

Examensarbete

**Stability of SBP schemes on overlapping domains**

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# Stability of SBP schemes on overlapping domains

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# Abstract

Using finite difference methods for partial differential equations, this thesis focuses on the problem of connecting overlapping solution domains in the context of a first order hyperbolic problem. Especially the stability properties of such constructions is studied, and a stable general implementation of the the test problem is proposed. However, no energy estimate could be found, and indeed proven not to exist in the natural norm. Finally, an example is also put forward where the interface conditions derived are, for stability considerations, incompatible with the boundary conditions in a coupled system of hyperbolic equations.

**Keywords:** Summation-by-parts, Simultaneous-approximation-term, overlapping domains, stability, energy stability, weak constraints, interpolation conditions, accuracy, energy conserving system, numerical simulations.

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# Nomenclature

## Symbols

- $M$  System matrix.
- $A$  Energy matrix.
- $\mathcal{A}$  Interface matrix.

## Abbreviations

- FD Finite Difference
- SBP Summation-By-Parts
- SAT Simultaneous-Approximation-Term
- ODE Ordinary Differential Equation
- PDE Partial Differential Equation
- IBVP Initial and Boundary Value Problem





# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	The scalar test problem . . . . .	1
1.2	Semidiscrete approximation . . . . .	1
1.3	Stability and energy stability . . . . .	2
1.4	One-domain SBP formulation . . . . .	3
<b>2</b>	<b>Two-domain formulation</b>	<b>5</b>
2.1	Example of interface implementation . . . . .	5
2.2	General interface implementation . . . . .	6
2.3	Consistency . . . . .	7
<b>3</b>	<b>Stability</b>	<b>9</b>
3.1	Energy stability in the natural norm . . . . .	9
3.2	Energy stability in other norms . . . . .	14
3.3	Conventional stability . . . . .	15
3.3.1	Example - second order accuracy . . . . .	16
3.3.2	Arbitrary accuracy . . . . .	18
<b>4</b>	<b>A coupled system of hyperbolic equations</b>	<b>21</b>
4.1	Single domain . . . . .	21
4.2	Two domains . . . . .	23
4.3	Conclusion . . . . .	25
<b>5</b>	<b>Numerical tests</b>	<b>27</b>
5.1	Single equation . . . . .	27
5.2	Coupled system . . . . .	30
<b>6</b>	<b>Conclusions</b>	<b>35</b>



# Chapter 1

## Introduction

Numerical treatment of IBVP:s for time dependent equations is generally a task of constructing a stable semidiscrete approximation to an original continuous problem. Using FD methods, the solution domain is represented as a grid with discrete points, and all spatial derivatives are replaced with difference approximations. This results in a system of ODE:s, to be solved using standard numerical techniques, such as the Runge-Kutta class of methods. Sometimes it is convenient to split the solution domain and apply the standard FD methods on the respective, sometimes overlapping, parts separately. In this thesis, general overlapping two-domain techniques are studied for a scalar first order IBVP, using the framework of SBP first-derivative operators. For the case of non-overlapping domains, an energy stable method for this problem was presented in [1].

### 1.1 The scalar test problem

The continuous scalar test problem, studied throughout most of this paper, is a one-dimensional advection IBVP (a right-moving wave):

$$\begin{aligned} U_t + U_x &= 0 & 0 \leq x \leq 1, t > 0 \\ U(x, 0) &= f(x) & 0 \leq x \leq 1 \\ U(0, t) &= g(t) & t > 0 \end{aligned} \tag{1.1}$$

### 1.2 Semidiscrete approximation

When using FD methods to discretize the continuous problem (1.1), the spatial derivative will be represented by a difference operator working on a discrete solution vector. The physical boundary condition and any domain boundaries can moreover be enforced by adding weak penalty terms, so called simultaneous-approximation-terms (SAT). The resulting semi-discrete problem is then a linear time-dependent system of ODE:s. We begin here by simply writing such a system in general terms as follows:

$$\mathbf{u}_t = M\mathbf{u} + \mathbf{b}(t), \tag{1.2}$$

Here,  $\mathbf{u}$  is the discrete solution vector. We call  $M$  the *system matrix*, specified by both the finite difference approximation of the spatial derivative and any weak impositions of interfaces and boundary conditions. The time dependent vector  $\mathbf{b}$  comes from the weak imposition of the boundary condition. We will exemplify all of these parts later, but first we will briefly address the concept of stability for linear ODE:s.

### 1.3 Stability and energy stability

A famous result in FD analysis states that a consistent method converges if and only if it has a stable solution [3]. Moreover, the solution to an ODE system like (1.2) is stable if and only if all eigenvalues of  $M$  is in the left half-plane [4]. A very useful (and often used in FD application) class of sufficient conditions for this is given by the following theorem:

**Theorem 1.3.1.** *Let  $M$  be any square matrix, and let  $H$  be a symmetric, positive definite matrix of the same dimension as  $M$ . Then, if the symmetric matrix  $HM + M^T H$  is negative semidefinite, it follows that all eigenvalues of  $M$  have non-positive real parts.*

*Proof.* Suppose that  $\lambda$  is an eigenvalue of  $M$ , and that  $\mathbf{v}$  is a corresponding eigenvector. We first partition both into their respective real and an imaginary parts:  $\lambda = \lambda_R + i\lambda_I$ , and  $\mathbf{v} = \mathbf{v}_R + i\mathbf{v}_I$ . Then by definition,

$$\begin{aligned} M\mathbf{v} &= M(\mathbf{v}_R + i\mathbf{v}_I) = (\lambda_R + i\lambda_I)(\mathbf{v}_R + i\mathbf{v}_I) = \lambda_R\mathbf{v}_R - \lambda_I\mathbf{v}_I + i(\lambda_R\mathbf{v}_I + \lambda_I\mathbf{v}_R) \\ &\Rightarrow M\mathbf{v}_R = \lambda_R\mathbf{v}_R - \lambda_I\mathbf{v}_I, \quad M\mathbf{v}_I = \lambda_R\mathbf{v}_I + \lambda_I\mathbf{v}_R. \end{aligned}$$

This gives, due to negative semidefiniteness:

$$\begin{aligned} \mathbf{v}_R^T(HM + M^T H)\mathbf{v}_R &= \mathbf{v}_R^T H(\lambda_R\mathbf{v}_R - \lambda_I\mathbf{v}_I) + (\lambda_R\mathbf{v}_R^T - \lambda_I\mathbf{v}_I^T)H\mathbf{v}_R = \\ &= 2\lambda_R\mathbf{v}_R^T H\mathbf{v}_R - \lambda_I\mathbf{v}_R^T H\mathbf{v}_I - \lambda_I\mathbf{v}_I^T H\mathbf{v}_R \leq 0 \quad (i), \end{aligned}$$

$$\begin{aligned} \mathbf{v}_I^T(HM + M^T H)\mathbf{v}_I &= \mathbf{v}_I^T H(\lambda_R\mathbf{v}_I + \lambda_I\mathbf{v}_R) + (\lambda_R\mathbf{v}_I^T + \lambda_I\mathbf{v}_R^T)H\mathbf{v}_I = \\ &= 2\lambda_R\mathbf{v}_I^T H\mathbf{v}_I + \lambda_I\mathbf{v}_I^T H\mathbf{v}_R + \lambda_I\mathbf{v}_R^T H\mathbf{v}_I \leq 0 \quad (ii). \end{aligned}$$

Adding (i) and (ii) together finally gives:

$$2\lambda_R(\mathbf{v}_R^T H\mathbf{v}_R) + \mathbf{v}_I^T H\mathbf{v}_I \leq 0 \Rightarrow \lambda_R \leq 0$$

□

This theorem also has a physical interpretation for the relation between (1.1) and (1.2). With  $g(t) = 0$  (implying  $\mathbf{b}(t) = \mathbf{0}$ ), the  $L_2$ -norm of the continuous

problem satisfies:

$$\begin{aligned} \frac{d}{dt} \|U\|_{L_2}^2 &= \frac{d}{dt} \int_0^1 U^2 dx = \int_0^1 \frac{d}{dt} U^2 dx = \int_0^1 2U_t U dx = \\ &= - \int_0^1 2U_x U dx = - \int_0^1 \frac{d}{dx} U^2 dx = U(0, t)^2 - U(1, t)^2 = -U(1, t)^2 \leq 0. \end{aligned}$$

In the discrete case, a corresponding relation can be found for a matrix induced norm of the solution vector. Let  $H$  be a symmetric, positive definite matrix. Then:

$$\frac{d}{dt} \|U\|_H^2 = \frac{d}{dt} \mathbf{u}^T H \mathbf{u} = 2\mathbf{u}^T H \mathbf{u}_t = 2\mathbf{u}^T H M \mathbf{u} = \mathbf{u}^T (H M + M^T H) \mathbf{u}$$

This means that if  $M$  and  $H$  meet the requirements of theorem 1.3.1, then  $\frac{d}{dt} \|U\|_H^2 \leq 0$ . Thus the induced norm imitates the behavior of the continuous norm in that the solution vector exhibits no non-physical growth. This property has led to an alternative definition of stability, called *energy stability*.

**Definition 1.3.1.** *The semi-discrete system (1.2) is called energy stable in the  $H$ -norm if the symmetric matrix  $A = MH + M^T H$  is negative semi-definite.*

We will henceforth refer to  $A = MH + M^T H$  as the *energy matrix* in the  $H$ -norm. Remember that, in view of theorem 1.3.1, energy stability in any norm implies stability in the conventional sense. However, energy stability is also a property that is very useful on its own, as it leads to specific estimates of convergence rate for the FD methods used [2].

