omega} (X(\omega) - \mathbb{E}\left[X(\omega)\right])^2 dP(\omega) \tag{C.3}$$

Appendix D

Hermite Polynomials

We will here let $H_k(\cdot)$ denote the k^{th} Hermite polynomial. This polynomial is the defined as the solution to the ordinary differential equation (ODE) [8]

$$\frac{d}{dx}\left(e^{-\frac{x^2}{2}}\frac{dH_k}{dx}(x)\right) + ke^{-\frac{x^2}{2}}H_k(x) = 0. \tag{D.1}$$

The only boundary condition to the ODE is that the solution is polynomially bounded at infinity and that the first Hermite polynomial is equal to a constant 1. Hence the solution (D.1) can be found recursively by

$$\begin{cases} H_{k+1}(x) = xH_k(k) - \dfrac{dH_k}{dx}(x) \\ H_0 = 1. \end{cases} \tag{D.2}$$

So the first polynomials looks like

$$\begin{aligned} &H_0(x) = 1, \\ &H_1(x) = x, \\ &H_2(x) = x^2 - 1, \\ &H_3(x) = x^3 - 3x, \\ &H_4(x) = x^4 - 6x^2 + 3, \\ &\vdots \end{aligned} \tag{D.3}$$