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Mathematical Optimization Models and Methods for Open-Pit Mining

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To Alberta, Grace and Felicia (my mother)

Abstract

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Open-pit mining is an operation in which blocks from the ground are dug to extract the ore contained in them, and in this process a deeper and deeper pit is formed until the mining operation ends. Mining is often a highly complex industrial operation, with respect to both technological and planning aspects. The latter may involve decisions about which ore to mine and in which order. Furthermore, mining operations are typically capital intensive and long-term, and subject to uncertainties regarding ore grades, future mining costs, and the market prices of the precious metals contained in the ore. Today, most of the high-grade or low-cost ore deposits have already been depleted, and to obtain sufficient profitability in mining operations it is therefore today often a necessity to achieve operational efficiency with respect to both technological and planning issues.

In this thesis, we study the open-pit design problem, the open-pit mining scheduling problem, and the open-pit design problem with geological and price uncertainty. These problems give rise to (mixed) discrete optimization models that in real-life settings are large scale and computationally challenging.

The open-pit design problem is to find an optimal ultimate contour of the pit, given estimates of ore grades, that are typically obtained from samples in drill holes, estimates of costs for mining and processing ore, and physical constraints on mining precedence and maximal pit slope. As is well known, this problem can be solved as a maximum flow problem in a special network. In a first paper, we show that two well known parametric procedures for finding a sequence of intermediate contours leading to an ultimate one, can be interpreted as Lagrangian dual approaches to certain side-constrained design models. In a second paper, we give an alternative derivation of the maximum flow problem of the design problem.

We also study the combined open-pit design and mining scheduling problem, which is the problem of simultaneously finding an ultimate pit contour and the sequence in which the parts of the orebody shall be removed, subject to mining capacity restrictions. The goal is to maximize the discounted net profit during the life-time of the mine. We show in a third paper that the combined problem can also be formulated as a maximum flow problem, if the mining capacity restrictions are relaxed; in this case the network however needs to be time-expanded.

In a fourth paper, we provide some suggestions for Lagrangian dual heuristic and time aggregation approaches for the open-pit scheduling problem. Finally, we study the open-pit design problem under uncertainty, which is taken into account by using the concept of conditional value-at-risk. This concept enables us to incorporate a variety of possible uncertainties, especially regarding grades, costs and prices, in the planning process. In real-life situations, the resulting models would however become very computationally challenging.

Populärvetenskaplig sammanfattning

Den enklaste formen av brytning av malmkroppar är i dagbrott, vilket innebär att man från jordytan skapar en gruva i form av en stor hålighet i berggrunden. Dagbrott kan vara många hundra meter djupa och kilometervida, och även om brytning i dagbrott är enkel jämfört med underjordsbrytning är det ändå en komplex industriell verksamhet. Detta gäller både med avseende på tekniska lösningar, till exempel val av metallurgiska processer för utvinning av metall ur malmen, och planering av brytningen, till exempel vilken malm som ska brytas och i vilken ordningsföljd. Gruvindustrin är kapitalintensiv och för att skapa lönsamhet krävs både kostnadseffektiva tekniska lösningar och god planering. Eftersom den malm som bryts idag ofta är mindre värdefull, exempelvis har lägre guldhalter, eller är mer kostnadskrävande än tidigare, exempelvis på grund av större brytningsdjup, ställs idag ofta höga krav på teknik och planering för att uppnå lönsamhet. Denna avhandling behandlar teori och metoder för några optimeringsfrågeställningar som uppkommer vid planering av gruvbrytning i dagbrott.

Inför gruvdrift i dagbrott görs provborrningar i berggrunden, ofta i ett rutnät och med några hundra meters djup, för att kartlägga malmkroppens utsträckning och för att uppskatta värdet av malmen i dess olika delar. Utifrån dessa uppgifter görs sedan en diskretisering av malmkroppen och berget runtomkring, genom att hela volymen delas in i kuber, eller block, typiskt med en sida på några tiotal meter. För varje block uppskattas förtjänsten som fås om det skulle brytas genom skattningar av värdet av malmen i blocket och kostnader för brytning och processering av malmen.

Den första frågeställningen som behandlas i avhandlingen är det mest grundläggande planeringsproblemet vid optimering av brytning i dagbrott, nämligen att bestämma vilka block som ska brytas för att gruvan ska bli maximalt lönsam. För att besvara denna fråga används skattningarna av förtjänsten för varje block, om det bryts, för att konstruera en matematisk optimeringsmodell som avgör vilka block som ska brytas för att maximera totala förtjänsten. Modellen innehåller två typer av restriktioner. För det första får dagbrottets väggar inte ha mer än en given maximal lutning, vanligen 45 grader, eftersom de annars rasar. För det andra kan ett block brytas endast om blocken direkt ovanför också bryts. Denna matematiska modell ger den optimala slutliga formen på dagbrottet.

För att göra planeringssituationen mer realistisk är det önskvärt att även ta hänsyn till att brytningen i praktiken är utsträckt över en lång tidsrymd, typiskt 10-20 år, och att man varje år inte kan bryta och processera mer än en viss begränsad mängd malm. Utsträckningen i tid gör vidare att det är nödvändigt att beräkna lönsamheten utifrån diskonterade nuvärden av framtida förtjänster. Detta ger upphov till en mer komplicerad optimeringsmodell som avgör både vilka block som ska brytas och när. I avhandlingen ges teori och lösningsmetoder för denna frågeställning.

Gruvbrytning är ofta förknippat med ett stort ekonomiskt risktagande. En orsak till detta är geologisk osäkerhet vad gäller värdet på den malm som ska brytas, beroende på att provborrning endast ger stickprov på värdet. En ytterligare osäkerhet är de framtida priserna på den metall som utvinns. Eftersom en gruva kräver stora investeringar, ofta under flera år innan brytningen ska påbörjas, kan dessa osäkerheter leda till en avsevärd ekonomisk risk. Det är därför önskvärt att redan på planeringsstadiet ta hänsyn till och kompensera för osäkerheten. I avhandlingen konstrueras optimeringsmodeller som åstadkommer detta med hjälp av ett riskmått som har sitt ursprung inom finansmatematiken.

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Linköping, September 15, 2011
Henry Amankwah

Contents

1

Introduction and Overview

Open-pit mining is a surface mining operation whereby ore, or waste, is excavated from the surface of the land. In the process of digging the surface of the land, a deeper and deeper pit is formed until the mining operation ends. It is usually suitable to determine the final shape of the pit before the mining operation starts.

The shape and size of an open-pit depend on certain factors which must be understood in the planning of the open-pit operation. Some of these factors are bench height, ore recovery, geology, grade and localization of the mineralization, extent of the deposit, topography, property boundaries, production rate, mining cost, processing cost, cutoff grade and pit slopes (Armstrong, 1990). For example, the bench height, which is the vertical distance between each horizontal level of the pit, should be set as high as possible within the limits of the size and type of equipment selected for the desired production. The cutoff grade is any grade that for any specified reason is used to separate any two courses of action. At any stage of the mining, the operator has to decide whether the next block of material should be mined or not. The grade of the block is usually used to make this decision. The pit slope is one of the major factors that determines the amount of waste to be removed so as to mine the ore. For more explanations of these mining terms, see Hustrulid and Kuchta (2006).

The limit of the open-pit must be set at the planning stage. This limit defines the amount of ore that is mined, the metal content and the amount of waste to be removed during the period of operation. Knowledge gained from designing the *ultimate pit limit* that maximizes profit is important to all open-pit mining endeavours. Other terms like *pit outline*, *pit contour*, and *maximum closure* are used by different authors to describe the final pit limit of a mine.

To design an optimal pit, the entire mining volume is partitioned into fixed-size blocks. By using geological information from drill cores, the value of the ore in each block is estimated. In addition, the cost of mining each block is determined. A profit value can thus be assigned to each block in the mine. The open-pit mine *design problem* is thus, deciding the blocks of an ore deposit to mine in order to maximize the total profit of the mine, while obeying digging constraints on pit slope and constraints that allow underlying blocks to be mined only after blocks on top of them.

Any feasible pit outline has a cash value which can be computed. To compute the value, we must decide on a *mining sequence* and then mine out the pit, progressively accumulating the revenues and costs as we go (Whittle, 1990). If we are able to fix the block values and the slopes then an optimal outline can be determined. An increase in block values yields that the optimal pit becomes bigger, while an increase in slopes implies the optimal pit gets deeper. It is important to calculate the accurate block values for any optimization, because wrong calculations will lead to wrong optimal pit outline. In fact, it is interesting to know that the pit outline and the mining sequence have to be known in order to compute the block values, particularly if the time value of money is essential, and on the other hand, the block values should be known in order to find the optimal outline.

For the purpose of optimization, there are two basic rules to be followed when calculating the block values. Firstly, a block value is calculated on the assumption that it has been uncovered and that it will be mined. Secondly, any ongoing cost which would stop if mining were stopped should be included (Whittle, 1990).

Production scheduling has an effect on the optimal outline. Scheduling an open-pit means determining the sequence in which materials of the pit should be removed over the lifetime of the mine and the time interval in which each material is to be mined. This has an effect on the value of the mine since it determines when various items of revenue and expenditure will occur. This is essential because the money we have today might lose its value in the coming times.

We usually have “worst case mining” and “best case mining” that are some kind of simple mining principles, not based on optimization. The worst case mining is when each bench is mined completely before the next bench is started. Here, the waste at the periphery of the pit is mined early, and the cost is discounted less than the revenue from the corresponding ore which is mined much later. The optimal pit for worst case mining is generally smaller than is indicated by optimization, by using the present costs and revenues. The best case mining is when each bench is mined in turn so that the related ore and waste is mined approximately the same time period. The optimal pit in this case is close to the one obtained by optimization. The worst case mining can be used when dealing with small pits, whereas the best case mining is preferred when dealing with larger pits. There are more opportunities for creative sequencing for the latter.

The selection of a mine design, as described by Abdel Sabour *et al.* (2008), is based on estimating net present values of all possible, technically feasible mine plans so as to select the one with the maximum value. In practice, mine planners cannot know with certainty the quantity and quality of ore in the ground. This, they term the *geological uncertainty*. It is recognized among practitioners that mining is a high risk business and the geological uncertainty is a major source of risk. There are however also other sources of uncertainties. The future market behaviour of metal prices and foreign exchange rates are impossible to be known with certainty, therefore, they are sources of risks affecting mine project profitability. Abdel Sabour *et al.* (2008) use the term *market uncertainty* for these sources of risks and classify them as the second major source of risk. The existence of uncertainties can thus lead to a high probability that the actual cash flows throughout the lifetime of the mining project will be different from those expected. An optimization model that maximizes expected return while minimizing risk is therefore important for the mining sector, as this will help make better decisions on the blocks of ore to mine at a particular point in time.