## 1.4 One-domain SBP formulation

We begin by defining the spatial discretization and discrete solution vector for the one-domain case. The interval  $(0, 1)$  is divided using a finite number of equidistant points, and a solution is sought at each of these points:

$$\begin{aligned} \mathbf{x} &= (0 = x_1 \ x_2 \ \dots \ x_{N-1} \ x_N = 1)^T \\ \mathbf{u} &= (u_1 \ u_2 \ \dots \ u_{N-1} \ u_N)^T \end{aligned}$$

Using the SBP formulation, a mapping approximating the spatial derivative is defined by two matrices,  $P$  and  $Q$ , such as:

$$\mathbf{U}_x =: (U_x(x_1) \ \dots \ U_x(x_N))^T \approx P^{-1} Q \mathbf{U}$$

For example, an FD scheme using first order forward- and backward differences at the boundaries, and second order central differences in the interior, is given by the following matrix formulation:

$$P = \Delta x \begin{pmatrix} \frac{1}{2} & 0 & \dots & & \\ 0 & 1 & \ddots & & \\ \vdots & \ddots & \ddots & & \\ & & & 1 & \\ & & & & \frac{1}{2} \end{pmatrix}, \quad Q = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & \dots & \\ -\frac{1}{2} & 0 & \frac{1}{2} & \ddots & \\ 0 & \ddots & \ddots & \ddots & \\ \vdots & \ddots & -\frac{1}{2} & 0 & \frac{1}{2} \\ & & & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad (1.3)$$

where  $\Delta x$  is the grid size given by the distance between two neighbouring points in  $x$ . In general, the formal requirement of a  $p$ :th order SBP operator is given by the following properties:

- (i)  $P^{-1}Q\mathbf{U} = \mathbf{U}_x + \mathbf{T}$ , where  $T_i = O(\Delta x^p)$  for  $i = 1, \dots, N$
- (ii)  $Q$  is nearly skew-symmetric :  $Q + Q^T = \text{diag}(-1, 0, \dots, 0, 1)$ .
- (iii)  $P$  is symmetric, positive definite.

In this framework, a semidiscretization of (1.1) can be written:

$$P\mathbf{u}_t + Q\mathbf{u} = \sigma(u_1 - g(t))\mathbf{e}_1,$$

with the SAT treatment of the boundary condition on the right hand side. Due to the specific properties of the SBP formulation, stability can be assured in a straightforward manner using the concept of energy stability (see definition 1.3.1). The system matrix is here given by:

$$M = P^{-1}(\text{diag}(\sigma, 0, \dots) - Q)$$

Especially, we consider energy stability in the  $P$ -norm. The energy matrix is then the following:

$$A = PM + MP^T = \text{diag}(2\sigma, 0, \dots) - (Q + Q^T) = \text{diag}(2\sigma + 1, 0, \dots, 0, -1).$$

Thus, energy stability in the  $P$ -norm is fulfilled if  $\sigma \leq -1/2$ .

## Chapter 2

# Two-domain formulation

We now consider a case where two different, partially overlapping grids, are used to discretize (1.1). We define the grids as follows:

$$\mathbf{x} = (0 = x_1 \ x_2 \ \dots \ x_{L-1} \ x_L)^T, \ \mathbf{y} = (y_1 \ y_2 \ \dots \ y_{R-1} \ y_R = 1)^T, \quad (2.1)$$

with corresponding discrete solution vectors:

$$\mathbf{u} = (u_1 \ u_2 \ \dots \ u_{L-1} \ u_L)^T, \ \mathbf{v} = (v_1 \ v_2 \ \dots \ v_{R-1} \ v_R = 1)^T, \quad (2.2)$$

Suppose moreover that the two grids have maximum interval distances  $\Delta x$  and  $\Delta y$ , with some proportional bound  $\Delta y \leq C \Delta x$ . We assume that  $C$  is independent of  $\Delta x$ , which gives the identity  $O(\Delta y) = O(\Delta x)$ .

On each of the grids, a separate SBP formulation is applied to approximate the continuous equation. The boundary condition is implemented as before in the left grid. To connect the two grids, an interface treatment needs to be added. This can be done similarly to the boundary treatment using SAT terms, which will be exemplified.

### 2.1 Example of interface implementation

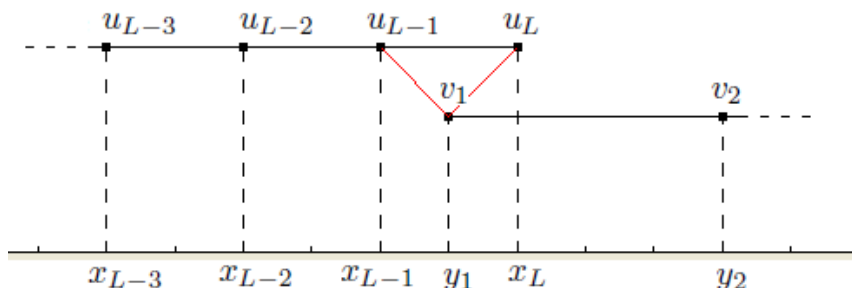
Consider the grid overlap in Figure 2.1. In order to connect the two FD methods, one can try to weakly impose a linear estimation for  $v_1$  based on points in the left grid. Remember that, if this can be done in a stable and consistent manner, convergence is assured. For example, let

$$\hat{v}_1 = \beta u_{L-1} + (1 - \beta)u_L, \quad \beta = \frac{x_L - y_1}{x_L - x_{L-1}}$$

Assuming that  $\mathbf{u}$  and  $\mathbf{v}$  are sampled from the exact solution of (1.1), then by Taylor's formula:

$$v_1 - \hat{v}_1 = v_1 - \beta u_{L-1} - (1 - \beta)u_L =$$

Figure 2.1:



$$\begin{aligned}
 &= v_1 - \beta(v_1 - U_x(x_{L-1}, t)(v_1 - u_{L-1})) - (1 - \beta)(v_1 + U_x(x_L, t)(x_L - v_1)) + O(\Delta x^2) = \\
 &= \dots = O(\Delta x^2).
 \end{aligned}$$

Guided by this, we can construct the following *consistent* implementation that connects two SBP treatments defined on the respective grids:

$$\begin{pmatrix} P_l & \mathbf{0} \\ \mathbf{0} & P_r \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}_t + \begin{pmatrix} Q_l & \mathbf{0} \\ \mathbf{0} & Q_r \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} \sigma(u_0 - g(t)) \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \tau(v_1 - \hat{v}_1) \\ \mathbf{0} \\ \vdots \end{pmatrix}, \quad (2.3)$$

where  $\tau$  is some penalty parameter. The question remaining is whether one can choose  $\tau$  such that the solution to (2.3) is stable. Before trying to answer this question, we will study a more general approach of imposing weak interface conditions of this kind.

## 2.2 General interface implementation

By weakly implementing linear estimations like in (2.3) to connect the two grids, the general formulation of the semidiscrete problem on two domains can be written as follows:

$$\begin{pmatrix} P_l & \mathbf{0} \\ \mathbf{0} & P_r \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}_t + \begin{pmatrix} Q_l & \mathbf{0} \\ \mathbf{0} & Q_r \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \mathcal{A} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} + \begin{pmatrix} \sigma(u_1 - g(t)) \\ \mathbf{0} \\ \vdots \end{pmatrix} \quad (2.4)$$



The interface treatment is here included in the matrix  $\mathcal{A}$ , and the boundary treatment is unchanged from the one-domain case. We call  $\mathcal{A}$  the *interface matrix*, and for future reference we define the elements and row vectors of this matrix as follows:

$$\mathcal{A} = \begin{pmatrix} a_1^1 & \cdots & a_{L+R}^1 \\ \vdots & & \vdots \\ a_1^{L+R} & \cdots & a_{L+R}^{L+R} \end{pmatrix}, \quad \mathbf{a}^i = (a_1^i \quad \cdots \quad a_{L+R}^i)^T \text{ for } i = 1, \dots, L+R. \quad (2.5)$$

Finally, we can also write down the system matrix of (2.4):

$$M = \begin{pmatrix} P_l^{-1} & \mathbf{0} \\ \mathbf{0} & P_r^{-1} \end{pmatrix} (\mathcal{A} + \text{diag}(\sigma, 0, \dots) - \begin{pmatrix} Q_l & \mathbf{0} \\ \mathbf{0} & Q_r \end{pmatrix}) \quad (2.6)$$

Remember that stability of the semidiscrete system is determined by the eigenvalues of  $M$ .

## 2.3 Consistency

In this section we derive a minimum requirement in order for the interface treatment (2.4) to preserve consistency. This will later be used when studying energy stability in the next chapter. Similar to the second order method exemplified in (1.3), SBP operators derived from FD methods must always be scaled with the grid size  $\Delta x$ . Using this FD scheme as an example, (2.4) can then be written out as follows:

$$\begin{aligned} (u_l)_t + (u_l)_x &= \frac{d_l}{\Delta x} (\mathbf{a}^l)^T \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} + O(\Delta x), & l = 1, \dots, L \\ (u_r)_t + (u_r)_x &= \frac{d_r}{\Delta y} (\mathbf{a}^{L+r})^T \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} + O(\Delta x), & r = 1, \dots, R \end{aligned}, \quad (2.7)$$

where  $d_l = 2$  for  $l = 1$  and  $l = L$ , and  $d_l = 1$  for all other  $l$  - and similarly for the right grid. This can be generalized for other (diagonal) forms of  $P$ . The order terms should then be interpreted as denoting *least* local order of accuracy for this class of methods (this order of accuracy is met with forward and backward differences at domain boundaries for the 2nd order FD method). In order to preserve consistency, i.e. at least first order accuracy of the system, we must thus demand that

$$(\mathbf{a}^i)^T \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = O(\Delta x^2), \quad \forall i = 1, \dots, L+R \quad (2.8)$$

Consider e.g. the right hand side of a row corresponding to the left grid:

$$(\mathbf{a}^l)^T \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = a_1^l u_1 + \cdots + a_L^l u_L + a_{L+1}^l v_1 + \cdots + a_{L+R}^l v_R \quad (2.9)$$

where  $1 \leq l \leq L$ . Expanding each term above using Taylor's formula around  $u_l$

gives:

$$\begin{aligned}
u_1 &= u_l + (u_l)_x(x_1 - x_l) + O((x_1 - x_l)^2) \\
&\vdots \\
u_L &= u_l + (u_l)_x(x_L - x_l) + O((x_L - x_l)^2) \\
v_1 &= u_l + (u_l)_x(y_1 - x_l) + O((y_1 - x_l)^2) \\
&\vdots \\
v_R &= u_l + (u_l)_x(y_R - x_l) + O((y_R - x_l)^2)
\end{aligned} \tag{2.10}$$

