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Optimal Resource Allocation for IR-HARQ

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Abstract—In this work, we first provide an exact closed-form expression for the packet drop probability (PDP) in incremental redundancy (IR) based hybrid automatic repeat request (HARQ) schemes with a limit on the maximum number of transmissions. In the later part, we extend our work in [5], and consider the optimal resource allocation problem of minimizing the PDP under the constraints of an average transmit power and total number of channel uses. We provide two approaches to solve this optimization problem and compare their performance with the solution given in [5].

I. INTRODUCTION

Hybrid automatic repeat request (HARQ) schemes are the retransmission schemes in which conventional ARQ mechanisms are combined with forward error correction (FEC) schemes to improve the throughput efficiency [1]. In type-I HARQ schemes, the receiver discards the information content of a data packet if it is received in error (using error detection methods such as cyclic redundancy check). In type-II HARQ schemes, the receiver combines the information it received across different transmissions about a data packet. Type-II HARQ schemes are mainly classified into:

- *Chase combining (CC) schemes*: In these schemes, the same codeword is transmitted to the destination during the retransmissions. The receiver uses maximal-ratio-combining (MRC) to decode a data packet.
- *Incremental redundancy (IR) schemes*: In these schemes, additional parity bits are sent to the destination in each retransmission. The receiver uses code combining to decode a data packet.

A. Related Work

Coding performance of both CC and IR based HARQ schemes was presented in [2]. In [3], a cross-layer optimization problem was solved for both CC and IR based HARQ schemes with outdated channel state information at the transmitter. A fixed outage probability analysis of HARQ schemes in block-fading channels was presented in [4]. However in [4], the authors assumed that the same number of channel uses as well as the same transmission power is used in different transmission rounds. In our previous work [5], and in [6], the optimization problems of minimizing the packet drop probability (PDP) and the bit error rate (BER) under an average transmit power constraint for IR-HARQ schemes were considered, respectively. In the works of [2], [5] and [6],

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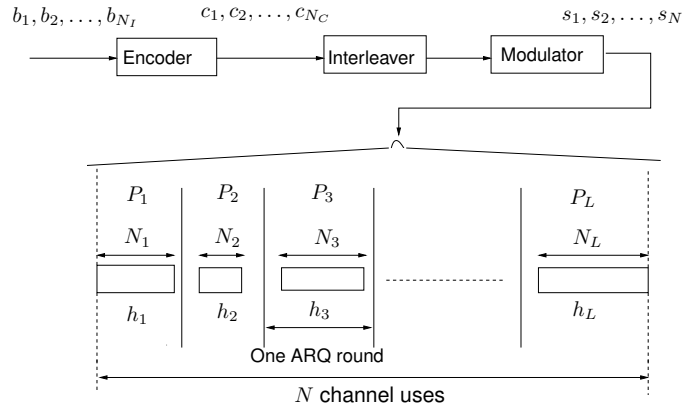


Figure 1. System model for the IR-HARQ scheme considered in this work.

the assumption of the same number of channel uses per transmission round was used.

B. Contributions

In this work, we generalize the results of [5] to the case when the number of channel uses may differ for different ARQ rounds. We specifically provide an exact closed-form expression for the PDP of IR-HARQ schemes when the total number of transmissions is limited. The closed-form expression for the PDP is in terms of the upper incomplete Fox's H-function [8]. Since these special functions are not mathematically tractable, we provide an approximate PDP expression for the case when only two transmissions are allowed. We also extend our previous work in [5] and formulate an optimization problem of minimizing PDP under an average power constraint as well as a limit on the total number of channel uses. We provide two approaches to solve this optimization problem and compare their performance with the solution provided in [5], in which the optimization problem was solved under the average transmit power constraint only.