Lerchs and Grossmann (1965) are the first to put forward a method to solve the open-pit mine problem. Many researchers have since tried to formulate various optimization models and have developed algorithms to solve the problem. Some researchers have also made efforts to solve the problem by applying existing optimization techniques. However, in spite of all the work done, researchers are still looking for better models and algorithms in this field of study. Newman *et al.* (2010) give a literature review based on several decades of researches in the field of mine planning. They place emphasis on more recent work, suggestions for emerging areas, and highlights of successful industry applications.

The purpose of this research work is to develop optimization models and methods in the area of open-pit mining. The thesis is given in two parts with the outline as follows. The first part is devoted to the introduction and overview of the subject area, followed by a summary of appended papers. The second part is comprised of five appended papers. In Paper I, we give some pitfalls and counterexamples on the use of parametric open-pit design models for mine scheduling. A duality-based derivation of the maximum flow formulation of the open-pit design problem is the focus of Paper II. This is followed by a multi-parametric maximum flow characterization of the open-pit scheduling problem in Paper III. In Paper IV, we provide some suggestions for Lagrangian dual heuristic and time aggregation approaches for the open-pit production scheduling problem. In Paper V, the last paper, we study a Conditional Value-at-Risk approach to uncertainty associated with open-pit mining.

1 Open-Pit Design Problem

Picard and Smith (2004) describe the open-pit design problem as a problem of choosing an ultimate contour whose total profit, that is, the sum of the profits of all the blocks in the contour, is maximal among all possible contours. In 1965, Lerchs and Grossmann made an earliest attempt of solving this problem. They associate a directed node-weighted graph, called the *mine graph*, with the three-dimensional grid of blocks. They note that the maximum profit open-pit mine contour corresponds to a maximum closure in the graph. A closure in the graph is a subset of the nodes such that if a node belongs to this set then all its successors also belong to the set, and the closure is maximal if the sum of node-weights is maximum.

In order that the walls of an open-pit shall not collapse during mining operations, miners are always mindful of how to dig the blocks from the ground. The slope requirement is the main physical constraint in that all blocks on top and preventing the mining of a given block must be removed. For example, in Figure 1.1, if the safe slope angles are assumed to be $45°$ and block 6 is to be removed, then we have to remove blocks 1, 2, and 3 as well.

It should be noted that we are only interested in mining the profitable blocks. For each block in Figure 1.2, let the values on top be block values and the corresponding number below represent the block label. Then blocks 2, 3, 4, 5, 6, 9, 10, 11, and 16 represent an

1	2	3	4
5	6	7	8
9	10	11	12

Figure 1.1: *A 2-D block model of a mine.*

ultimate pit of blocks where the maximal pit angle is $45°$ with a value of 10. Block 15 is not profitable to mine because by mining this block we must also mine blocks 1 and 8, and we see that the total value of these blocks is -1. Blocks 2, 3, 4, 5, 9, and 10 represent an alternative, and smallest, ultimate pit, since the value of removing blocks 6, 11, and 16 is zero and, therefore, contributes nothing. Since the blocks inside an ultimate pit contain a profitable amount of ore, they are to be mined.

−1 1	+1 2	+2 3	−1 4	+1 5	−1 6
−2 7	−1 8	+3 9	+4 10	−1 11	−2 12
−3 13	−2 14	+1 15	+2 16	−1 17	−2 18

Figure 1.2: *A 2-D block model of a mine with given block values.*

The open-pit design problem can be represented as a graph problem defined on a directed graph $G = (V, A)$, where V is the set of nodes and A is the set of arcs. Each block corresponds to a node and is assigned a weight. This weight, which can either be positive or negative, represents the profit value of mining the block (Lerchs and Grossmann, 1965). There is a directed arc from node i to node j if block i cannot be extracted before block j, which is on a layer immediately above block i. To decide on the blocks to mine in order to maximize profit, a maximum weight set of nodes in the graph such that all successors of all nodes in the set are also included in the set is to be found. As mentioned earlier, this set is the maximum closure of the graph, G.

In order to give a mathematical optimization formulation of the open-pit design problem, we give the following notations. Let

$$
\begin{array}{rcl}
V & = & \text{set of all blocks that can be mined.} \\
A & = & \text{set of pairs } (i,j) \text{ of blocks such that block } j \text{ is a neighbouring} \\
 & & \text{block to } i \text{ that must be removed before block } i \text{ can be mined.} \\
c_i & = & \text{cost of mining and processing block } i. \\
r_i & = & \text{revenue obtained from block } i. \\
p_i & = & \text{profit obtained by mining and processing block } i \text{ (i.e., } p_i = r_i - c_i\text{).} \\
x_i & = & \begin{cases} 1, & \text{if block } i \text{ is mined} \\ 0, & \text{otherwise.} \end{cases}
\end{array}
$$

The revenues r_i are estimated by using sampling information from drill holes. The mathematical optimization model for maximizing the total profit of the mine is therefore given by

$$
\begin{aligned}
& \max \sum_{i \in V} p_i x_i \\
& \text{subject to} \\
& x_i \le x_j, \qquad (i,j) \in A \\
& x_i \in \{0,1\}, \qquad i \in V.
\end{aligned}
\tag{1.1}
$$

The condition $x_i \le x_j$ implies that block j has to be mined before block i, while obeying the pit slope. The set A captures both the slope and the immediate precedence constraints. The constraints of the linear programming relaxation of Model (1.1), where the condition $x_i \in \{0,1\}$ is replaced by $0 \le x_i \le 1$, form a unimodular matrix. The linear programming relaxation therefore possesses the integrality property, in that all the extreme points are integral (Rhys, 1970). The condition $x_i \in \{0,1\}$ can therefore be replaced by $0 \le x_i \le 1$ without changing the optimal objective value.

Consider, for example, the pit in Figure 1.2 and let the block values represent the profits. Then the associated model is

$$
\begin{aligned}
& \max -x_1 + x_2 + 2x_3 + \ldots + 3x_9 + 4x_{10} + \ldots - x_{17} - 2x_{18} \\
& \text{subject to} \\
& x_7 \le x_1 \\
& x_7 \le x_2 \\
& x_8 \le x_1 \\
& x_8 \le x_2 \\
& x_8 \le x_3 \\
& \quad \vdots \\
& x_{15} \le x_8 \\
& x_{15} \le x_9 \\
& x_{15} \le x_{10} \\
& \quad \vdots \\
& x_{18} \le x_{11} \\
& x_{18} \le x_{12} \\
& x_i \in \{0,1\}, \qquad i = 1, \ldots, 18.
\end{aligned}
\tag{1.2}
$$

When we solve the general model (1.1), we get an optimal pit. The outline of this optimal pit is the optimal pit design. So, for the pit in Figure 1.2, the pit contour is composed of blocks 2, 3, 4, 5, 6, 9, 10, 11, and 16. However, we often also want information about scheduling (which we treat in Section 4) so that we know the order in which the blocks shall be mined.

2 Maximum Flow Formulation of the Design Problem

In this section, we give a general description of the maximum flow problem and a reformulation of the open-pit design problem as a maximum flow problem. Furthermore, we give an example of how to solve the design problem as a maximum flow problem. Finally, we discuss some solution methods.

2.1 General Maximum Flow Problem

For a given directed graph, or network, $G = (V, A)$, with V being the set of nodes and A the set of arcs, a source s and sink t are added. The source is the start node for a flow through the network and the sink is the end node for the flow. Let every arc $(i, j) \in A$ be associated with a capacity $u(i, j) \geq 0$ and let $v(i, j)$ be the flow along the arc. It is required to associate the value $v(i, j)$ satisfying $v(i, j) \leq u(i, j)$ for each arc, $(i, j) \in A$, such that for every node other than s and t, the sum of the values associated to the arcs that enter it must equal the sum of the values associated to the arcs that leave it. It is further required to maximize the sum of the values associated to the arcs leaving the source, and also entering the sink, which is the total flow in the network. The *maximum flow problem* is to find a feasible flow through the network that is maximal.

Now, let $N = (V', A')$ be the network associated with G, where $V' = V \cup \{s\} \cup \{t\}$. A *cut* (S, T) in the network N separating s and t is a partition of the nodes of V' with $s \in S$ and $t \in T$, $S \cup T = V'$, and $S \cap T = \phi$. The *capacity* $C(S, T)$ of the cut (S, T) is the sum of the capacities of the arcs over the cut. Thus,

$$C(S, T) = \sum_{i \in S, j \in T} u(i, j). \tag{2.1}$$

A *finite cut* is a cut with finite capacity and a *minimum cut* is a cut of minimum capacity among all cuts (S, T). Determining the minimum cut of the network N is equivalent to finding the maximum flow in N, from s to t. Ford and Fulkerson (1962) define this as

$$\max_{v} f = \min_{(S,T)} C(S, T), \tag{2.2}$$

where f is the value of flow.

Ford and Fulkerson (1956) develop an algorithm that computes the maximum flow in a network. As long as there is a path from s to t, with available capacity on all arcs in the path, flow is sent along one of these paths. Then another path is found, and so on. A path with available capacity is called an augmenting path. To describe the algorithm, it should be noted that the following is maintained after every step in the process:

- $v(i, j) \leq u(i, j)$, that is, the flow from i to j does not exceed the capacity.

- $v(i,j) = -v(j,i)$, that is, maintaining the net flow between i and j. If say a units are going from i to j, and b units from j to i, then $v(i,j) = a-b$ and $v(j,i) = b-a$.
- $\sum_{j \in V} v(i,j) = 0$, unless $i = s$ or $j = t$. That is, the net flow to a node is zero, except for the source, which "produces" flow, and the sink, which "consumes" flow.

This means that the flow through the network is a *legal flow* after each round in the algorithm. They define the *residual network* $N_v = (V, A'_v)$ as the network with capacity $u_v(i,j) = u(i,j) - v(i,j)$ and no flow. It should be noted that it is not certain that $A' = A'_v$, as sending flow on arc (i,j) might close (i,j), but open a new arc (j,i) in the residual network. The algorithm is as follows:

Inputs: Graph N with flow capacity u, source s and sink t

Output: A flow from s to t which is maximum

1. $v(i,j) \leftarrow 0$ for all arcs $(i,j) \in A'$
2. While there is a path p from s to t in N_v with $u_v(i,j) > 0$ for all $(i,j) \in p$
   1. Find $u_v(p) = \min\{u_v(i,j) \mid (i,j) \in p\}$
   2. For each arc $(i,j) \in p$
      1. $v(i,j) \leftarrow v(i,j) + u_v(p)$, that is, send flow along the arc
      2. $v(j,i) \leftarrow v(j,i) - u_v(p)$, that is, decrease flow along the arc, or, equivalently, send flow in the backward direction.