Inserting this into (2.9), we get:

$$\begin{aligned}
\mathbf{a}^l \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} &= u_l(a_1^l + \dots + a_L^l + a_{L+R}^l) + \\
&+ (u_l)_x((a_1^l(x_1 - x_l) + \dots + a_L^l(x_L - x_l) + a_{L+1}^l(y_1 - x_l) + \dots + a_{L+R}^l(y_R - x_l)) + \\
&+ O(a_1^l(x_1 - x_l)^2 + \dots + a_L^l(x_L - x_l)^2 + a_{L+1}^l(y_1 - x_l)^2 + \dots + a_{L+R}^l(y_R - x_l)^2).
\end{aligned} \tag{2.11}$$

Terms lower than second order can thus be avoided by setting:

$$\begin{aligned}
a_1^l + \dots + a_{L+R}^l &= 0 \\
a_1^l(x_1 - x_l) + \dots + a_{L+R}^l(y_R - x_l) &= 0 \quad .
\end{aligned}$$

By collecting the terms including  $x_l$  in the second equation and inserting into the first equation, this system can be simplified into the following:

$$\begin{aligned}
a_1^l + \dots + a_{L+R}^l &= 0 \\
a_1^l x_1 + \dots + a_{L+R}^l y_R &= 0 \quad .
\end{aligned}$$

This result leads us to make the following definition:

**Definition 2.3.1.** *The interface matrix  $\mathcal{A}$  in (2.5) is strictly of order 2, if the following holds:*

$$\begin{aligned}
a_1^i + \dots + a_{L+R}^i &= 0, & \forall 1 \leq i \leq L + R \\
a_1^i x_1 + \dots + a_L^i x_L + a_{L+1}^i y_1 + \dots + a_{L+R}^i y_R &= 0, & \forall 1 \leq i \leq L + R
\end{aligned} \tag{2.12}$$

Remember that, as we found above, this property is needed in order for the interface treatment to preserve consistency.

# Chapter 3

## Stability

Since stability as well as consistency is needed to guarantee convergence, what we would like is to find a strictly second order interface matrix  $\mathcal{A}$  such that the semi-discrete two-domain system (2.4) is *stable*. A common approach is to prove stability through energy stability in the natural norm. This is especially useful as it allows for using different types of SAT terms simultaneously, e.g. when solving coupled systems of hyperbolic equations. However, in Section 3.1 below we prove that energy stability in the natural norm is indeed not possible to achieve. Section 3.2 briefly addresses the problem of proving energy stability in other norms than the natural, while in section 3.3 we present an interface treatment that provably leads to stability in the conventional sense. Even though this is an encouraging result, the lack of energy stability will still lead to trouble when applying this interface treatment to an energy conserving system, a problem that is later exemplified in Chapter 4.

### 3.1 Energy stability in the natural norm

In order to show energy stability, we must find an interface matrix  $\mathcal{A}$  and a positive symmetric matrix  $H$  such that the energy matrix  $A$  is negative semi-definite, where  $A$  is given by:

$$\begin{aligned} A = HM + M^T H = H \begin{pmatrix} P_l^{-1} & \mathbf{0} \\ \mathbf{0} & P_r^{-1} \end{pmatrix} (\mathcal{A} + \text{diag}(\sigma, 0, \dots) - \begin{pmatrix} Q_l & \mathbf{0} \\ \mathbf{0} & Q_r \end{pmatrix}) + \\ + (\mathcal{A}^T + \text{diag}(\sigma, 0, \dots) - \begin{pmatrix} Q_l^T & \mathbf{0} \\ \mathbf{0} & Q_r^T \end{pmatrix}) \begin{pmatrix} P_l^{-T} & \mathbf{0} \\ \mathbf{0} & P_r^{-T} \end{pmatrix} H. \end{aligned} \quad (3.1)$$

In order to simplify this expression so that it is no longer dependent on the specific SBP treatment, a natural choice of norm to study is the following:

$$H = \begin{pmatrix} P_l & \mathbf{0} \\ \mathbf{0} & P_r \end{pmatrix}. \quad (3.2)$$

We call this choice the *natural norm*, and it now allows for the following simplification of the energy matrix  $A$ :

$$\begin{aligned} A &= \mathcal{A} + \mathcal{A}^T + \text{diag}(1, 0, \dots, -1, 1, 0, \dots, -1) + \text{diag}(2\sigma + 1, 0, \dots) = \\ &= \mathcal{A} + \mathcal{A}^T + \text{diag}(0, \dots, -1, 1, 0, \dots, 0) + \text{diag}(2\sigma + 1, 0, \dots, 0, \dots, -1). \end{aligned}$$

Moreover, assuming that the boundary treatment in itself is energy stable in this norm, see section 1.4, we can here ignore the corresponding terms and consider the following matrix including the contributions from the interface treatment:

$$\begin{aligned} \tilde{A} &= (\mathcal{A} + \mathcal{A}^T + \text{diag}(0, \dots, 0, -1, 1, 0, \dots, 0)) = \\ &= \begin{pmatrix} 2a_1^1 & a_2^1 + a_1^2 & \dots & & & a_{L+R}^1 + a_1^{L+R} \\ \vdots & & & & & \vdots \\ a_1^L + a_L^1 & \dots & 2a_L^L - 1 & & \dots & a_{L+R}^L + a_L^{L+R} \\ a_1^{L+1} + a_{L+1}^1 & \dots & & 2a_{L+1}^{L+1} + 1 & \dots & a_{L+R}^{L+1} + a_{L+1}^{L+R} \\ \vdots & & & & & \vdots \\ a_1^{L+R} + a_{L+R}^1 & \dots & & & a_{L+R-1}^{L+R} + a_{L+R}^{L+R-1} & 2a_{L+R}^{L+R} \end{pmatrix}. \end{aligned} \quad (3.3)$$

The next theorem states that this can not be achieved assuming that the interface matrix is strictly of second order, which was needed for consistency. We begin however with a lemma which gives a parametric formulation of the set of strictly second order interface matrices.

**Lemma 3.1.1.** *Let  $\mathcal{A}$  be an interface matrix of strictly second order. Then any row  $\mathbf{a}^i$  of  $\mathcal{A}$  can be written as*

$$(\mathbf{a}^i)^T = \mathbf{X} \mathbf{t}^i,$$

where

$$\mathbf{X} = \begin{pmatrix} x_3 - x_2 & \dots & x_L - x_2 & y_1 - x_2 & \dots & y_R - x_2 \\ -(x_3 - x_1) & \dots & -(x_L - x_1) & -(y_1 - x_1) & \dots & -(y_R - x_1) \\ x_2 - x_1 & 0 & \dots & & \dots & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & & & & \vdots \\ & & x_2 - x_1 & & & \vdots \\ & & & x_2 - x_1 & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \dots & & \dots & 0 & x_2 - x_1 \end{pmatrix}.$$

*Proof.* Consider the equation system (2.12), and subtract  $x_1$  times the first equation from the second. This gives:

$$\begin{aligned} a_1^i + \dots + a_{L+R}^i &= 0 \\ a_2^i(x_2 - x_1) + \dots + a_{L+R}^i(y_R - x_1) &= 0 \end{aligned}$$

Now introduce the following parametrization:

$$\begin{aligned} t_1^i &= \frac{a_3^j}{x_2 - x_1} \\ &\vdots \\ t_{L+R-2}^i &= \frac{a_{L+R}^i}{x_2 - x_1} \end{aligned}$$

This gives:

$$\begin{aligned} a_1^i &= (x_3 - x_2)t_1^i + \dots + (x_L - x_2)t_{L-2}^i + (y_1 - x_2)t_{L-1}^i + \dots + (y_R - x_2)t_{L+R-2}^i \\ a_2^i &= -(x_3 - x_1)t_1^i - \dots - (x_L - x_1)t_{L-2}^i - (y_1 - x_1)t_{L-1}^i - \dots - (y_R - x_1)t_{L+R-2}^i \\ a_3^i &= (x_2 - x_1)t_1^i \\ &\vdots \\ a_{L+R}^i &= (x_2 - x_1)t_{L+R-2}^i \end{aligned}$$

On matrix form, this can indeed be written as the lemma states:

$$(\mathbf{a}^i)^T = \begin{pmatrix} x_3 - x_2 & \dots & x_L - x_2 & y_1 - x_2 & \dots & y_R - x_2 \\ -(x_3 - x_1) & \dots & -(x_L - x_1) & -(y_1 - x_1) & \dots & -(y_R - x_1) \\ x_2 - x_1 & 0 & \dots & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & & & & \vdots \\ & & x_2 - x_1 & & & \vdots \\ & & & x_2 - x_1 & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & x_2 - x_1 \end{pmatrix} \begin{pmatrix} t_1^i \\ \vdots \\ t_{L+R-2}^i \end{pmatrix}.$$

□

**Theorem 3.1.1.** *Consider the system (2.4), and assume that  $x_L \neq y_1$ , i.e. that the grids do not meet without overlap. Then, if the interface matrix  $\mathcal{A}$  is strictly of second order, the system is not energy stable in the norm (3.2).*

*Proof.* We need to show that the symmetric matrix  $\tilde{A}$  as defined by (3.3) is strictly indefinite for  $x_L \neq y_1$ . We start by finding a rotation matrix  $R$  so that  $R^T \tilde{A} R$  has one diagonal element which is zero. This will give a set of sufficient condition for indefiniteness. A rotation matrix with this property is the following:

$$R = \begin{pmatrix} 1 & 0 & \dots & 0 & 1 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & & \\ 0 & \dots & & 0 & 1 \end{pmatrix} \Rightarrow R^T \tilde{A} R =$$

$$= \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \dots & \tilde{A}_{1,L+R-1} & c_1 \\ \vdots & & & & \vdots \\ \tilde{A}_{L+R-1,1} & \tilde{A}_{L+R-1,2} & \dots & \tilde{A}_{L+R-1,L+R-1} & c_{L+R-1} \\ c_1 & c_2 & \dots & c_{L+R-1} & d \end{pmatrix}, \quad (3.4)$$

where

$$\begin{aligned} c_i &= \tilde{A}_{i1} + \dots + \tilde{A}_{i,L+R} \\ d &= c_1 + \dots + c_{L+R} \end{aligned}$$