II. SYSTEM MODEL

The system model considered for the IR-HARQ scheme is shown in Fig. 1. N_I information bits b_1, b_2, \dots, b_{N_I} are encoded to obtain a code-block with coded bits c_1, c_2, \dots, c_{N_C} , where N_C denotes the number of coded bits. These N_C coded bits are then mapped onto a constellation \mathcal{S} to obtain N modulation symbols s_1, s_2, \dots, s_N . The N modulation symbols are then transmitted to the destination in an incremental redundancy fashion. We assume that the modulation symbols have unit average energy, i.e., $\mathbb{E}|s_n|^2 = 1$. The number of transmission rounds for the IR-HARQ scheme is limited to L and the l th transmission round consists of N_l channel uses. We also assume that one modulation symbol

is transmitted per channel use in each ARQ round, i.e., we have $\sum_{l=1}^L N_l = N$. We consider a block-fading Rayleigh channel in which the channel gain remains constant in each ARQ round and varies independently between different ARQ rounds. Let h_l denote the channel gain for the l th ARQ round, and let $\lambda_l = \mathbb{E}|h_l|^2, \forall l = 1, 2, \dots, L$. We assume that the transmitter has statistical knowledge of the channel gains i.e., it has the knowledge about the distribution of channel gains. This assumption is practical as it does not require instantaneous feedback from the receiver. We also assume that the same transmission power is used in all the channel uses of the l th ARQ round and it is equal to P_l .

With the above described system model, we can write the received signal at the destination during the i th channel use of the l th ARQ round as:

$$y_{l,i} = \sqrt{P_l} h_l s_{l,i} + e_{l,i}, \quad l = 1, 2, \dots, L, i = 1, 2, \dots, N_l \quad (1)$$

where $s_{l,i} \in \mathcal{S}$ and $e_{l,i}$ denote the modulated symbol and the additive white Gaussian noise sample in the i th channel use of the l th ARQ round, respectively. We assume that $e_{l,i}, \forall l = 1, 2, \dots, L$ and $i = 1, 2, \dots, N_l$ are i.i.d. with distribution $\mathcal{CN}(0, 1)$. Assuming Gaussian signaling and that N is large, we can write the total accumulated mutual information at the destination till the l th ARQ round as¹:

$$\begin{aligned} I_l &= \sum_{m=1}^l N_m \log \left(1 + P_m |h_m|^2 \right) \\ &= \sum_{m=1}^l N_m \log (1 + P_m \alpha_m) \end{aligned} \quad (2)$$

where $\alpha_m = |h_m|^2$ with $\alpha_m \sim \exp(\lambda_m), \forall m = 1, 2, \dots, l$.

Now we provide a closed form expression for the PDP of the IR-HARQ scheme with the system model described above. We assume that the feedback signaling from the receiver is based on the total accumulated mutual information at the destination. The receiver sends an acknowledgement (ACK) signal after the l th ARQ round if $I_l > N_I$ and a negative ACK (NACK) signal otherwise. We also assume that the ACK/NACK feedback signaling from the receiver to the transmitter is instantaneous and error-free. So the probability that the packet is in outage after l ARQ rounds can be written as [7]:

$$p_{\text{out},l} \triangleq \Pr(I_l < N_I \cap \dots \cap I_1 < N_I) = \Pr(I_l < N_I) \quad (3)$$

and the PDP, which is the probability that the packet is in outage after L ARQ rounds is given by:

$$p_{\text{drop}} \triangleq p_{\text{out},L} = \Pr(I_L < N_I) \quad (4)$$

Proposition 1. $p_{\text{out},l}$ for $1 \leq l \leq L$ can be expressed in closed-form using the upper incomplete Fox's H -function [8] (see Appendix A for its mathematical definition) as

$$p_{\text{out},l} = 1 - K_l \times \left[\frac{2}{M_l} \left| \begin{array}{c} (1, 1, 0) \\ (0, 1, 0), (1, \frac{1}{\delta_1}, \frac{1}{P_1 \lambda_1}), \dots, (1, \frac{1}{\delta_l}, \frac{1}{P_l \lambda_l}) \end{array} \right. \right] \quad (5)$$

¹All logarithms used in this work are with base 2 unless otherwise specified.