The concept described here can be used to solve the open-pit design problem.

2.2 Reformulation of the Design Problem

Picard (1976), using the mine graph augmented with source and sink nodes, finds the maximum closure by solving a maximum flow problem.

To find a maximum closure $Y \subset V$, for any given directed graph $G = (V, A)$, where V represents the set of nodes and A the set of arcs, Picard formulates the problem as a 0-1 programming given by

$$\begin{aligned} &\max z = \sum_{i \in V} p_i x_i \\ \text{subject to} \quad & \\ &x_i \leq x_j, \qquad (i,j) \in A \\ &x_i \in \{0,1\}, \qquad i \in V, \end{aligned} \tag{2.3}$$

where p_i is a weight associated to node i and x_i is a binary variable, which is equal to 1 if $i \in Y$ and 0 otherwise. In the context of open-pit design the node weight is the net profit of a block. The condition $x_i \leq x_j$ for $(i,j) \in A$ is equivalent to the relation $a_{ij} x_i (x_j - 1) = 0$, where a_{ij} is the element (i,j) of the incident matrix of the graph G, that is, $a_{ij} = 1$ if $(i,j) \in A$ and 0 otherwise. Problem (2.3) can then be written as

$$
\begin{aligned}
&\max z = \sum_{i \in V} p_i x_i \\
&\text{subject to} \\
&\sum_{i \in V} \sum_{j \in V} a_{ij} x_i (x_j - 1) = 0 \\
&x_i \in \{0,1\}, \qquad i \in V.
\end{aligned}
\tag{2.4}
$$

Since $a_{ij}x_i(x_j - 1) \leq 0$ for all $x_i \in \{0,1\}$ and $x_j \in \{0,1\}$, Picard deduces that (2.4) is equivalent to

$$
\begin{aligned}
&\max z = \sum_{i \in V} p_i x_i + \lambda \sum_{i \in V} \sum_{j \in V} a_{ij} x_i (x_j - 1) \\
&\text{subject to} \\
&x_i \in \{0,1\}, \qquad i \in V,
\end{aligned}
\tag{2.5}
$$

where λ is a positive number large enough to ensure that an optimal solution of (2.5) satisfies $\sum_{i \in V} \sum_{j \in V} a_{ij} x_i (x_j - 1) = 0$, that is, that we have a closure of G. Picard then replaces the maximization problem by a minimization problem and finally arrives at an equivalent problem

$$
\begin{aligned}
&\min f(x) = \sum_{i \in V} (-p_i x_i) + \sum_{i \in V} \sum_{j \in V} \lambda a_{ij} x_i (1 - x_j) \\
&\text{subject to} \\
&x_i \in \{0,1\}, \qquad i \in V.
\end{aligned}
\tag{2.6}
$$

According to Picard, solving (2.6) is equivalent to finding a minimum cut in a related network. Let the digging constraints of a given mine graph $G = (V, A)$ be such that block $j \in V$ has to be removed before block $i \in V$. The closure in G is a set of nodes $Y \subset V$ such that if $i \in Y$ and $(i,j) \in A$, then $j \in Y$, that is, it is a subset Y of nodes such that if a node belongs to the closure then all its successors also belong to the set Y. The *value* of a closure Y is given by

$$
v(Y) = \sum_{i \in Y} p_i. \tag{2.7}
$$

The closure of maximum value is thus a maximum closure. A feasible contour of the open-pit mine corresponds to a closure in the mine graph, and the open-pit mine design problem then becomes that of determining the maximum closure in the mine graph (see also Picard and Smith, 2004). The set of arcs in A' of the network $N = (V', A')$, as defined in Section 2.1, are such that:

Arcs (s,i) with capacity $u(s,i) = p_i$ for all i with $p_i > 0$.
Arcs (j,t) with capacity $u(j,t) = -p_j$ for all j with $p_j \leq 0$.
Arcs (k,l) with capacity $u(k,l) = \infty$ for all arcs $(k,l) \in A$.

Picard establishes that a maximum closure in the graph G corresponds to a minimum cut in the network N. Further, the relation

$$v(Y) = \sum_{i:p_i>0} p_i - C(S,T) \tag{2.8}$$

was established between the value of the closure and the capacity of the corresponding cut.

To solve the open-pit design problem, we form a network related to the mine graph of the mine deposit and then solve the maximum flow problem in the network (Picard and Smith, 2004).

2.3 An Example

In this section, we give an example of the maximum flow formulation of the open-pit design problem. The derivation of the maximum flow formulation does not follow Picard (1976). Instead, we apply the duality-based derivation presented in Paper II.

$+1$ x_1	$+1$ x_2	-1 x_3	-1 x_4
	$+2$ x_5	-1 x_6	

Figure 2.1: *A 2-D block model of a mine with given profit values.*

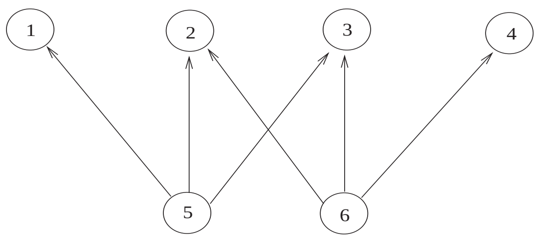

Figure 2.2: *Mine graph of the block model in Figure 2.1.*

Consider the two-dimensional block model of a mine with given profit values in Figure 2.1. The mine graph corresponding to this block model is given in Figure 2.2. If we think of maximizing profit, then we have to solve the associated problem

$$
\begin{aligned}
&\max x_1 + x_2 - x_3 - x_4 + 2x_5 - x_6 \\
\text{subject to}\quad & \\
& x_5 \le x_1 \\
& x_5 \le x_2 \\
& x_5 \le x_3 \\
& x_6 \le x_2 \\
& x_6 \le x_3 \\
& x_6 \le x_4 \\
& x_i \in \{0,1\}, \qquad i = 1,\dots,6.
\end{aligned}
\tag{2.9}
$$

By relaxing the integrality restrictions on the variables and the lower and upper bound constraints on the variables having positive and negative profit values, respectively, we obtain the problem

$$
\begin{aligned}
&\max x_1 + x_2 - x_3 - x_4 + 2x_5 - x_6 \\
\text{subject to}\quad & \\
&\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
& x_1 \le 1, x_2 \le 1, x_5 \le 1 \\
& -x_3 \le 0, -x_4 \le 0, -x_6 \le 0.
\end{aligned}
\tag{2.10}
$$

Letting y_{ij}, w_i, and z_i, be the dual variables for the three respective sets of constraints, the linear programming dual is given by

$$
\begin{aligned}
&\min w_1 + w_2 + w_5 \\
\text{subject to}\quad & \\
& -y_{51} + w_1 = 1 \\
& -y_{52} - y_{62} + w_2 = 1 \\
& -y_{53} - y_{63} - z_3 = -1 \\
& -y_{64} - z_4 = -1 \\
& y_{51} + y_{52} + y_{53} + w_5 = 2 \\
& y_{62} + y_{63} + y_{64} - z_6 = -1 \\
& y_{51} \ge 0,\ y_{52} \ge 0,\ y_{53} \ge 0,\ y_{62} \ge 0,\ y_{63} \ge 0,\ y_{64} \ge 0 \\
& w_i \ge 0, \qquad i = 1,2,5 \\
& z_i \ge 0, \qquad i = 3,4,6,
\end{aligned}
\tag{2.11}
$$

which can be reformulated as

$$
\begin{aligned}
&\max f \\
&\text{subject to} \\
&v_{s1} + v_{s2} + v_{s5} = f \\
&-v_{s1} - y_{51} = 0 \\
&-v_{s2} - y_{52} - y_{62} = 0 \\
&-y_{53} - y_{63} + u_{3t} = 0 \\
&-y_{64} + u_{4t} = 0 \\
&-v_{s5} + y_{51} + y_{52} + y_{53} = 0 \\
&y_{62} + y_{63} + y_{64} + u_{6t} = 0 \\
&-u_{3t} - u_{4t} - u_{6t} = -f \\
&y_{51} \geq 0,\ y_{52} \geq 0,\ y_{53} \geq 0,\ y_{62} \geq 0,\ y_{63} \geq 0,\ y_{64} \geq 0 \\
&v_{si} \geq 0, \qquad i = 1, 2, 5 \\
&u_{it} \geq 0, \qquad i = 3, 4, 6.
\end{aligned}
\tag{2.12}
$$

This is a maximum flow problem as shown in Figure 2.3, where f is total flow, v_{si} is a flow from a source node s to a node corresponding to block i with positive profit value, u_{it} is a flow from a node corresponding to block i with non-positive profit value to a sink node t, and y_{ij} is a flow from node i to node j. The values given on the arcs are their respective capacities. Solving the maximum flow problem, we get the result in Figure 2.4, with the optimal flow and capacity shown on each arc.

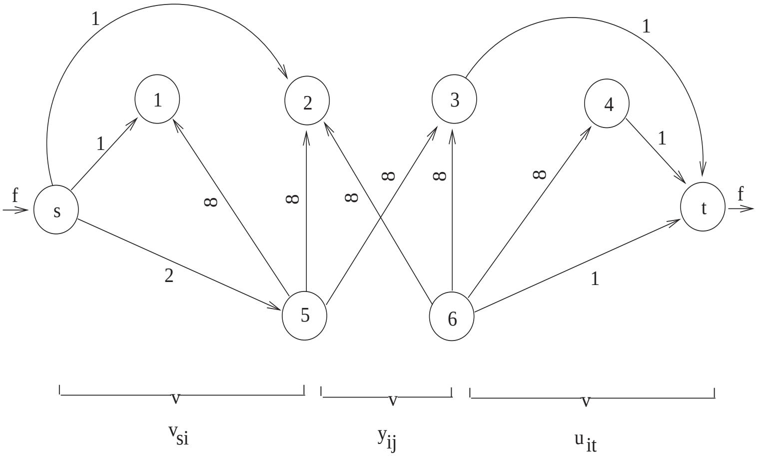

Figure 2.3: *A network for the maximum flow problem.*

Here, the capacity of the cut separating s from t has the same value as the flow, which equals 1. This means that (2.2) is fulfilled, so that the flow is maximal and the cut is minimal. To maximize profit for the pit, we therefore have to mine blocks 1, 2, 3, and 5 as shown in Figure 2.5. The maximum profit is 3, while $p_1 + p_2 + p_5 = 4$ and the minimum cut capacity is 1, which illustrates the relationship (2.8).