We will indeed soon find that  $d = 0$ . First, the  $c_i$  expression above can for  $i \neq L, L+1$  be written out and simplified as follows:

$$\begin{aligned} c_i &= \tilde{A}_{i1} + \dots + \tilde{A}_{i,L+R} = \\ &= a_1^i + a_i^1 + \dots + a_{L+R}^i + a_i^{L+R} = \\ &= (a_1^i + \dots + a_{L+R}^i) + (a_i^1 + \dots + a_i^{L+R}) = a_i^1 + \dots + a_i^{L+R}, \quad i \neq L, L+1. \end{aligned}$$

Note that the last equality above is due to the first property in (2.12). The two rows  $c_L$  and  $c_{L+1}$  only differs from this by an additional 1- or  $-1$ -term:

$$c_i = \begin{cases} a_i^1 + \dots + a_i^{L+R} & i \neq L, L+1 \\ a_i^1 + \dots + a_i^{L+R} - 1 & i = L \\ a_i^1 + \dots + a_i^{L+R} + 1 & i = L+1 \end{cases}$$

Now we can similarly write out  $d$ , rearrange the terms and once again use the first equation in (2.12) to simplify:

$$\begin{aligned} d &= c_1 + \dots + c_{L+R} = \\ &= a_1^1 + \dots + a_1^{L+R} + \\ &+ a_2^1 + \dots + a_2^{L+R} + \\ &+ \dots + a_{L+R}^1 + \dots + a_{L+R}^{L+R} + 1 - 1 = \\ &= a_1^1 + \dots + a_{L+R}^1 + \\ &+ a_1^2 + \dots + a_{L+R}^2 + \\ &+ a_1^{L+R} + \dots + a_{L+R}^{L+R} = 0 \end{aligned}$$

Due to the fact that  $d = 0$ ,  $A$  is indefinite unless  $c_i = 0$  for all  $i = 1, \dots, L+R$ . Now, using the parametrization given by lemma 3.1.1, this condition can be formulated on vector form as:

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_{L+R} \end{pmatrix} = \begin{pmatrix} a_1^1 \\ \vdots \\ a_{L+R}^1 \end{pmatrix} + \dots + \begin{pmatrix} a_1^{L+R} \\ \vdots \\ a_{L+R}^{L+R} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ -1 \\ 1 \\ \vdots \\ 0 \end{pmatrix} =$$

$$= (\mathbf{a}^1)^T + \dots + (\mathbf{a}^{L+R})^T + \begin{pmatrix} 0 \\ \vdots \\ -1 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \mathbf{X}(\mathbf{t}^1 + \mathbf{t}^2 + \dots + \mathbf{t}^{L+R}) + \begin{pmatrix} 0 \\ \vdots \\ -1 \\ 1 \\ \vdots \\ 0 \end{pmatrix}. \quad (3.5)$$

Now, let:

$$\tilde{\mathbf{t}} = \mathbf{t}^1 + \dots + \mathbf{t}^{L+R}.$$

Then (3.5) can be written as the following equation system:

$$\mathbf{X}\tilde{\mathbf{t}} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix}, \quad \text{or:}$$

$$\begin{pmatrix} x_3 - x_2 & \dots & x_L - x_2 & y_1 - x_2 & \dots & y_R - x_2 \\ -(x_3 - x_1) & \dots & -(x_L - x_1) & -(y_1 - x_1) & \dots & -(y_R - x_1) \\ x_2 - x_1 & 0 & \dots & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \dots & \dots & \vdots \\ 0 & \ddots & \ddots & \dots & \dots & \vdots \\ \vdots & \ddots & x_2 - x_1 & 0 & \dots & \vdots \\ 0 & \dots & 0 & x_2 - x_1 & \dots & \vdots \\ \vdots & \dots & \dots & \dots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & 0 \\ \vdots & \dots & \dots & \dots & \dots & \vdots \\ 0 & \dots & \dots & \dots & 0 & x_2 - x_1 \end{pmatrix} \begin{pmatrix} \tilde{t}_1 \\ \vdots \\ \tilde{t}_{L+R-2} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix}.$$

The 3:rd through the  $L + R$ :th row of this equation system gives the following solution:

$$\tilde{t}_i = \begin{cases} 0 & i \neq L, L + 1 \\ \frac{1}{x_2 - x_1} & i = L \\ \frac{-1}{x_2 - x_1} & i = L + 1 \end{cases}$$

With this inserted, the first equation becomes:

$$-\frac{x_L - x_2}{x_2 - x_1} + \frac{y_1 - x_2}{x_2 - x_1} = 0 \Rightarrow x_L = y_1. \quad (3.6)$$

This clearly contradicts the assumption, so the proof is complete.  $\square$

### 3.2 Energy stability in other norms

The result of the previous section does rule out the possibility of stability in the conventional sense. Remember that energy stability in a certain norm is merely a sufficient, not a necessary, condition for stability.

In particular one could ask whether there could be other norms than the natural (3.2) in which an interface implementation can be shown to be energy stable. We note first that the natural norm studied so far has a specific property which makes it easier to study energy stability, namely that the energy matrix, is not affected by the specific choice of matrices  $P_l$ ,  $P_r$ ,  $Q_l$  and  $Q_r$  defining the SBP operators. We restrict the discussion in this section to the question whether there are other norms with the same property. Consider the structure of the system matrix (2.6). In order for the energy matrix  $A = HM + M^T H$  to be independent of the choice of  $P_l$  and  $P_r$ , the norm  $H$  must clearly be of the following form:

$$H = N \begin{pmatrix} P_l & \mathbf{0} \\ \mathbf{0} & P_r \end{pmatrix}, \quad (3.7)$$

where  $N$  is a matrix chosen so that  $H$  is symmetric positive definite. This leads to the following simplification of the energy matrix (3.1):

$$\begin{aligned} A &= HM + M^T H = \\ &= N(\mathcal{A} + \text{diag}(\sigma, 0, \dots) - \begin{pmatrix} Q_l & \mathbf{0} \\ \mathbf{0} & Q_r \end{pmatrix}) + (\mathcal{A}^T + \text{diag}(\sigma, 0, \dots) - \begin{pmatrix} Q_l^T & \mathbf{0} \\ \mathbf{0} & Q_r^T \end{pmatrix})N^T. \end{aligned}$$

Consider especially the terms including the remaining matrices from the SBP treatment:

$$-N \begin{pmatrix} Q_l & \mathbf{0} \\ \mathbf{0} & Q_r \end{pmatrix} + \begin{pmatrix} Q_l^T & \mathbf{0} \\ \mathbf{0} & Q_r^T \end{pmatrix} N^T$$

From this, in order to make the energy matrix independent of  $Q_l$  and  $Q_r$ , the SBP properties  $Q_l + Q_l^T = \text{diag}(-1, 0, \dots, 0, 1)$  and  $Q_r + Q_r^T = \text{diag}(-1, 0, \dots, 0, 1)$  can be taken advantage of by factoring these expressions out, if they commute with  $N$  in the following way:

$$N \begin{pmatrix} Q_l & \mathbf{0} \\ \mathbf{0} & Q_r \end{pmatrix} = \begin{pmatrix} Q_l & \mathbf{0} \\ \mathbf{0} & Q_r \end{pmatrix} N$$

Without knowing the structure of  $Q_l$  and  $Q_r$ , for this to happen there is now no other choice but setting  $N$  to

$$N = \text{diag}(1, \dots, 1, \alpha, \dots, \alpha) \Rightarrow H = \begin{pmatrix} P_l & \mathbf{0} \\ \mathbf{0} & \alpha P_r \end{pmatrix}, \quad \text{where } \alpha > 0 \quad (3.8)$$

Allowing the upper left block in the diagonal matrix above to have a different scaling than 1 would only change the scaling of the energy matrix  $A$ , and not its potential negative definiteness. With this norm, one can however not as easily rotate  $A$  as in theorem 3.1.1 to find necessary conditions on the interface

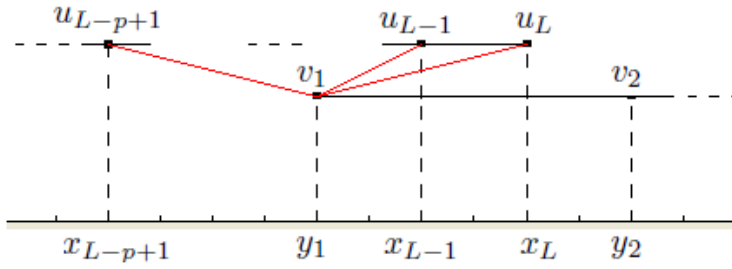


parameters. This type of norm will however prove to be useful in the next section, allowing us to find implementations that are stable in the conventional sense using a limit approach.