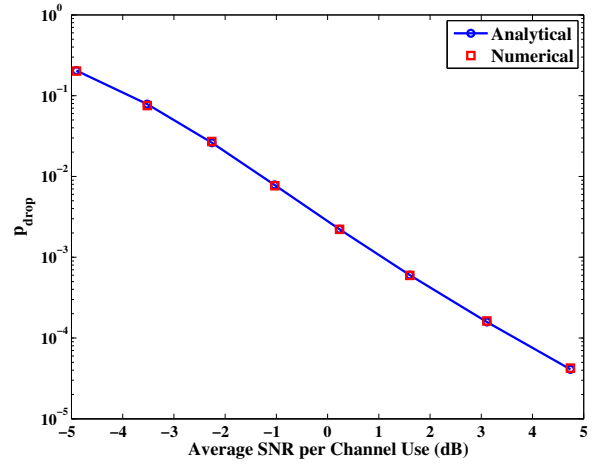


Figure 2. Analytical and numerical packet drop probability p_{drop} comparison. Parameters for the simulation are $L = 3, P_1 = P_2 = P_3 = 0$ dB, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda, \delta_1 = 1$ bpcu, $\delta_2 = 0.5$ bpcu, and $\delta_3 = 1$ bpcu. The average SNR per channel use is varied by varying λ .

where $K_l \triangleq \exp \left(\sum_{m=1}^l P_m \lambda_m \right), \delta_m \triangleq \frac{N_I}{N_m}$, for $m = 1, 2, \dots, l$, and $M_l \triangleq \prod_{m=1}^l (P_m \lambda_m)^{-\delta_m}$.

Proof: See Appendix B. ■

With the convention that $p_{\text{out},0} = 1$, we define the average transmit power per channel use as:

$$P_{\text{avg}} \triangleq \frac{\sum_{l=1}^L P_l N_l p_{\text{out},l-1}}{N} \quad (6)$$

and the average signal-to-noise ratio (SNR) per channel use in dB as $10 \log_{10} \left(\frac{\sum_{l=1}^L P_l N_l \lambda_l p_{\text{out},l-1}}{N} \right)$. Fig. 2 shows the analytical and numerical performance comparison of PDP for an IR-HARQ system with $L = 3$. The parameters used for the performance comparison are $P_1 = P_2 = P_3 = 0$ dB, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda, \delta_1 = 1, \delta_2 = 0.5$, and $\delta_3 = 1$ bits per channel use (bpcu) respectively. The analytical result in Fig. 2 is obtained by evaluating the expression in (5) by inserting the parameter values.² The numerical result curve in Fig. 2 is obtained by generating random channel gains and empirically evaluating the probability in (4). We can see from the figure that the analytical expression result matches closely with the numerical result.

III. THE SPECIAL CASE OF $L = 2$

The closed-form PDP expression provided in (5) does not give much direct insight into the variation of $p_{\text{out},l}$ as a function of the resource parameters P_l and δ_l . Since we are interested in the optimization of resources, we now provide an approximate PDP expression, which allows us to optimize the resources for the special case of two transmissions. Using the approach given in Appendix A of [5], we can write:

²The expression in (5) can be evaluated in MATHEMATICA using the program given in Appendix B of [8].

$$\mathcal{L}(P_1, P_2, N_1, \gamma_1, \gamma_2, \mu_1, \mu_2) = \frac{\left(2^{\frac{N_I}{N_{\text{given}} - N_1}} - 2^{\frac{N_I}{N_1}}\right)}{P_1 P_2 \lambda_1 \lambda_2 \left(2 - \frac{N_{\text{given}}}{N_1}\right)} - \frac{\left(2^{\frac{N_I}{N_{\text{given}} - N_1}} - 1\right)}{P_1 P_2 \lambda_1 \lambda_2} + \gamma_1 P_1 + \gamma_2 P_2 + \mu_1 (N_1 - N_{\text{given}}) + \mu_2 \left(\frac{P_1 N_1 + (N_{\text{given}} - N_1) P_2 p_{\text{out},1}}{N_{\text{given}}} - P_{\text{avg}}^{\text{given}}\right). \quad (10)$$