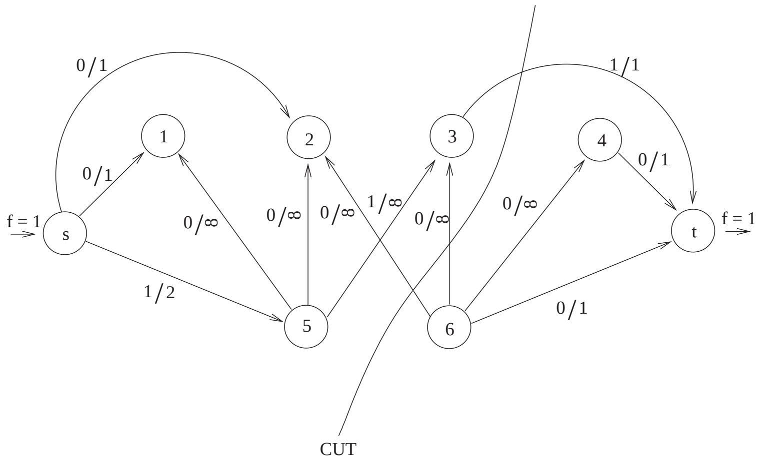

Figure 2.4: *A maximum flow and a minimum cut.*

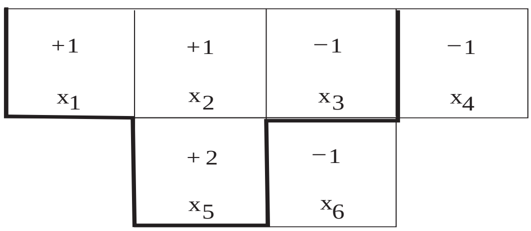

Figure 2.5: *Final pit shape.*

3 Solution Methods for the Design Problem

Lerchs and Grossmann (1965) present an algorithm to determine the optimum design for an open-pit mine. The aim is to design the contour of a pit that maximizes the difference between total mine value of the extracted ore and the total cost of extraction of ore and waste materials. They give the mine graph model of the problem and show that an optimal solution of the ultimate pit problem is the same as finding the maximum closure of their model.

The mine graph G is first augmented with a dummy node x_0 and dummy arcs (x_0, x_i). The node x_0 is assigned a negative weight so that it cannot be part of the maximum closure. A tree T with one distinguished node x_0 (called the *root* of T) is known as a *rooted tree*. The algorithm starts with the construction of a tree T^0 in G. The tree is then transformed into successive trees $T^1, T^2, \ldots, T^n$ following given rules, until no further transformation is possible. The maximum closure is then given by the nodes of a set of well identified branches of the final tree.

Each arc a_i of a tree defines a *branch* T_i. The arc a_i is said to *support* T_i. The weight p_i of a branch is the sum of the weights of all nodes of the branch. This weight is associated with a_i and we say that a_i *supports* a weight p_i. In a tree T with root x_0, a branch T_i is characterized by the orientation of the arc a_i with respect to x_0. The arc a_i is called a *p-arc* (plus-arc) if it points toward T_i, that is, if the terminal node of a_i is part of T_i. The branch T_i then is called a *p-branch*. If a_i points away from the branch then it is called an *m-arc* (minus-arc) and the branch is an *m-branch*. A *p-arc* (branch) is *strong* if it supports a weight that is strictly positive. An *m-arc* (branch) is strong if it supports a weight that is zero or negative. Arcs (branches) that are not strong are said to be *weak*. A node x_i is said to be strong if there exists at least one strong arc on the (unique) path of T joining x_i to the root x_0. Nodes that are not strong are said to be weak. Finally, a tree is *normalized* if the root x_0 is common to all strong arcs. The maximum closure of a normalized tree is the set of its strong nodes.

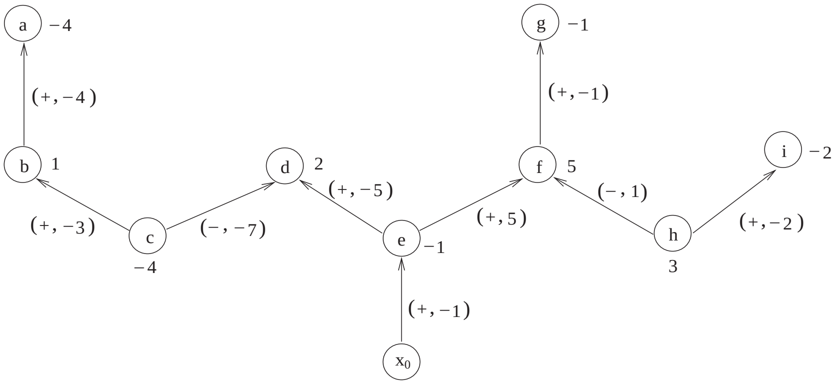

Figure 3.1: *A mine graph augmented with a dummy node* x_0.

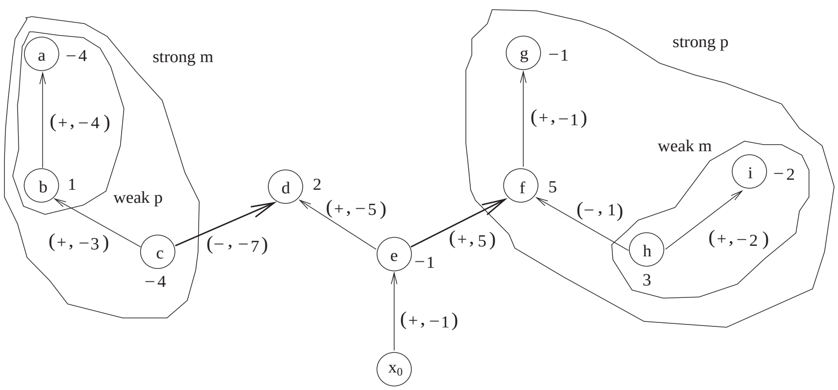

Figure 3.2: *Strong and weak branches.*

To illustrate these concepts, we consider the augmented mine graph in Figure 3.1, which has been used by Lerchs and Grossmann and also further explained by Caccetta and Giannini (1988). The value assigned to each node is the weight of that node. Each arc is labelled in the form $(\pm, p_i)$ to identify its status. Here, a $+$ or a $-$ sign represents a p-arc or an m-arc respectively, and p_i is the support of the arc. By examining the label of an

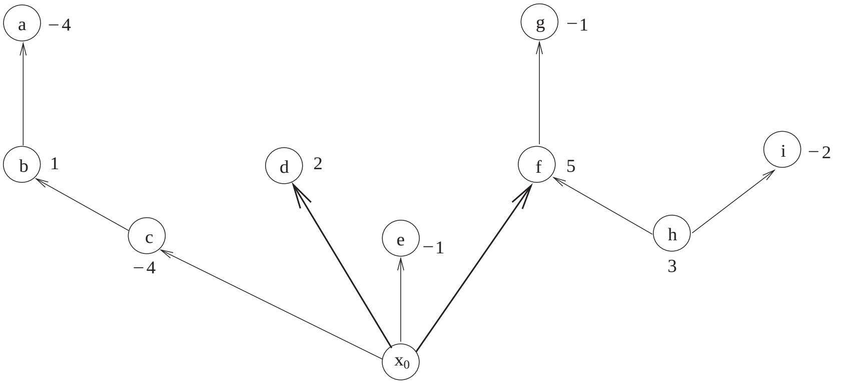

Figure 3.3: *The tree obtained after normalization.*

arc we can identify it as strong or weak. The arcs (c, d) and (e, f) are strong, and all the others are weak. Therefore, nodes a, b, c, f, g, h, and i are strong while d and e are weak. An illustration of strong and weak branches is given in Figure 3.2. Figure 3.3 is the tree obtained after normalization. We note here that as all dummy arcs are p-arcs, all strong arcs of a normalized tree will also be p-arcs.

The Steps of the algorithm are summarized by Caccetta and Giannini as follows.

Step 1 (**Initialization**):
Construct a normalized tree T^0; T^0 is taken as the spanning tree (a tree of maximal set of arcs that contains no cycle) whose arc set is $\{(x_0, x_i) : x_i \in V\}$. Identify the set Y^0 of strong nodes of T^0 (which are those nodes with positive weight). Set $i = 0$ and proceed to Step 2.

Step 2 (**Optimality test**):
Search for an arc (x_k, x_l) in G such that $x_k \in Y^i$ and $x_l \notin Y^i$ then proceed to Step 3. If no such arc is found, stop; Y^i is a maximum closure of G.

Step 3 (**Update**):
Identify the unique (x_0, x_k)-*path* P in T^i. Suppose x_0 is joined to x_m in P. Construct the tree $T^{'i}$ by replacing the arc (x_0, x_m) of T^i with the arc (x_k, x_l) and proceed to Step 4.

Step 4 (**Normalization**):
Normalize the tree $T^{'i}$. This yields a tree T^{i+1}. Identify the set Y^{i+1} of strong vertices of T^{i+1}. Set $i + 1 = i$ and go to Step 2.

It should be noted that

$$T^{'i} = T^i + (x_k, x_l) - (x_0, x_m) \tag{3.1}$$

The normalized tree T^{i+1} is obtained by moving along the unique (x_m, x_0)-*path* and when a strong arc is encountered this arc is deleted and x_0 is joined to its branch root.

Picard and Queyranne (1982) give a binary quadratic programming formulation of the minimum cut problem and mention that it is of considerable importance in the determination of the optimal contour of an open-pit mine.

Zhao and Kim (1992) present a graph theory oriented algorithm for optimum pit design. Their algorithm produces an optimal solution and maximizes the total undiscounted net profit for a given 3-D block mine model. In terms of the reduction in computation time and computer memory requirements, they view their algorithm to be of better performance as compared to the well known Lerchs and Grossmann Algorithm.

Underwood and Tolwinski (1998) develop a network flow algorithm based on the dual to solve the problem of finding a maximum closure in a mine graph, as studied by Lerchs and Grossmann (1965). They demonstrate that the algorithm is closely related to that of Lerchs and Grossmann and show how the steps in the algorithm of the latter can be viewed in mathematical programming terms. They show that at any stage of the dual simplex algorithm the nodes in the union of all strong branches in a tree correspond to the nodes being 'removed'. The algorithm, they describe, will move from one basic solution of the dual to another. When considering new arcs to bring into the basis, it is needed only to look at arcs joining strong nodes to weak nodes. The goal of their algorithm and that of Lerchs and Grossmann is to find a collection of strong nodes which are feasible. A strong node is feasible when there are no weak blocks above it which must first be removed in order to allow access to the block, otherwise, there is a strong node which depends on a weak node. The major difference between the two algorithms is the manner a cycle is broken for the purpose of attaining a spanning tree. In the case of Underwood and Tolwinski, a cycle is broken by analyzing the change in flow around the cycle. This value is so chosen to increase this flow as much as possible while maintaining dual feasibility. Lerchs and Grossmann instead use a simple rule for removing the arc. For a detailed description of how this is done, see Underwood and Tolwinski (1998).