### 3.3 Conventional stability

From the results and discussion in the previous chapter, it seems that energy stability of a sufficiently accurate interface implementation is hard to prove or disprove, except in the case of the natural norm where it is provably impossible. There is however still the possibility to prove stability in the conventional sense. In this chapter we present a class of methods implementing a single estimation that can be shown to provide stability. We consider the following linear estimation of  $v_1$  based on an arbitrary number of points in the left grid:

Figure 3.1: A general overlap



$$\hat{v}_1 = a_1 u_L + a_2 u_{L-1} + \dots + a_p u_{L-p+1} \quad (3.9)$$

Note that, if the overlap is big, as many coefficients  $a_1, a_1, \dots$  as needed can be set to zero without loss of generality. Moreover, consider the following implementation:

$$\begin{pmatrix} P_l & \mathbf{0} \\ \mathbf{0} & P_r \end{pmatrix} \mathbf{w}_t + \begin{pmatrix} Q_l & \mathbf{0} \\ \mathbf{0} & Q_r \end{pmatrix} \mathbf{w} = \begin{pmatrix} \sigma(u_0 - g(t)) \\ 0 \\ \vdots \\ 0 \\ \tau(v_1 - \hat{v}_1) \\ 0 \\ \vdots \end{pmatrix}, \quad (3.10)$$

where  $\hat{v}_1$  is implemented on the row corresponding to  $v_1$ , i.e. on row  $L + 1$ . Then the system matrix is given by:

$$M = \begin{pmatrix} P_l^{-1} & \mathbf{0} \\ \mathbf{0} & P_r^{-1} \end{pmatrix} \left( \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & S \end{pmatrix} + \begin{pmatrix} \sigma & 0 & \dots \\ 0 & 0 & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix} - \begin{pmatrix} Q_l & \mathbf{0} \\ \mathbf{0} & Q_r \end{pmatrix} \right),$$

where

$$S = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ -a_p\tau & \dots & -a_0\tau & \tau \end{pmatrix}$$

In section 3.3.1, stability of this system is discussed for grids with single overlap and a 2nd order interpolation condition. Section 3.3.2 provides a proof of stability in the general case.

### 3.3.1 Example - second order accuracy

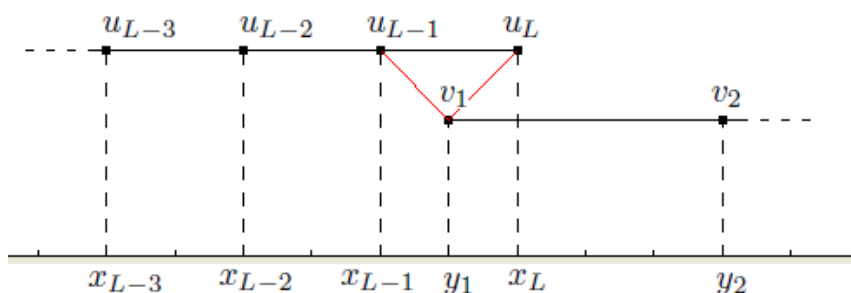
We will begin with a simple case, first discussing in some more depth the difficulty of proving energy stability. After this a solution to the problem is proposed that only proves stability in the conventional sense.

Consider the case where a single point laps over into the opposite grid, according to Figure 2.1, for convenience repeated in Figure 3.2 below. A second order accurate interpolation estimate for  $v_1$ , already presented in section 2.1, is the following:

$$\hat{v}_1 = \beta u_{L-1} + (1 - \beta)u_L, \quad \text{where} \quad \beta = \frac{x_L - y_1}{x_L - x_{L-1}}$$

Since this interface treatment is strictly of second order, it is not energy stable in the natural norm. Instead we consider energy stability in the class of modified norms (3.8) proposed earlier, i.e. we let

Figure 3.2:



$$H = \begin{pmatrix} P_l & \mathbf{0} \\ \mathbf{0} & \alpha P_r \end{pmatrix}, \quad \alpha > 0$$

The energy matrix is then given by:

$$A = HM + M^T H = \begin{pmatrix} 2\sigma + 1 & 0 & \dots \\ 0 & 0 & \\ \vdots & & \ddots \\ & & & -1 & & \\ & & & & \alpha & & \\ & & & & & 0 & \\ & & & & & & \ddots \\ & & & & & & & -\alpha \end{pmatrix} + \begin{pmatrix} \mathbf{0} & & \mathbf{0} \\ \alpha(S + S^T) & & \mathbf{0} \\ \mathbf{0} & & \mathbf{0} \end{pmatrix}.$$

Now consider the 3-by-3 submatrix of  $A$  that corresponds to the interface treatment:

$$A_{L-1,L,L+1} = \alpha(S + S^T) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \alpha \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\alpha\tau(1-\beta) \\ 0 & -1 & -\alpha\tau\beta \\ -\alpha\tau(1-\beta) & -\alpha\tau\beta & 2\tau + \alpha \end{pmatrix}.$$

Due to the zero diagonal element, this matrix is indefinite unless all other elements on the first row is zero. This means setting  $\tau = 0$ , and in which case the matrix is still clearly indefinite.

Instead, we will show that energy stability can be achieved for a class of matrices in a neighbourhood of  $M$ . Then  $M$  must also have non-positive eigenvalues because of continuity, proving stability in the conventional sense. Thus, consider the following sequence of matrices:

$$M(\epsilon) = \begin{pmatrix} P_l^{-1} & \mathbf{0} \\ \mathbf{0} & P_r^{-1} \end{pmatrix} \left( \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ S(\epsilon) & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \sigma & 0 & \dots \\ 0 & 0 & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix} \right) - \begin{pmatrix} Q_l & \mathbf{0} \\ \mathbf{0} & Q_r \end{pmatrix},$$

where

$$S(\epsilon) = \begin{pmatrix} -\epsilon & 0 & 0 \\ 0 & 0 & 0 \\ -\tau(1-\beta) & -\tau\beta & \tau \end{pmatrix}, \quad \epsilon > 0.$$

Note especially that  $M(0) = M$ . For each  $\epsilon > 0$ , define:

$$H(\epsilon) = \begin{pmatrix} P_l & \mathbf{0} \\ \mathbf{0} & \epsilon P_r \end{pmatrix}$$

Now, let  $\mathbf{x}$  be any vector of the same length as the dimension of  $M$ . Then we have:

$$\begin{aligned} \mathbf{x}^T A(\epsilon)\mathbf{x} &= \mathbf{x}^T (H(\epsilon)M(\epsilon) + M(\epsilon)^T H(\epsilon))\mathbf{x} = (2\sigma + 1)x_1^2 - \epsilon x_{L+R}^2 + \\ &+ \begin{pmatrix} x_{L-1} & x_L & x_{L+1} \end{pmatrix} \begin{pmatrix} -2\epsilon & 0 & -\epsilon\tau(1-\beta) \\ 0 & -1 & -\epsilon\tau\beta \\ -\epsilon\tau(1-\beta) & -\epsilon\tau\beta & \epsilon(2\tau+1) \end{pmatrix} \begin{pmatrix} x_{L-1} \\ x_L \\ v_1 \end{pmatrix}. \end{aligned}$$

If we now set  $\sigma = -1$  and  $\tau = -1$ , then this becomes:

$$\mathbf{x}^T A(\epsilon)\mathbf{x} = -\epsilon x_{L+R}^2 + \begin{pmatrix} x_{L-1} & x_L & x_{L+1} \end{pmatrix} \begin{pmatrix} -2\epsilon & 0 & \epsilon(1-\beta) \\ 0 & -1 & \epsilon\beta \\ \epsilon(1-\beta) & \epsilon\beta & -\epsilon \end{pmatrix} \begin{pmatrix} x_{L-1} \\ x_L \\ v_1 \end{pmatrix}.$$

Since  $0 \leq \beta \leq 1$  (due to the simple overlap), the matrix in the expression above is diagonally dominant, and thus negative semi-definite, if  $0 \leq \epsilon \leq 1$ . From theorem 1.3.1 now follows that  $M(\epsilon)$  has eigenvalues with non-positive real parts for  $0 < \epsilon \leq 1$ . Moreover, the unordered set of eigenvalues to any matrix is continuous with respect to its entries, since they are roots to a polynomial with coefficients given by these entries. From this continuity property then follows that  $M = M(0)$  has eigenvalues with non-positive real parts as well.

### 3.3.2 Arbitrary accuracy

For the general case with arbitrary overlap and also an arbitrary interpolation condition, a proof of stability of (3.10) is slightly more complicated and gives a slightly different bound on  $\tau$ . However the general proof idea is the same as for the simpler case.

**Theorem 3.3.1.** *With  $\hat{v}_1$  as in (3.9), the system (3.10) is stable if  $\sigma \leq -\frac{1}{2}$  and  $\tau < -\frac{1}{2}$ .*

*Proof.* We prove the theorem directly by showing that all eigenvalues of the system matrix lie in the left half-plane. In analogy with the discussion in section 3.3.1, we consider the following sequence of matrices in a neighbourhood of  $M$ :

$$M(\epsilon) = \begin{pmatrix} P_l^{-1} & \mathbf{0} \\ \mathbf{0} & P_r^{-1} \end{pmatrix} \left( \begin{pmatrix} \mathbf{0} & & \mathbf{0} \\ & S(\epsilon) & \\ \mathbf{0} & & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \sigma & 0 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & & \ddots & \\ 0 & \dots & & 0 \end{pmatrix} - \begin{pmatrix} Q_l & \mathbf{0} \\ \mathbf{0} & Q_r \end{pmatrix} \right),$$

where

$$S(\epsilon) = \begin{pmatrix} -\epsilon & 0 & \dots & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & -\epsilon & & \\ 0 & \dots & 0 & 0 & 0 \\ -a_p\tau & \dots & & -a_1\tau & \tau \end{pmatrix}$$

Note again that  $M(0) = M$ . We first show all eigenvalues of  $M(\epsilon)$  lie in the left half-plane for sufficiently small  $\epsilon > 0$ , implying by continuity that the eigenvalues of  $M = M(0)$  are all in the left half-plane as well. To this end, we define  $H(\epsilon)$  as follows:

$$H(\epsilon) = \begin{pmatrix} P_l & \mathbf{0} \\ \mathbf{0} & \alpha(\epsilon)P_r \end{pmatrix}, \quad \alpha(\epsilon) > 0$$