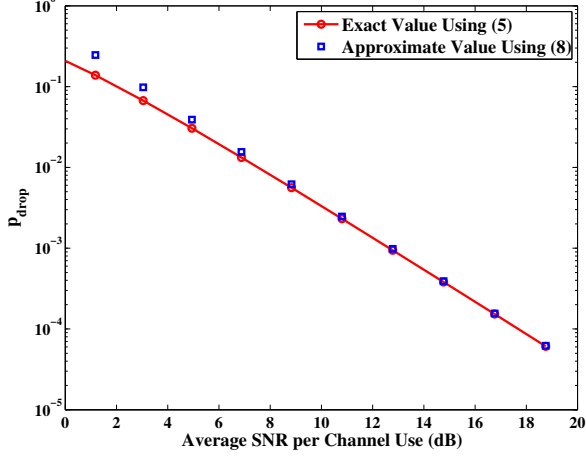


Figure 3. P_{drop} comparison using exact Fox's H-function evaluation of (5) and the approximate simplified expression of (8). Parameters for the simulation are $L = 2$, $P_1 = P_2 = 0$ dB, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, $\delta_1 = \frac{2}{3}$ bpcu and $\delta_2 = 2$ bpcu. The average SNR per channel use is varied by varying λ .

$$P_{\text{out},1} = 1 - \exp\left(-\frac{2^{\delta_1} - 1}{P_1 \lambda_1}\right), \quad (7)$$

$$P_{\text{drop}} = \begin{cases} \frac{\delta_2^\delta \ln 2 - 2^{-\delta} + 1}{P_1 P_2 \lambda_1 \lambda_2} + O\left(\frac{1}{P_3}\right), & \text{if } \delta_1 = \delta_2 = \delta \\ \frac{(2^{\delta_2} - 2^{\delta_1})}{P_1 P_2 \lambda_1 \lambda_2 \left(1 - \frac{\delta_1}{\delta_2}\right)} - \frac{(2^{\delta_2} - 1)}{P_1 P_2 \lambda_1 \lambda_2} + O\left(\frac{1}{P_3}\right), & \text{if } \delta_1 \neq \delta_2 \end{cases} \quad (8)$$

Fig. 3 shows the comparison of the P_{drop} for the special case of $L = 2$ using the exact expression given in (5) and the expression given in (8) by neglecting $O\left(\frac{1}{P_3}\right)$ terms. We can see from the figure that the approximation using (8) is tight for average SNR values higher than 6 dB.

The optimization results in [5] were obtained under the assumption that the same number of channel uses is used in both ARQ rounds. In this work, we remove that restriction and consider the case with $\delta_1 \neq \delta_2$ or equivalently $N_1 \neq N_2$. We neglect the $O\left(\frac{1}{P_3}\right)$ terms of (8) in the further analysis.

A. Optimization Problem

For $L = 2$, we consider the following optimization problem of minimizing the PDP under the constraints of given average transmit power per channel use ($P_{\text{avg}}^{\text{given}}$), and the total number of available channel uses (N_{given}).

$$\min_{(P_1, P_2, N_1)} \frac{\left(2^{\frac{N_I}{N_{\text{given}} - N_1}} - 2^{\frac{N_I}{N_1}}\right)}{P_1 P_2 \lambda_1 \lambda_2 \left(2 - \frac{N_{\text{given}}}{N_1}\right)} - \frac{\left(2^{\frac{N_I}{N_{\text{given}} - N_1}} - 1\right)}{P_1 P_2 \lambda_1 \lambda_2}$$

subject to $C_1 : 0 \leq P_1$

$$C_2 : 0 \leq P_2$$

$$C_3 : 0 \leq N_1 \leq N_{\text{given}} \quad (9)$$

$$C_4 : \frac{P_1 N_1 + (N_{\text{given}} - N_1) P_2 p_{\text{out},1}}{N_{\text{given}}} \leq P_{\text{avg}}^{\text{given}}$$

The objective function in (9) is obtained from the PDP expression in (8) and using the fact that $N_1 + N_2 = N_{\text{given}}$.