Hochbaum and Chen (2000) view the open-pit mining problem as a problem to determine the contours of a mine, based on economic data and engineering feasibility requirements in order to yield maximum possible net income. They mention that this problem needs to be solved for very large data sets. They note that in practice, it is necessary to test multiple scenarios taking into account a variety of realizations of geological predictions and forecasts of ore value. Even though they view the problem to be equivalent to the minimum cut or maximum flow problem they realize that the industry was experiencing computational difficulties in solving the problem. They recognize that the most widely commonly used algorithm by the mining industry has been the one devised by Lerchs and Grossmann in 1965. They develop a maximum flow "push-relabel" algorithm and compared it to that of Lerchs and Grossmann. They demonstrate that their algorithm has a superior performance over the Lerchs and Grossmann algorithm for all types of mine data they tested. However, they admit that the Lerchs and Grossmann algorithm has an advantage with its more economic use of space and memory, which reduces overhead.

4 Open-Pit Scheduling Problem

Once the open-pit design problem is solved and the final pit shape is determined, we want to know the order in which to mine the blocks in the pit. The open-pit mine production scheduling can be viewed as the sequence by which the blocks of the mine should be removed over the lifetime of the mine in order to maximize the total profit from the mine, subject to a set of operational and physical constraints. The optimum pit design plays an important role in mine scheduling, and it should be at constant review at all stages of the life of an open-pit.

Lerchs and Grossmann (1965) use a parametric analysis concept with the aim of generating an extraction sequence of a mine. They use an undiscounted model and introduce an additional, artificial, cost, $\lambda \geq 0$, as a parameter to change the economic value of each block i from p_i to $(p_i - \lambda)$. By increasing the value of λ, a set of corresponding nested pit outlines are generated. Let $S(\lambda)$ denote the pit outline generated for a given λ. As shown in the example in Figure 4.1, smaller and smaller pits are formed as λ is increased, that is, $S(\lambda_4) \supseteq S(\lambda_3) \supseteq S(\lambda_2) \supseteq S(\lambda_1)$ as $\lambda_4 < \lambda_3 < \lambda_2 < \lambda_1$. These pits can be used to produce a production schedule of a mine. However, since time is not an explicit input factor, the pits produced may be unpredictable with respect to mining time needed for each of the pits in the sequence.

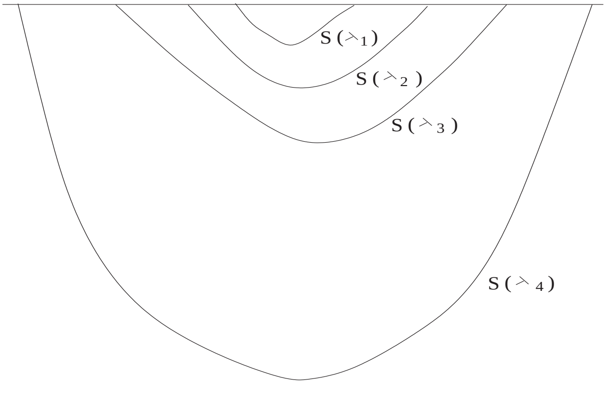

Figure 4.1: *Nested pit outlines for different parameters.*

Picard and Smith (2004) define an economic criterion of evaluating intermediate contours on the way to the optimal ultimate contour, and also show how to calculate these contours. They apply the technique of Picard (1976) and a parametric maximum flow algorithm. They show with this parametric analysis how to choose intermediate contours in open-pit mine design so as to proceed to the optimal ultimate contour while maximizing the benefit-cost ratio of the intermediate contours. As in the parametric method of Lerchs and Grossmann, described above, the intermediate contours can be viewed as a mining sequence and thus, a kind of mine scheduling.

In an attempt to determine a production schedule which is optimum, Dagdelen and Johnson (1986) develop an algorithm which, in theory, is based on the Lagrangian concepts of mathematics, and in practice is based on the parameterization concept of mining. They describe the fundamentals of the algorithm and further programmed and applied it to a small three-dimensional block model. Applications of programs to different scheduling conditions showed fast convergence.

Both methods by Lerchs and Grossmann (1965) and Underwood and Tolwinski (1998), presented in Section 3, provide a sequence of mining an open-pit. However, these approaches of mining sequencing are not enough to solve the open-pit mine production scheduling problem. In practice, we want to construct intermediate optimal pits and if possible, we want to control how these intermediate optimal pits change from one nested pit to another. If in the construction we have a rapid change in the size of the pit at some stage then we cannot control the sequence of intermediate optimal pit structure. This possibility of a large increment in the size of the pit from one nested pit to the next is known as the 'gapping phenomenon'.

4.1 Models for Scheduling

To avoid the situation of encountering the gapping phenomenon, the time factor, as well as mine capacity constraints on the tonnage to be mined can be introduced explicitly in the formulation of the open-pit scheduling problem. We begin by defining the following notations to be used in the problem formulation. In addition to the notations V and A in Section 1, let

$$
\begin{array}{rcl}
T & = & \text{number of time periods.} \\
p_i^t & = & \text{profit derived from mining and processing block } i \text{ in time period } t. \\
b_i & = & \text{tonnage of block } i. \\
u^t & = & \text{upper bound on tonnage mined in time period } t. \\
x_i^t & = & \begin{cases} 1, & \text{if block } i \text{ is mined in time period } t\text{, or earlier} \\ 0, & \text{otherwise.} \end{cases}
\end{array}
$$

A mathematical model for the open-pit scheduling problem then is

$$
\max \sum_{t=1}^{T} \sum_{i \in V} p_i^t \left(x_i^t - x_i^{t-1} \right)
$$

subject to

$$
\begin{aligned}
& \sum_{i \in V} b_i \left(x_i^t - x_i^{t-1} \right) \le u^t, \qquad t = 1, \dots, T \\
& x_i^t \le x_j^t, \qquad (i,j) \in A,\ t = 1, \dots, T \\
& x_i^{t-1} \le x_i^t, \qquad t = 1, \dots, T,\ i \in V \\
& x_i^0 = 0, \qquad i \in V \\
& x_i^t \in \{0,1\}, \qquad i \in V,\ t = 1, \dots, T.
\end{aligned}
\tag{4.1}
$$

As above, the set A captures both the slope and the immediate precedence constraints, so that the condition $x_i^t \le x_j^t$ implies that block j has to be mined if block i is to be mined, in time period t.

As an example, consider the pit model in Figure 4.2, which is taken from Lerchs and Grossmann (1965). For each block, the value on top is the profit and the number at the bottom is the block index. Taking the profits as the initial profits, p_i^0, and solving the scheduling problem for $T = 5$, $u^t = 9$, for all t, and $b_i = 1$, for all i, we obtain the result in Figure 4.3. A discount factor of 0.90 is assumed for all the time periods and so the profit for period t is computed as $p_i^t = 0.90^{t-1} p_i^0$. In Figure 4.3, the number in a block indicates the time period in which that block is mined.

Caccetta and Hill (2003) present a mixed integer linear programming scheduling formulation that distinguishes between ore and waste blocks and that incorporates constraints

−4 71	−4 72	−4 73	−4 74	8 75	12 76	12 77	0 78	−4 79	−4 80	−4 81	−4 82	−4 83	−4 84	−4 85	−4 86	−4 87	−4 88
	−4 55	−4 56	−4 57	0 58	12 59	12 60	8 61	−4 62	−4 63	−4 64	−4 65	−4 66	−4 67	−4 68	−4 69	−4 70	
		−4 41	−4 42	−4 43	8 44	12 45	12 46	0 47	−4 48	−4 49	−4 50	−4 51	−4 52	−4 53	−4 54		
			−4 29	−4 30	0 31	12 32	12 33	8 34	−4 35	−4 36	−4 37	−4 38	−4 39	−4 40			
				−4 19	−4 20	8 21	12 22	12 23	0 24	−4 25	−4 26	−4 27	−4 28				
					−4 11	0 12	12 13	12 14	8 15	−4 16	−4 17	−4 18					
						−4 5	8 6	12 7	12 8	0 9	−4 10						
							0 1	12 2	12 3	8 4							

Figure 4.2: *A pit model.*

		4	2	1	1	1	1	1	2	3	3	5					
			4	2	1	1	1	2	3	3	5						
				4	2	1	2	3	3	5							
					4	2	3	3	5								
						4	3	5									
							5										

Figure 4.3: *Scheduling result for $T = 5$ and $u^t = 9$, for all t.*

on mill throughput (mill feed and mill capacity) and volume of material extracted. The formulation is given as

$$\max z = \sum_{t=2}^{T} \sum_{i \in V} \left(p_i^{t-1} - p_i^t\right) x_i^{t-1} + \sum_{i \in V} p_i^T x_i^T \tag{4.2a}$$

subject to

$$\sum_{i \in V_o} b_i x_i^1 - m^1 = 0 \tag{4.2b}$$

$$\sum_{i \in V_o} b_i \left(x_i^t - x_i^{t-1}\right) - m^t = 0, \qquad t = 2, \ldots, T \tag{4.2c}$$

$$\sum_{i\in V_w} b_i x_i^1 \le u_w^1 \tag{4.2d}$$

$$\sum_{i\in V_w} b_i \left(x_i^t - x_i^{t-1}\right) \le u_w^t, \qquad t = 2,\dots,T \tag{4.2e}$$

$$x_i^{t-1} \le x_i^t, \qquad t = 2,\dots,T,\ i \in V \tag{4.2f}$$

$$x_i^t \le x_j^t, \qquad i \in V,\ (i,j) \in A,\ t = 1,\dots,T \tag{4.2g}$$

$$l_o^t \le m^t \le u_o^t, \qquad t = 1,\dots,T \tag{4.2h}$$

$$x_i^t \in \{0,1\}, \qquad i \in V,\ t = 1,\dots,T, \tag{4.2i}$$

where T is again the number of time periods over which the mine is being scheduled, V is the set of blocks (ore or waste), V_o is the set of ore blocks, V_w is the set of waste blocks, A is the set of precedence arcs, p_i^t is the profit (net present value) resulting from the mining of block i in period t, b_i is the tonnage of block i, m^t is the tonnage of ore milled in period t, l_o^t is the lower bound on the amount of ore that is milled in period t, u_o^t is the upper bound on the amount of ore that is milled in period t, u_w^t is the upper bound on the amount of waste that is mined in period t, and x_i^t is a binary variable, which is equal to 1 if block i is mined in periods 1 to t and 0 otherwise.