Now, as before let  $\mathbf{x}$  be any vector of length  $L + R$ . Then the energy estimate is given by

$$\begin{aligned} \mathbf{x}^T A(\epsilon) \mathbf{x} &= \mathbf{x}^T (H(\epsilon)M(\epsilon) + M(\epsilon)^T H(\epsilon)) \mathbf{x} = \\ &= (1+2\sigma)x_1^2 - \tau x_L^2 + \begin{pmatrix} x_{L-p+1} & \dots & x_{L-1} \end{pmatrix} \begin{pmatrix} -2\epsilon & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ 0 & \dots & 0 & -2\epsilon \end{pmatrix} \begin{pmatrix} x_{L-p+1} \\ \vdots \\ x_{L-1} \end{pmatrix} - \\ &- 2\alpha(\epsilon)\tau x_{L+1} \begin{pmatrix} a_p & \dots & a_1 \end{pmatrix} \begin{pmatrix} x_{L-p+1} \\ \vdots \\ x_L \end{pmatrix} + \alpha(\epsilon)(2\tau + 1)x_{L+1}^2 - \alpha(\epsilon)x_{L+R}^2 \leq \\ \{\sigma \leq -\frac{1}{2}, \alpha(\epsilon) > 0\} &\leq -2\epsilon \|\bar{\mathbf{x}}\|^2 + 2\alpha(\epsilon)|\tau| |x_{L+1}| \|\mathbf{a}\| \|\bar{\mathbf{x}}\| + \alpha(\epsilon)(2\tau + 1)x_{L+1}^2 = \end{aligned}$$

$$\left( \|\bar{\mathbf{x}}\| \quad |x_{L+1}| \right) \begin{pmatrix} -2\epsilon & \frac{\alpha(\epsilon)|\tau| \|\mathbf{a}\|}{2} \\ \frac{\alpha(\epsilon)|\tau| \|\mathbf{a}\|}{2} & \alpha(\epsilon)(2\tau + 1) \end{pmatrix} \begin{pmatrix} \|\bar{\mathbf{x}}\| \\ |x_{L+1}| \end{pmatrix},$$

where  $\bar{\mathbf{x}} = (x_{L-p+1} \dots x_L)^T$ , and  $\mathbf{a} = (a_p \dots a_1)^T$ . Thus  $A(\epsilon)$  is negative semidefinite if the matrix  $F$  is negative semidefinite, where

$$F = \begin{pmatrix} -2\epsilon & \frac{\alpha(\epsilon)|\tau| \|\mathbf{a}\|}{2} \\ \frac{\alpha(\epsilon)|\tau| \|\mathbf{a}\|}{2} & \alpha(\epsilon)(2\tau + 1) \end{pmatrix}$$

Since  $F$  has negative diagonal elements (remember that  $\tau < -\frac{1}{2}$ ), it follows that  $F$  is negative semidefinite if and only if  $\det F \geq 0$ :

$$\det F = -2\epsilon\alpha(\epsilon)(2\tau + 1) - \frac{\alpha(\epsilon)|\tau| \|\mathbf{a}\|}{4} \geq 0 \quad \Leftrightarrow \quad \alpha(\epsilon) \leq \frac{-8\epsilon(2\tau + 1)}{|\tau| \|\mathbf{a}\|}.$$

Since  $\frac{-8\epsilon(2\tau+1)}{|\tau| \|\mathbf{a}\|}$  is positive whenever  $\epsilon > 0$ , one can indeed choose  $\alpha(\epsilon) > 0$  such that  $A(\epsilon)$  is negative semi-definite. By lemma (1.3.1), the eigenvalues of  $M(\epsilon)$  thus have non-positive real parts for  $\epsilon > 0$ .  $\square$



## Chapter 4

# A coupled system of hyperbolic equations

A system of hyperbolic equations that is especially sensitive to unstable methods, and thus providing a hard test for the interface implementation we have proposed, is the following coupled system:

$$\begin{aligned} U_t + U_x &= 0 \\ V_t - V_x &= 0 \quad 0 \leq x \leq 1, t \geq 0, \end{aligned} \tag{4.1}$$

subject to the following initial and boundary conditions:

$$\begin{aligned} U(x, 0) &= f_1(x) \\ V(x, 0) &= f_2(x) \\ U(0, t) &= V(0, t) \\ V(1, t) &= U(1, t) \end{aligned}$$

Note that all initial energy of this system is preserved over time. It is not at all clear that the method presented in section 3.3 can be directly applied to this problem in a way that preserves stability. Remember that we have so far only considered the single equation problem 1.1.

### 4.1 Single domain

We begin by studying the one-domain implementation of (5.2), for which stability can indeed be ensured using a proper SAT boundary implementation. We define the grid and the two discrete solution vectors as follows:

$$\begin{aligned} \mathbf{x} &= (0 = x_1 \ x_2 \ \dots \ x_{N-1} \ x_N = 1)^T \\ \mathbf{u}^1 &= (u_1^1 \ u_2^1 \ \dots \ u_{N-1}^1 \ u_N^1)^T \\ \mathbf{u}^2 &= (u_1^2 \ u_2^2 \ \dots \ u_{N-1}^2 \ u_N^2)^T \end{aligned}$$

In other words,  $\mathbf{u}^1$  is the discrete solution corresponding to  $U$ , whereas  $\mathbf{u}^2$  corresponds to  $V$ . The straightforward way to implement (5.2) using SBP operators and SAT boundary implementation is then the following:

$$\begin{pmatrix} P & \mathbf{0} \\ \mathbf{0} & P \end{pmatrix} \begin{pmatrix} \mathbf{u}_t^1 \\ \mathbf{u}_t^2 \end{pmatrix} + \begin{pmatrix} Q & \mathbf{0} \\ \mathbf{0} & -Q \end{pmatrix} \mathbf{w} = \begin{pmatrix} \sigma_1(u_1^1 - u_1^2) \\ 0 \\ \vdots \\ \sigma_2(u_N^1 - u_N^2) \\ \sigma_3(u_1^2 - u_1^1) \\ 0 \\ \vdots \\ \sigma_4(u_N^2 - u_N^1) \end{pmatrix} \quad (4.2)$$

where  $P^{-1}Q$  is an SBP operator. This results in the following system matrix:

$$M = \begin{pmatrix} P^{-1} & \mathbf{0} \\ \mathbf{0} & P^{-1} \end{pmatrix} \left( \begin{pmatrix} -Q & \mathbf{0} \\ \mathbf{0} & Q \end{pmatrix} + \begin{pmatrix} \sigma_1 & 0 & \dots & -\sigma_1 & \dots & 0 \\ 0 & 0 & & & & \vdots \\ \vdots & & \ddots & & & \\ -\sigma_3 & & & \sigma_2 & & -\sigma_2 \\ \vdots & & & & \sigma_3 & \\ 0 & \dots & -\sigma_4 & & 0 & \dots \\ \vdots & & & & & \ddots \\ 0 & \dots & -\sigma_4 & & & \sigma_4 \end{pmatrix} \right).$$

We prove energy stability using the natural norm given by  $P$ :

$$H = \begin{pmatrix} P & \mathbf{0} \\ \mathbf{0} & P \end{pmatrix},$$

The energy matrix is then given by:

$$A = HM + M^T H = \begin{pmatrix} 2\sigma_1 + 1 & 0 & \dots & -\sigma_1 - \sigma_3 & \dots & 0 \\ 0 & 0 & & & & \vdots \\ \vdots & & \ddots & & & \\ -\sigma_1 - \sigma_3 & & & 2\sigma_2 - 1 & & -\sigma_2 - \sigma_4 \\ \vdots & & & & 2\sigma_3 - 1 & \\ 0 & \dots & -\sigma_2 - \sigma_4 & & 0 & \dots \\ \vdots & & & & & \ddots \\ 0 & \dots & -\sigma_2 - \sigma_4 & & & 2\sigma_4 + 1 \end{pmatrix}.$$

In order to produce a set of necessary and sufficient conditions of energy stability in this norm, we introduce the following rotation matrix:



$$R = \begin{pmatrix} 1 & 0 & \dots & 1 & \dots & 0 \\ 0 & \ddots & & \vdots & & \vdots \\ \vdots & & \ddots & & & \\ & & & 1 & & \\ & & & 0 & 1 & \\ & & & \vdots & & \ddots \\ 0 & \dots & 1 & & 0 & 1 \end{pmatrix}$$

Negative semidefiniteness of  $A$  is then equivalent to semidefiniteness of  $B$ , where

$$B = R^T A R = \begin{pmatrix} 2\sigma_1 + 1 & 0 & \dots & & \sigma_1 - \sigma_3 + 1 & \dots & 0 \\ 0 & 0 & & & & & \vdots \\ \vdots & & \ddots & & & & \\ \sigma_1 - \sigma_3 + 1 & & & 0 & & & \sigma_4 - \sigma_2 + 1 \\ \vdots & & & & 0 & & \\ 0 & \dots & \sigma_4 - \sigma_2 + 1 & & & \ddots & 2\sigma_4 + 1 \end{pmatrix}.$$

Thus the necessary and sufficient conditions are:

$$\sigma_1 \leq -\frac{1}{2}, \quad \sigma_3 = \sigma_1 + 1, \quad \sigma_4 \leq -\frac{1}{2}, \quad \sigma_2 = \sigma_4 + 1. \quad (4.3)$$

## 4.2 Two domains

With two overlapping domains, we define the grid and solution vectors as follows:

$$\begin{aligned} \mathbf{x} &= (0 = x_1 \ x_2 \ \dots \ x_{L-1} \ x_L)^T \\ \mathbf{y} &= (y_1 \ y_2 \ \dots \ y_{R-1} \ y_R = 1)^T \\ \mathbf{u}^1 &= (u_1^1 \ u_2^1 \ \dots \ u_{L-1}^1 \ u_L^1)^T \\ \mathbf{v}^1 &= (v_1^1 \ v_2^1 \ \dots \ v_{R-1}^1 \ v_R^1)^T \\ \mathbf{u}^2 &= (u_1^2 \ u_2^2 \ \dots \ u_{L-1}^2 \ u_L^2)^T \\ \mathbf{v}^2 &= (v_1^2 \ v_2^2 \ \dots \ v_{R-1}^2 \ v_R^2)^T \end{aligned}$$

The straightforward way to implement boundary and interface conditions is then to combine (4.2) and (3.10) in the following way:



where

$$T_1(\epsilon) = \begin{pmatrix} -2\epsilon & 0 & \dots & & -\alpha_1 a_p \tau_1 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & -2\epsilon & & \\ 0 & \dots & 0 & -1 & -\alpha_1 a_1 \tau_1 \\ -\alpha_1 a_p \tau_1 & \dots & & -\alpha_1 a_1 \tau_1 & \alpha_1 (2\tau_1 + 1) \end{pmatrix},$$

$$T_2(\epsilon) = \begin{pmatrix} \alpha_2 (2\tau_2 + 1) & -\alpha_2 b_1 \tau_2 & \dots & & -\alpha_2 b_q \tau_2 \\ -\alpha_2 b_1 \tau_2 & -1 & 0 & \dots & 0 \\ \vdots & & -2\epsilon & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ -\alpha_2 b_q \tau_2 & & \dots & 0 & -2\epsilon \end{pmatrix}.$$