Constraints (4.2b), (4.2c) and (4.2h) ensure that the milling capacities hold. Constraints (4.2d) and (4.2e) ensure that the tonnage of waste mined does not exceed the prescribed upper bounds. Constraint (4.2f) ensures that a block is mined in one period only. Constraint (4.2g) describes the wall slope restrictions. Clearly, this formulation has a close resemblance to the model to be presented next.

Now, we define auxiliary variables y_i^t as

$$y_i^t = \begin{cases} 1, & \text{if block } i \text{ is being mined in period } t \\ 0, & \text{otherwise.} \end{cases}$$

Then $y_i^t = x_i^t - x_i^{t-1}$ and the objective becomes $\max \sum_t \sum_i p_i^t y_i^t$. The constraint (4.2f) means $0 = x_i^0 \le x_i^1 \le x_i^2 \le \dots \le x_i^T$, which is the same as $\sum_t y_i^t \le 1$. This means each block is mined at most once. The fact that y_i^t is 0 or 1, implies that $y_i^t \ge 0$ and thus $x_i^t \ge x_i^{t-1}$, which is constraint (4.2f). Also, we have that $\sum_{\tau=1}^t y_i^\tau = x_i^t$, $t = 1,\dots,T$, and so constraint (4.2g) turns to $\sum_{\tau=1}^t y_i^\tau \le \sum_{\tau=1}^t y_j^\tau$. This implies that $y_i^t \le \sum_{\tau=1}^t y_j^\tau$, $t = 1,\dots,T$, $(i,j) \in A$. The formulation (4.2) can thus be restated as

$$
\begin{aligned}
&\max \sum_{t=1}^{T} \sum_{i \in V} p_i^t y_i^t \\
&\text{subject to} \\
&\sum_{i \in V_o} b_i y_i^t - m^t = 0, \qquad t = 1, \ldots, T \\
&\sum_{i \in V_w} b_i y_i^t \le u_w^t, \qquad t = 1, \ldots, T \\
&\sum_{\tau=1}^{t} y_j^\tau - y_i^t \ge 0, \qquad t = 1, \ldots, T,\ (i,j) \in A \\
&\sum_{t=1}^{T} y_i^t \le 1, \qquad i \in V \\
&y_i^0 = 0, \qquad i \in V \\
&l_o^t \le m^t \le u_o^t, \qquad t = 1, \ldots, T \\
&y_i^t \in \{0,1\}, \qquad i \in V,\ t = 1, \ldots, T.
\end{aligned}
\tag{4.3}
$$

Ferland *et al.* (2007) consider the capacitated open-pit mine problem by looking at the sequential extraction of blocks in order to maximize the total discounted profit under an extraction capacity during each period of a planning horizon. As above, let V be the set of blocks in the mining site and A the set of precedence arcs. Further, let p_i denote the net value of extracting block i, where $i \in V$, b_i denote the tonnage of block i, $i \in V$, u^t the maximal weight that can be extracted during time period t, $1 \le t \le T$, and $1/(1+\alpha)$ the discount rate per period. Also, let a binary variable x_i^t be associated with each block i for each period t:

$$
x_i^t = \begin{cases} 1, & \text{if block } i \text{ is being mined in period } t \\ 0, & \text{otherwise.} \end{cases}
$$

Then their formulation of the scheduling block extraction problem is as follows.

$$\max \sum_{t=1}^{T} \sum_{i \in V} \frac{p_i}{(1+\alpha)^{t-1}} x_i^t \tag{4.4a}$$

subject to

$$\sum_{t=1}^{T} x_i^t \le 1, \qquad i \in V \tag{4.4b}$$

$$\sum_{l=1}^{t} x_j^l - x_i^t \ge 0, \quad t = 1, \ldots, T,\ (i,j) \in A \tag{4.4c}$$

$$\sum_{i \in V} b_i x_i^t \le u^t, \qquad t = 1, \ldots, T \tag{4.4d}$$

$$x_i^t \in \{0,1\}, \qquad i \in V,\ t = 1, \ldots, T. \tag{4.4e}$$

The objective function (4.4a) accounts for the discount factor in evaluating the net values of the blocks when they are extracted. Constraint (4.4b) guarantees that each block i is extracted at most once during the planning horizon. The extraction precedence is enforced by constraint (4.4c), and the constraint (4.4d) describes the extraction capacity during each period of the horizon.

To initiate the first extraction period, $t = 1$, we first remove the block among those having no predecessor (i.e., in the top layer) having the largest priority. During any period t, the next block to be removed is one of those with the highest priority among those having all their predecessors already extracted such that the capacity u^t is not exceeded by its extraction. If no such block exists, then a new extraction period $(t + 1)$ is initiated.

4.2 Adaptations to Large-Scale Mining

Researchers usually encounter computational difficulties when solving the scheduling problem of realistic size, because most often, there are too many blocks (over 100,000) within the final pit limits to determine the optimum annual production schedule.

Zhang (2006) approaches the problem of large number of blocks in a mine by combining a genetic algorithm and topological sorting to find the extraction schedule of a mine. For a given orebody, the approach simultaneously determines an ultimate pit of a mine and an optimal block extraction schedule that maximizes the net present value by specifying whether a block should be extracted and if yes, when to dig it out and where to send it (i.e., the waste dump or the processing plant), subject to a number of constraints including maximum wall slope, mining and processing capacities.

Genetic algorithms are stochastic, parallel search algorithms based on the theory of natural selection and the process of evolution. These algorithms begin with a set of possible solutions randomly selected in the solution space as a population. Each possible solution is represented as a chromosome and evaluated for its fitness calculated using an objective function. Genetic operators such as selection, crossover and mutation are then applied to individuals in the current population so as to generate a new population. The process is repeated until an optimal solution, which is the best fitness of any generation, is obtained. A topological sort is an ordering of the nodes of a directed acyclic graph such that every edge from a node numbered i to a node numbered j satisfies $i < j$. A mine graph is clearly an acyclic graph. Topological sorting is used to randomly select a block from a block model and put it in a queue. The approach starts with an initial population of chromosomes being created. For each chromosome in the population, topological sorting is performed and a schedule is determined. Next, the chromosomes are evaluated for their fitness to determine whether or not they have all been processed. If so, then a stopping criteria is reached and the process ends, otherwise, genetic algorithm operators to generate a new population are applied and the process is repeated.

Before using such a procedure for production schedule, the blocks of a mine are first aggregated. Spatially connected blocks with similar properties are predisposed to belong to the same aggregation. Blocks in an aggregation are classified into bins according to their grades, with blocks in each bin being assumed to be homogeneous in quality. Each chromosome is comprised of a number of genes, where a gene represents a block aggregate. A block aggregate is encoded using the following:

- A random real number which stands for the topological sort preference.
- A decision array where each element represents a bin in the block aggregate and is

encoded using a random integer standing for a selected destination and a random real number in $(0, 1]$ representing the proportion of the bin to be sent to the selected destination.

Here, destination is where a block bin is sent, whether to the waste dump or to the processing plant.

A schedule is determined based on the number encoded in the corresponding chromosome. Based on the topological sort preference in each block aggregate, topological sorting is performed to obtain a feasible extraction sequence of block aggregates satisfying the maximum wall slope constraints. Information from the encoded destination and the fraction of a block bin to be extracted in each block aggregate determine the total tonnage for each block aggregate directed to the waste dump and processing plant respectively. The extraction time for block aggregates are determined by cutting the extraction sequence of block aggregates into a number of smaller groups retaining the extraction sequence. To evaluate the fitness value of each schedule in a population, Zhang uses the objective function

$$NPV = \sum_t \sum_k \sum_j \sum_i V_{ijk} x_{ijk} d_t, \tag{4.5}$$

where V_{ijk} is the value of bin i of block aggregation j to destination k, x_{ijk} is the fraction of bin i of block aggregation j to destination k, and d_t is the discount factor at time t.

By this approach, Zhang concludes that the computational time can effectively be reduced whilst maintaining near optimal mine plans. This conclusion was arrived at after an implementation and an evaluation of the approach against BHP-Billiton's existing industrial benchmark achieved by the commercial optimizer ILOG CPLEX. Zhang notes that a difficulty in the use of genetic algorithm to solve the mine planning problem is that of dealing with the constraints. Two ways are suggested to handle this difficulty. Firstly, the constraints are relaxed and then the violations of constraints are penalized in the objective function. Secondly, the constraints are embedded in chromosome coding and the chromosomes are forced to be feasible in the population. Newman *et al.* (2010) point out that Zhang does not mention the practical consequences of aggregation or how to subsequently disaggregate.

Ramazan (2007) proposes an algorithm called the "Fundamental Tree Algorithm", to reduce the number of binary variables required in mixed integer programming (MIP) formulations and the number of constraints within MIP for optimizing annual production scheduling in open-pit mines. The size of the problem can be reduced by partitioning the blocks within the final pit limits into smaller volumes called "pushbacks". Mixed integer and linear programming models are recognized as having significant potential for optimizing production scheduling in open-pit mines, in which the objective is to maximize total discounted profit. However, an MIP formulation of the production scheduling problem for open-pit mines requires too many binary variables making it very difficult or impossible to solve. For example, if there are 10,000 mining blocks in a pushback to be scheduled over three years, it will require 30,000 binary variables to generate the MIP formulation. The algorithm of Ramazan involves a newly developed linear programming model formulation to combine blocks into Fundamental Trees. A "Fundamental Tree" is a term used to describe a set of combined blocks, on condition that the combined blocks have the following three properties.

1. The combined blocks can be mined without violating the slope constraints.
2. The total economic value of the combined blocks is positive.
3. A fundamental tree is maximal in that it cannot be partitioned into smaller trees without violating 1 or 2.

All the blocks within the pushback considered for optimization must belong to a fundamental tree. MIP scheduling formulations of the trees are done by treating each fundamental tree as a block having certain amount of ore tons with average grade commodities and possibly some waste tons. According to Ramazan, a substantial reduction in the MIP problem size by applying the fundamental tree algorithm reduced the required solution time significantly, making it possible to apply MIP models to large open-pit mines.