We remind ourselves of that the system is stable if  $A(\epsilon)$  is energy stable for positive  $\epsilon$  in a neighborhood of zero. First consider the following submatrix:

$$A_{1,L+R+1}(\epsilon) = \begin{pmatrix} 2\sigma_1 + 1 & -\sigma_1 - \alpha_2 \sigma_3 \\ -\sigma_1 - \alpha_2 \sigma_3 & \alpha_2 (2\sigma_3 - 1) \end{pmatrix}$$

In order for this to be negative semidefinite, we must have:

$$\begin{aligned} \det A_{1,L+R+1}(\epsilon) &= \alpha_2 (2\sigma_1 + 1)(2\sigma_3 - 1) - (\sigma_1 - \alpha_2 \sigma_3)^2 \geq 0 \\ &\Rightarrow (\sigma_1 - \alpha_2 \sigma_3)^2 \leq \alpha_2 (2\sigma_1 + 1)(2\sigma_3 - 1) \end{aligned} \quad (4.5)$$

However, the lower  $R+q$  times  $R+q$  block of  $A$  provides, once again analogous to the proof of theorem 3.3.1, with the following condition on  $\alpha_2$ :

$$\alpha_2 \leq \frac{-8\epsilon(2\tau_2 + 1)}{|\tau_2| \|\mathbf{b}\|}.$$

Clearly this bound approaches zero as  $\epsilon$  approaches zero. This means that we must set  $\sigma_1$  to zero to meet inequality (4.5), but then the first diagonal element of  $A_{1,L+R+1}(\epsilon)$  is positive. Thus  $A(\epsilon)$  can not be negative semi-definite. This means that the interface implementation derived for the single equation case can not be directly applied to this system with a similar proof of stability.

### 4.3 Conclusion

Without being able to prove stability of the two-domain implementation (4.4), it seems the best we can do is to propose choosing the penalty parameters according to theorem 3.3.1 and 4.6, and to test empirically if this indeed make the

system stable. Thus, we propose setting:

$$\begin{aligned} \tau_1 &\leq -\frac{1}{2}, & \tau_2 &\leq -\frac{1}{2} \\ \sigma_1 &\leq -\frac{1}{2}, & \sigma_3 &= \sigma_1 + 1 \\ \sigma_4 &\leq -\frac{1}{2}, & \sigma_2 &= \sigma_4 + 1 . \end{aligned} \tag{4.6}$$

# Chapter 5

## Numerical tests

In this chapter we apply the implementations derived in chapter 4 and 5 to test for stability and check convergence rate. For all numerical experiments using overlapping grid techniques, the number of grid points in the left and right grids were always kept the same. In particular, the following type of grids were used, with successive grid refinement:

$$\mathbf{x} = (0 = x_1 \dots x_N = 0.5)^T, \mathbf{y} = (0.4831926 = y_1 y_2 \dots y_N = 1)^T,$$

The lower bound in the right grid was chosen randomly to reduce the chance of getting systematically coinciding points in the two grids. The implementations were tested using second, fourth, sixth and eighth order FD operators. The least local order of accuracy at the boundaries was  $\frac{p}{2}$ , where  $p$  is the order at the interior of the operator (2,4,6 and 8). In order to preserve this order of consistency, any interpolation condition implemented was of order  $\frac{p}{2} + 1$ . This also means that energy stable implementations are expected to converge with rate  $\frac{p}{2} + 1$ , see [2].

### 5.1 Single equation

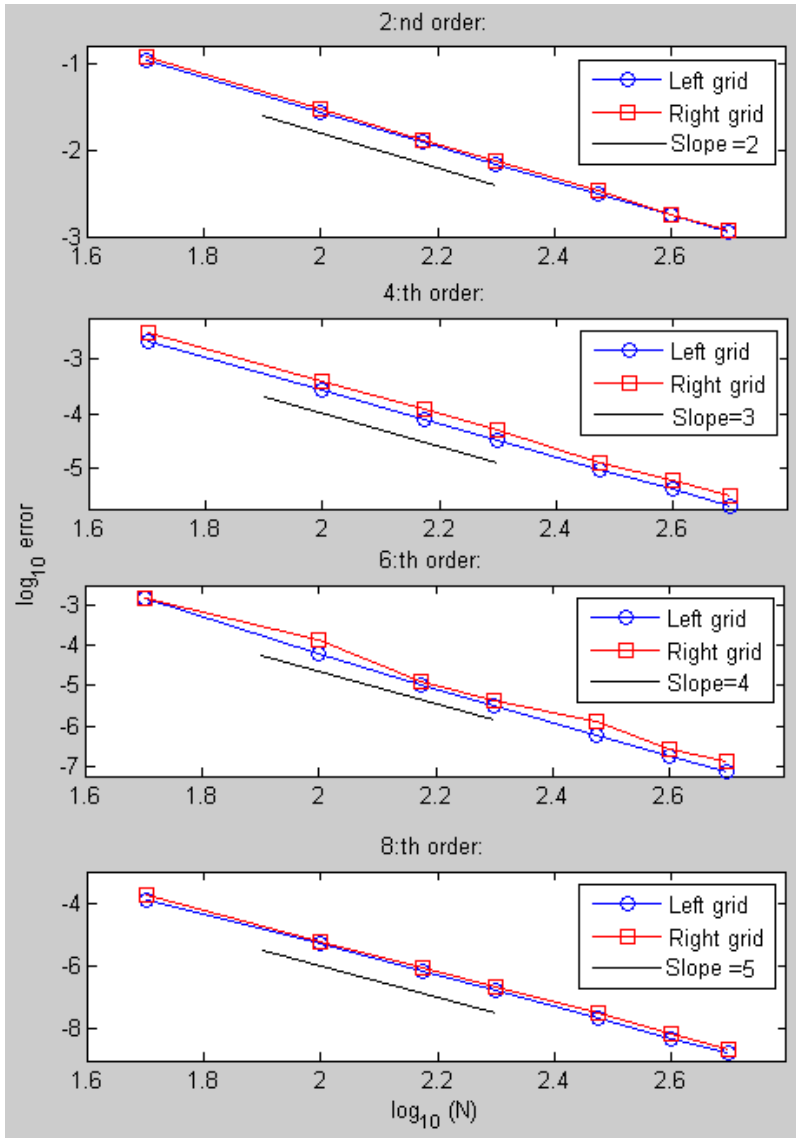
For the single equation case we used the following continuous test equation:

$$\begin{aligned} U_t + U_x &= 0 & 0 \leq x \leq 1, t \geq 0 \\ U(x, 0) &= \sin 6\pi x & t = 0 \\ U(0, t) &= \sin 6\pi t & x = 0 \end{aligned} \tag{5.1}$$

We applied the implementation (3.10) on (5.1), with  $\sigma = \tau = -1$ . Since we have a right-moving wave, the solution on the left grid is expected to be energy stable, i.e. to converge with rate  $p/2 + 1$ . Since the implementation is stable however, we expect the right grid to converge as well, but with unknown rate. Results in terms of  $L_2$  errors for the left and right grid is shown in Figure 5.2, while stability properties are shown in Figure 5.1. As can be seen in Figure 5.2, the solution on both grids consistently converges with rate  $p/2 + 1$ . Moreover, the error magnitude on the right grid does not seem to increase with longer time integration, as illustrated in Figure 5.5.

Figure 5.1: Maximum real parts of system eigenvalues for single equation implementations.

N	2 <sup>nd</sup> order	4 <sup>th</sup> order	6 <sup>th</sup> order	8 <sup>th</sup> order
<b>Single equation, overlapping grids</b>				
50	-0.0116	-0.0102	-0.0352	-0.0020
100	-0.0028	-0.0024	-0.0062	-0.0004
150	-0.0012	-0.0010	-0.0025	-0.0001
300	-0.0003	-0.0003	-0.0006	-0.0000
600	-0.0001	-0.0001	-0.0001	-0.0000

Figure 5.2: Convergence of single equation implementations at  $t=1$ .

## 5.2 Coupled system

For the coupled system we used the following system of equations for the tests:

$$\begin{aligned} U_t + U_x &= 0 \\ V_t - V_x &= 0 \quad 0 \leq x \leq 1, t \geq 0, \end{aligned} \tag{5.2}$$

subject to the following initial and boundary conditions:

$$\begin{aligned} U(x, 0) &= \sin(6\pi x) \\ V(x, 0) &= -\sin(6\pi x) \\ U(0, t) &= V(0, t) \\ V(1, t) &= U(1, t) \end{aligned}$$

Both the one-domain (4.2) and two-domain (4.4) implementations were tested, in the latter case using  $\tau_1 = \tau_2 = -1$ . Moreover, the boundary penalty terms were assigned three different sets of values:

$$\begin{aligned} 1 : \quad & \sigma_1 = \sigma_4 = -\frac{1}{2} \\ 2 : \quad & \sigma_1 = \sigma_4 = -1 \\ 3 : \quad & \sigma_1 = \sigma_4 = -2 \end{aligned}$$

Stability of the resulting systems are illustrated in Figure 5.3, while  $L_2$  errors for the case  $\sigma_1 = \sigma_4 = -1$  is shown in Figure 5.4. Again, solutions converge with rate  $p/2 + 1$  as was expected for an energy stable implementation. But as can be seen in Figure 5.3, the two-domain implementations are all unstable. However, in Figure 5.5 we see that the  $L_2$  error for the two-domain implementation increases with longer time-integration.



Figure 5.3: Maximum real parts of system eigenvalues for coupled system implementations. The small fluctuations around 0 in the single grid case is most likely due to computational rounding errors.

N	$\sigma_1 = \sigma_4$	2 <sup>nd</sup> order	4 <sup>th</sup> order	6 <sup>th</sup> order	8 <sup>th</sup> order
<b>Coupled system, single grid</b>					
50	-0.5	7.2e-15	1.8e-14	1.1e-14	6.4e-14
	-1	-5.2e-16	5.6e-16	1.0e-14	-9.6.e-15
	-2	-3.3e-15	-2.4e-14	-2.5e-15	-9.7e-15
100	-0.5	3.4e-14	4.7e-14	5.7e-14	7.2e-14
	-1	2.3e-14	2.6e-14	5.8e-16	-1.7e-16
	-2	-1.8e-14	-1.1e-14	-1.1e-14	8.9e-16
150	-0.5	3.4e-14	8.5e-14	7.1e-14	1.1e-13
	-1	-3.6e-15	-8.2e-15	-1.3e-14	1.2e-14
	-2	1.0e-15	3.6e-14	5.1e-15	5.2e-15
300	-0.5	1.4e-13	2.4e-13	3.0e-13	3.6e-13
	-1	1.1e-13	1.1e-14	2.4e-14	-2.7e-15
	-2	-1.8e-14	-1.4e-14	5.1e-14	2.0e-14
600	-0.5	3.13e-13	5.4e-13	5.1e-13	6.0e-13
	-1	-1.3e-13	4.8e-14	2.7e-14	7.8e-14
	-2	-2.1e-14	2.5e-14	8.2e-15	4.8e-14
1000	-0.5	5.2e-13	9.4e-13	1.2e-12	1.1e-12
	-1	8.9e-14	8.7e-14	7.8e-14	8.5e-14
	-2	2.0e-14	7.6e-14	7.9e-14	1.2e-13
<b>Coupled system, overlapping grids</b>					
50	-0.5	0.0692	0.0859	0.0005	0.0004
	-1	0.0221	0.0663	0.0001	0.0065
	-2	0.0285	0.0663	0.1444	0.0245
100	-0.5	0.1029	0.1925	5.7e-14	0.0437
	-1	0.1011	0.1444	4.9e-14	0.0078
	-2	0.0992	0.1444	2.35e-14	0.0302
150	-0.5	0.1674	0.2228	0.0413	0.0979
	-1	0.1732	0.2098	0.0435	0.0897
	-2	0.1766	0.2098	0.0509	0.1143
300	-0.5	0.1571	0.2578	0.3792	0.2215
	-1	0.1097	0.2745	0.2791	0.1517
	-2	0.1171	0.2745	0.2508	0.1689
600	-0.5	0.1227	0.1753	0.3009	0.1849
	-1	0.1122	0.1736	0.2068	0.1997
	-2	0.1131	0.1736	0.1898	0.2083
1000	-0.5	0.1211	0.1810	0.2941	0.1690
	-1	0.0806	0.1905	0.2113	0.1660
	-2	0.0712	0.1905	0.1916	0.1760

Figure 5.4: Convergence at  $t=1$  of coupled system implementations, with  $\sigma_1 = \sigma_4 = -1$

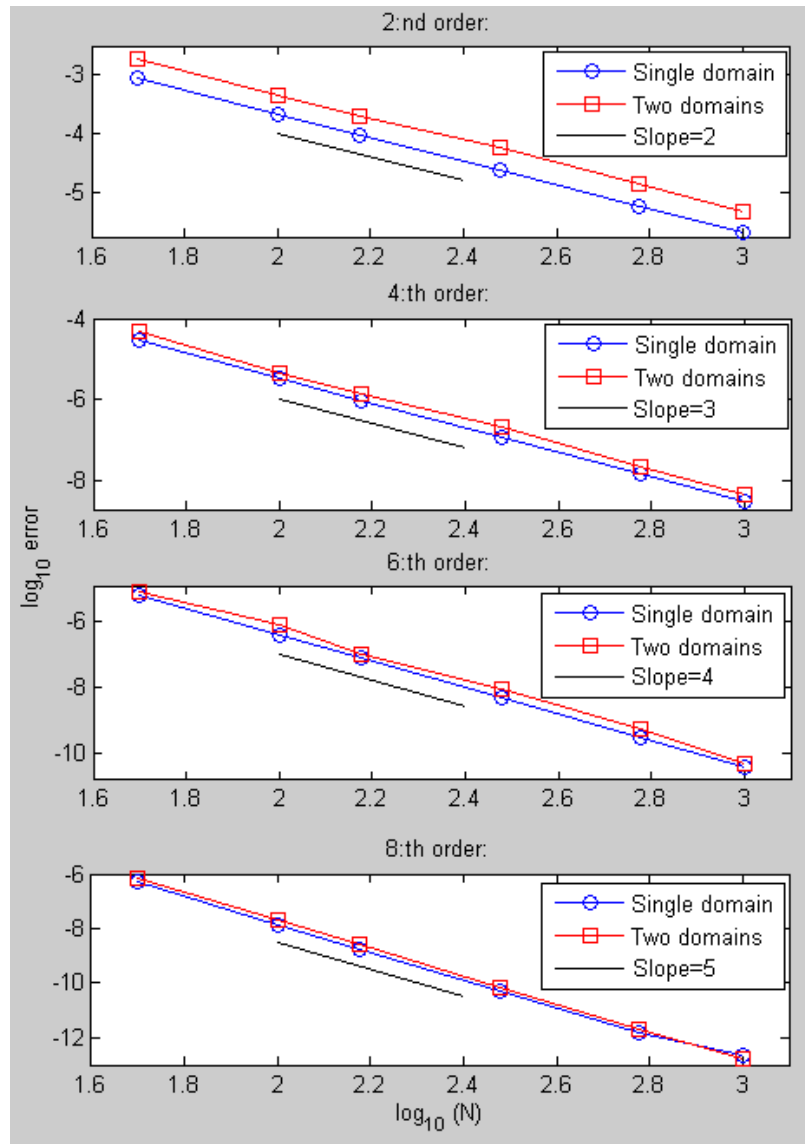
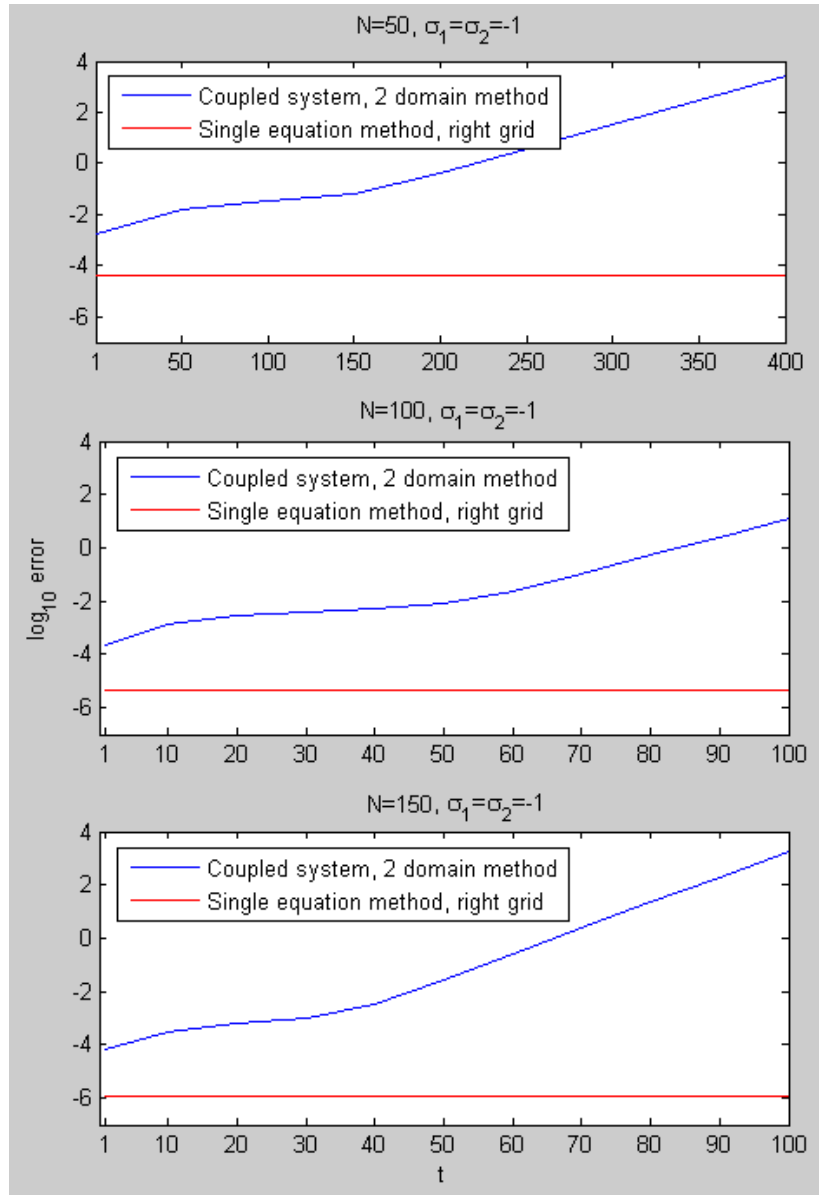


Figure 5.5: Comparison of  $L_2$  error between fourth order two-domain implementations for the single equation and the coupled system.





## Chapter 6

# Conclusions

A stable and consistent interface implementation was derived for the scalar test equation, even though energy stability in the natural norm proved not to be possible for a consistent interface implementation. Successfully proving energy stability in norms other than the natural is likely to be difficult.

Since energy stability in the natural norm was provably impossible to achieve for a consistent interface implementation, the proposed implementation may not work together with other SAT treatments where stability *is* proven in the natural norm. A coupled energy preserving system of PDE:s was analyzed together with the proposed interface implementation, but stability could not be proven in this case.

Numerical tests confirmed stability for the proposed single equation implementation, while the proposed implementation of the energy conserving system proved not to be stable.

For small time integrations, both implementations converged as if they were energy stable.

Longer time integration did not seem to effect the single equation implementation, while the error in the coupled system implementation eventually exploded, as would be expected for an unstable implementation.



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