Boland *et al.* (2009) use aggregation with respect to processing, and iterative disaggregation that refines aggregates up to the point where the refined aggregates defined for processing produce the same optimal solution for the linear programming relaxation of MIP as the optimal solution of the linear programming relaxation with individual block processing. They propose several strategies of creating refined aggregates for the MIP processing, using duality results and exploiting the problem structure. They are of the view that these refined aggregates allow the solution of very large problems in reasonable time with very high solution quality in terms of NPV. They demonstrate their approach using instances containing as many as 125 aggregates and almost 100,000 blocks.

Bley *et al.* (2010) consider the integer linear programming formulation presented by Caccetta and Hill (2003). For each time period and each attribute (tonnage of ore and waste contained within a block), the corresponding production constraint and the precedence constraints among the blocks combine to form a precedence constrained knapsack problem. By exploiting the special structure of such a problem, they derive several types of additional constraints, whose addition to the standard integer programming formulation leads to a tighter linear relaxation and consequently, a reduction of the computation times required to solve the production scheduling problems.

5 Uncertainty and Risk

There can be uncertainties in the field of mining since the ore values are estimated by the information from drill holes and it is after the mining process that the returns can be known. The uncertainty here is due to the fact that we only have samples of blocks. In this section, we look at the uncertainty associated with open-pit mining and also introduce Conditional Value-at-Risk (CVaR) as a measure to reduce the risk of high losses.

5.1 Open-Pit Mining Uncertainty

For long time horizon open-pit mining, large initial investments and operational budgets are required. In addition to the historic fluctuations of metal prices, the percentage of ore contained in each block of a deposit is uncertain. These two kinds of uncertainties are known as *price* and *geological uncertainty*, respectively. Significant costs are involved in determining ore content with accuracy (Newman *et al.*, 2010). Miners therefore have to take risk as the mining operations continue. One of the standard procedures to deal with uncertainty and risk is to perform the evaluation process for different scenarios of the project key variables (Abdel Sabour *et al.*, 2008).

Dimitrakopoulos *et al.* (2002) are the first to introduce the integration of geological uncertainty into open-pit mine planning (Abdel Sabour *et al.*, 2008). Dimitrakopoulos *et al.* are of the view that geological uncertainty as an element in key parameters of open-pit mining projects can be quantified by conditional simulation combined with open-pit optimization studies. They further claim that having an accurate assessment of uncertainty arising from grade variability in the ore reserve allows risk in a mining project to be quantified and considered in decision-making processes. In their opinion, further technical integration of uncertainty in optimization algorithms is needed to enhance the interaction and efficacy of open-pit optimization and risk assessment.

Ramazan and Dimitrakopoulos (2004) note that the mixed integer programming (MIP) approach for optimizing production schedules of open-pit mines has been found to be limited in terms of feasibility in generating optimal solutions with practical mining schedules and also in its ability to deal with *in-situ* variability of orebodies. They present a production scheduling method for multi-element deposits in open-pit mines and investigate the issues of infeasibility associated with the traditional MIP models and their limitations with respect to practical scheduling. They give an alternative formulation that considers the probability of blocks being scheduled in a given production period, thereby dealing with *in-situ* variability and practical issues in scheduling patterns. Their proposed model generates feasible scheduling patterns in terms of practical excavation of the blocks during the periods they are scheduled. The excavation requires significantly less movement of equipment compared with the schedules produced by the traditional MIP formulation. According to them, since the probability of the blocks being scheduled in a period is considered in the optimization, the new schedule is less risky than the traditional in terms of meeting production targets, when considering orebody uncertainty.

As compared to the linear programming (LP) model presented by Dimitrakopoulos and Ramazan (2003) on the same subject, Osanloo *et al.* (2008) support the fact that the approach by Ramazan and Dimitrakopoulos is able to maximize the net present value explicitly with the consideration of equipment movements and block access in such a way as to produce a schedule pattern that is less risky than the ones obtained with the traditional methods. The LP model by Dimitrakopoulos and Ramazan is noted not to generate maximum net present value in the presence of grade uncertainty, since the net present value is not maximized explicitly in their objective function. Osanloo *et al.* further support the claim that the approach used by Ramazan and Dimitrakopoulos is able to eliminate partial block mining as compared to the method by Dimitrakopoulos and Ramazan, which schedules some blocks partially and thus contributes to infeasibility or non-optimality of the design.

The method by Ramazan and Dimitrakopoulos (2004) however has disadvantages which are given by Osanloo *et al.* (2008) as follows:

- The generation of several scheduling patterns on simulated orebodies is complicated and costly.

- The direct integration of grade uncertainty in production planning has not been done. This contributes to the stochastic nature of the grade that in turn leads to violating some constraints some periods of time. As a matter of fact the method does not give the best profitable schedule with the minimum possible geological risk.

- Because there may be very significant local deviations between the true grade and simulated grade, especially in the situation that the drill grid is sparse and wide, detailed mine design on each simulation may result in generating an unrealistic scheduling in the final optimization stage.

- Like all other MIP models for long-term production planning problems, it cannot be implemented on large deposits.

Dimitrakopoulos and Godoy (2006) present a risk-based, stochastic, approach that integrates grade uncertainty into the optimization of long-term production scheduling. A general framework for long-term production scheduling is reviewed and extended through combinatorial optimization to allow effective minimization of risk in not achieving production targets due to geological uncertainty. Their approach first generates a series of mining schedules, each corresponding to a simulated realization of the spatial distribution of grades. These mining sequences are optimized within a common feasible domain and post processed to provide a single optimal mining sequence, which minimizes the chance of deviating from target production figures. The following is the outline of their approach.

1. Derive a stable solution domain of ore production and waste removal stable to all simulated models of the distribution of grades within the orebody.

2. Determine the optimum production schedule within the solution domain derived in the first stage above, using a linear programming formulation. This will generate optimum mining rates for the life-of-mine scheduling.

3. For each one of the simulated models generate a physical mining sequence constrained to the mining rate derived in the second stage (note that all are sub-optimal mining schedules).

4. Combine, using combinatorial optimization, the mining sequence generated in the third stage to produce a single optimal mining sequence that minimizes the chance of deviating from production targets.

Their approach has the ability to minimize deviations from production target variables to acceptable ranges. For the deterministic approach, the optimization formulation processes a single estimated orebody model to produce mining schedule. According to them, estimation errors, including the inevitable smoothness of “average” mining block grade estimates, are propagated to the optimization, because the result is based on imperfect geological knowledge. The final result, as indicated by Dimitrakopoulos *et al.* (2002), is a single and often biased forecast for the economic outcome of the production schedule. For the stochastic framework, geological uncertainties are characterized by a series of equally probable models of the orebody, as produced by conditional simulation techniques.

Ramazan and Dimitrakopoulos (2007) consider a stochastic integer programming (SIP) mathematical model that generates optimum long-term production schedules for open-pit mines for a defined objective function, considering the operational requirements at the mine. Their model takes multiple simulated orebody models, without averaging the grades, and maximizes the total net present value when considering geological uncertainty caused by grade variability. In the case of production scheduling, the objectives are to maximize the total net present value and to minimize unsatisfied demand for processed ore. They incorporate geologic risk discounting concept within the SIP model to control the risk distribution between production periods. In their implementation of the model, penalty parameters for deviations from targets are used to control the geological risk distribution in terms of magnitude and variability. The scheduling method developed in their work allows the decision-maker to define a risk profile based on the existing uncertainty quantified by simulated orebody models. In terms of solution time, they consider the SIP model to be more efficient compared to the traditional mixed integer programming mine scheduling models, as the SIP method contains substantially less binary variables than the latter. Although they used a hypothetical data set to illustrate the strength of the new SIP model, they claim that their model is applicable to large open-pit mines. From their viewpoint, stochastic programming and modeling concept is useful not only for optimizing the production scheduling process, but also for investigating various stages of the whole mining process, such as finding the value of an additional drilling.

A popular approach to deal with risk is to apply the conventional Monte Carlo simulation, in which case a distribution for the mine value is obtained rather than a single expected value. From this distribution, the risk associated with a long term production scheduling can be explored by defining the range for the expected value at a certain degree of confidence (Abdel Sabour *et al.*, 2008). According to Carneiro *et al.* (2010), in many cases it is better to maximize returns with risk constraints. They support the proposal of Krokhmal *et al.* (2002) that, it is better to specify a maximum level of risk. These approaches can be combined in a careful way to formulate design and investment models for the open-pit mine.

5.2 Concept of Conditional Value-at-Risk

We will in this section give a brief introduction to a measure of uncertainty and risk, known as Conditional Value-at-Risk (CVaR), which has its origin within finance. We give an example of its use, for portfolio optimization.

Measure of risk plays a critical role in the optimization of portfolios under the presence of uncertainties. Loss can be envisioned as a function $z = f(x, y)$ of a decision vector $x \in X \subset R^n$ representing what we may generally call a portfolio, with X expressing decision constraints, and $y \in Y \subset R^n$ representing the future values of a number of variables like interest rates or weather data (Rockafellar and Uryasev, 2002). Further, when y is taken to be random with known probability distribution, z comes out as a random variable having its distribution dependent on the choice of x. Any optimization problem involving z in terms of the choice of x should then take into account not just expectations, but also the "riskiness" of x.

Value-at-Risk (VaR) is an important measure of risk that is used in the financial industry. Among various risk criteria, is a popular measurement of risk representing the percentile of the loss distribution with a specified confidence level (Lim *et al.*, 2010). Let $\alpha \in (0, 1)$ denote the confidence level and $f(x, y)$ a loss function associated with a portfolio x (a vector indicating the fraction of instrument of some available budget in each of n financial instruments) and an instrument price (or return) vector $y \in R^n$. [It should be noted that $f(x, y) < 0$ means a positive return.] Then, the VaR function $\zeta(x, \alpha)$ is given by the smallest number satisfying $\Phi(x, \zeta(x, \alpha)) = \alpha$, where $\Phi(x, \zeta)$ is the probability that the loss $f(x, y)$ does not exceed a threshold value ζ, that is, $\Phi(x, \zeta) = \mathtt{Pr}[f(x, y) \leq \zeta]$. So, for any portfolio $x \in R^n$ and a confidence level α, VaR is interpreted as the value of ζ such that the probability of the loss not exceeding ζ is α. In simple terms, VaR is the maximum likely loss incurred over a specified period of time at a given confidence level (Lai *et al.*, 2009). The chosen confidence level depends on the purpose of the exercise and the risk tolerance level of management.

Although the VaR measure is a very popular measure of risk, it has drawbacks, among them is lack of consistency, as it violates subadditivity. Nonsubadditivity means that the risk of a portfolio can be higher than the sum of the risks of its individual components. Moreover, in practice, the VaR measure is nonconvex, as well as nondifferentiable, and hence, it is difficult to find global minimum via conventional optimization technique (Lim *et al.*). Furthermore, VaR is difficult to optimize when it is calculated from scenarios. It is coherent only when it is based on the standard deviation of normal distributions (for a normal distribution VaR is proportional to the standard deviation).

Criticisms of the VaR approach resulted in new proposals for ways to measure risk in portfolios. As an alternative measure, the CVaR was introduced by Rockafellar and Uryasev (2000) and was further developed by Rockafellar and Uryasev (2002). The CVaR measure gives rise to a convex problem and it is considered to be more consistent than the VaR. It is defined as the mean loss by which the VaR is exceeded. In other words, the CVaR is the conditional expected loss of a portfolio at a confidence level, given that the loss to be accounted for exceeds or equals the VaR (Carneiro *et al.*, 2010). By definition, the VaR at a given confidence level is never higher than the corresponding CVaR. CVaR is also known as *mean excess loss*, *mean shortfall* or *tail VaR*. Rockafellar and Uryasev (2000) place emphasis on CVaR as an approach to optimizing a portfolio so as to reduce the risk of high losses, even though VaR play a role in this approach. By definition with respect to a specified probability level β, the β-VaR of a portfolio is the lowest amount α such that, with probability β, the loss will not exceed α, whereas the β-CVaR is the conditional expectation of losses above that amount α.

Gaivoronski and Pflug (2004/05) present an empirical study of the properties of historic VaR and CVaR as functions of portfolio composition using stock market data. They note that VaR optimization is nonconvex, exhibits many local minima, and is of combinatorial character, that is, exhibit exponential growth in computational complexity. Even though they argue that computation of mean-VaR efficient portfolios based on historic data is a feasible task, they confirm that VaR optimization is more difficult than CVaR optimization.

Lim *et al.* (2010) are in favour of CVaR. They say that CVaR is a portfolio evaluation function having appealing features such as sub-additivity and convexity. Although the CVaR function is nondifferentiable, scenario-based CVaR minimization problems can be reformulated as linear programs that afford solutions via widely-used commercial softwares. However, finding solutions through linear programming formulations for problems having many financial instruments and a large number of price scenarios can be time consuming as the dimension of the problem greatly increases. Andersson *et al.* (2001) and

Mansini *et al.* (2007) support the use of CVaR over VaR.

For continuous distributions, CVaR is the conditional expected loss given that the loss exceeds VaR. That is, CVaR is given by

$$\Phi_\alpha(x) = (1-\alpha)^{-1} \int\limits_{f(x,y)>\zeta(x,\alpha)} f(x,y)p(y)dy, \tag{5.1}$$

where $p(y)$ is a probability density function of y. To avoid complications caused by an implicitly defined function $\zeta(x,\alpha)$, Rockafellar and Uryasev (2000) give an alternative function

$$F_\alpha(x,\zeta) = \zeta + (1-\alpha)^{-1} \int\limits_{f(x,y)>\zeta} [f(x,y)-\zeta]p(y)dy, \tag{5.2}$$

for which they show that minimizing $F_\alpha(x,\zeta)$ with respect to (x,ζ) yields the minimum CVaR and its solution.

When applied to portfolio optimization (Lim *et al.*), x_i is the portion of the total investment that is made in a certain security. If the probability distribution of y is not available we can exploit price scenarios, which can be obtained from past price data. Assume that this price data is equally likely (for example, random sampling from a joint price distribution). For a given price data y^j, $j = 1, \ldots, J$, we can approximate $F_\alpha(x,\zeta)$ by

$$\widetilde{F}_\alpha(x,\zeta) = \zeta + [(1-\alpha)J]^{-1} \sum_{j=1}^{J} \max\{f^j(x)-\zeta,\ 0\}, \tag{5.3}$$

where $f^j(x) \equiv f(x,y^j)$. It is well-known that $\widetilde{F}_\alpha(x,\zeta)$ is a convex function when each $f^j(x)$ is convex. The function $\widetilde{F}_\alpha(x,\zeta)$ is noted to be nondifferentiable at (x,ζ) such that $f^j(x)-\zeta=0$, because then both $(\nabla f^j(x)^T, -1)^T$ and 0 are subgradients of $\max\{f^j(x)-\zeta,\ 0\}$, and since $(\nabla f^j(x)^T, -1)^T \neq 0$, the subdifferential is not a singleton. A portfolio that minimizes CVaR can be found by considering the nondifferentiable optimization problem

$$\begin{aligned}
&\min \zeta + v \sum_{j=1}^{J} \max\{f^j(x)-\zeta,\ 0\} \\
&\text{subject to} \\
&\sum_{i=1}^{n} x_i = 1 \\
&x \geq 0, \quad \zeta \text{ unrestricted.}
\end{aligned} \tag{5.4}$$

Here, $v \equiv [(1-\alpha)J]^{-1}$.

When the loss function $f^j(x)$ is linear, then (5.4) can be solved by using linear programming. For a given vector of purchase prices $p \in R^n$, the linear loss function is given by $f^j(x) = (p-y^j)^T x$. Introducing an auxiliary variable vector $z \in R^J$, (5.4) can be formulated as the linear programming given by

$$\min \zeta + v \sum_{j=1}^{J} z_j$$

subject to

$$z_j \geq (p - y^j)^T x - \zeta, \qquad j = 1, \ldots, J \tag{5.5}$$

$$\sum_{i=1}^{n} x_i = 1$$

$$x \geq 0, \quad z \geq 0, \quad \zeta \text{ unrestricted.}$$

2

Summary of Papers

Paper I: On the use of Parametric Open-Pit Design Models for Mine Scheduling - Pitfalls and Counterexamples

In this paper, we give an interpretation of the open-pit mining model presented by Picard and Smith (2004), which finds a sequence of intermediate contours leading to an ultimate one, as a Lagrangian relaxation approach. Our aim is to relate their approach to a standard optimization procedure and standard results, and explain the properties and weaknesses of their approach. We give examples of worst case performance as well as best case performance. The worst case behaviour can be very poor in that we might not obtain any scheduling information at all. Our analysis also covers the parametric scheduling approach of Lerchs and Grossmann (1965), which is closely related to the Picard-Smith approach.

We show that the two approaches suggested by Lerchs and Grossmann (1965) and Picard and Smith (2004) as attempts to obtain scheduling information can be interpreted as Lagrangian relaxations of side-constrained versions of the classic open-pit design model. The unpredictable behaviour of the nested pits produced by these approaches, known as the gapping phenomenon, is observed to be nothing but a manifestation of the lack of controllability that is inherent in Lagrangian relaxation schemes when applied to linear optimization problems. We further show that it is not the Lagrangian relaxation step in itself that creates the gapping phenomenon, since it is still present in the explicitly side-constrained open-pit design model. Furthermore, the side-constrained model does not possess the nested pit property, which leads to the conclusion that the Lagrangian relaxation step is crucial for obtaining the nested pit property.

The important deduction from this work is that it seems difficult to avoid the gapping phenomenon and obtain proper scheduling information by manipulating the design model. Hence, we need to introduce *time* explicitly to get proper scheduling information for open-pit mining.

Paper II: A Duality-Based Derivation of the Maximum Flow Formulation of the Open-Pit Design Problem

In this paper, we formulate the open-pit design problem as a maximum flow problem in a certain capacitated network, as first shown by Picard in 1976. His derivation is based on a restatement of the problem as a quadratic binary program. We give an alternative derivation of the maximum flow formulation, which uses only linear programming duality.

Paper III: A Multi-Parametric Maximum Flow Characterization of the Open-Pit Scheduling Problem

We give, in this paper, a multi-parametric maximum flow characterization of the open-pit design and scheduling problem. This is the problem of finding an optimal mining schedule for an open-pit during a number of time periods, subject to mining capacity restrictions for the time periods. By applying Lagrangian relaxation to the capacities, a multi-parametric formulation is obtained. We show that this formulation can be restated as a maximum flow problem in a time-expanded network. This result extends the well known maximum flow formulation of the open-pit design problem of Picard from 1976 to the case of multiple time periods.

Paper IV: Open-Pit Production Scheduling - Suggestions for Lagrangian Dual Heuristic and Time Aggregation Approaches

We make suggestions for Lagrangian dual heuristic approaches for the open-pit scheduling problem. A Lagrangian relaxation of the problem, which is solvable as a maximum flow problem is presented. Two procedures for finding near-optimal solutions to the scheduling problem are presented. Linear programming relaxations are solved to find values of the multipliers for the capacity restrictions to be used in a Lagrangian relaxation of these constraints. It is noted that the solution to the relaxed problem will not in general satisfy the capacity restrictions. By adjusting the multiplier values of the Lagrangian relaxation the relaxed solution is forced to become feasible. In order to reduce the computational burden of solving the linear programming relaxations of large-scale problems, we suggest using aggregation of the time periods. We further derive a heuristic procedure for finding near-optimal solutions to the combined open-pit design and scheduling problem of several capacity constraints for each time period. This heuristic is based on newly developed conditions for optimality and near-optimality in general discrete optimization problems.

Paper V: Open-Pit Mining with Uncertainty - A Conditional Value-at-Risk Approach

The selection of a mine design is based on estimating net present values of all possible, technically feasible mine plans so as to select the one with the maximum value. In practice, mine planners cannot know with certainty the quantity and quality of ore in the ground, and this geological uncertainty is a major source of risk. Furthermore, the future market behaviour of metal prices and foreign exchange rates are impossible to be known with certainty, and therefore, they are also sources of risks affecting mine project profitability. Because of these uncertainties, it is recognized among practitioners that mining is a high risk business. An optimization model that maximizes expected return while minimizing risk is therefore important for the mining sector as this will help make better decisions on the blocks of ore to mine at a particular point in time.

We present in this paper a Conditional Value-at-Risk (CVaR) approach to open-pit mining. Value-at-Risk (VaR) is a measure that has been used by the financial institutions to minimize the loss caused by inadequate monitoring of risk. This measure is noted to have certain drawbacks such as lack of consistency, nonconvexity, and nondifferentiability. Rockafellar and Uryasev (2000) introduce the CVaR measure as an alternative to the VaR measure. The CVaR measure gives rise to a convex problem. This measure is further developed by Rockafellar and Uryasev (2002).

Our CVaR approach to open-pit mining can take into account both geological and price uncertainty in the planning. Investment and design models based on this approach are formulated for the open-pit mine. Furthermore, we give a nested pit scheduling model based on CVaR. Several numerical results are presented by using scenarios from simulated geological and price uncertainties.

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