Proposition 2. Let (P_1^*, P_2^*, N_1^*) denote the optimal solution to (9), then the constraint C_4 of (9) is active at the optimal solution.

Proof: We write the Lagrangian function of (9) as shown in (10) on top of the page. Let $(P_1^*, P_2^*, N_1^*, \gamma_1^*, \gamma_2^*, \mu_1^*, \mu_2^*)$ denote the optimal values of (10). Considering the Karush-Kuhn-Tucker (KKT) necessary conditions [9], we require:

$$\mu_2^* \left(\frac{P_1^* N_1^* + (N_{\text{given}} - N_1^*) P_2^* p_{\text{out},1}}{N_{\text{given}}} - P_{\text{avg}}^{\text{given}}\right) = 0 \text{ and}$$

$$\mu_2^* \geq 0. \text{ To prove that } C_4 \text{ is active, we need to prove that } \mu_2^* > 0. \text{ Now considering } \frac{\partial \mathcal{L}}{\partial P_1} \Big|_{(P_1^*, P_2^*, N_1^*, \gamma_1^*, \gamma_2^*, \mu_1^*, \mu_2^*)} = 0, \text{ we have}$$

$$\begin{aligned} & -\frac{\left(2^{\frac{N_I}{N_{\text{given}} - N_1}} - 2^{\frac{N_I}{N_1}}\right)}{P_1^2 P_2 \lambda_1 \lambda_2 \left(2 - \frac{N_{\text{given}}}{N_1}\right)} + \frac{\left(2^{\frac{N_I}{N_{\text{given}} - N_1}} - 1\right)}{P_1^2 P_2 \lambda_1 \lambda_2} + \gamma_1 + \mu_2 \times \\ & \left[\frac{N_1}{N_{\text{given}}} + \frac{(N_{\text{given}} - N_1) \left(1 - 2^{\frac{N_I}{N_1}}\right)}{N_{\text{given}} P_1^2 \lambda_1} \right] \times \\ & \left. (1 - p_{\text{out},1}) P_2 \right] \Big|_{(P_1^*, P_2^*, N_1^*, \gamma_1^*, \gamma_2^*, \mu_1^*, \mu_2^*)} = 0. \end{aligned} \quad (11)$$

Since P_1^* is bounded above by $\frac{N_{\text{given}} P_{\text{avg}}^{\text{given}}}{N_1^*}$ (as seen from C_4), we have $\gamma_1^* = 0$. Use this in (11) and note that $\mu_2^* = 0$ if and only if $N_1^* = N_2^* = \frac{N_{\text{given}}}{2}$. However by observing that $N_1^* = \frac{N_{\text{given}}}{2}$ cannot be the optimal solution, we have that $\mu_2^* > 0$. ■

Now using the result that C_4 is active at the optimal solution, we have

$$P_2 = \frac{N_{\text{given}} P_{\text{avg}}^{\text{given}} - N_1 P_1}{(N_{\text{given}} - N_1) p_{\text{out},1}}. \quad (12)$$

Using this, we can simplify the optimization problem in (9) to the following:

$$\min_{(P_1, N_1)} f(P_1, N_1)$$

subject to $0 \leq P_1$, and $0 \leq N_1 \leq N_{\text{given}}$ (13)

where

$$f(P_1, N_1) = \left[\frac{\left(2^{\frac{N_I}{N_{\text{given}} - N_1}} - 2^{\frac{N_I}{N_1}}\right)}{P_1 \lambda_1 \lambda_2 \left(2 - \frac{N_{\text{given}}}{N_1}\right)} - \frac{\left(2^{\frac{N_I}{N_{\text{given}} - N_1}} - 1\right)}{P_1 \lambda_1 \lambda_2} \right] \times \left[\frac{(N_{\text{given}} - N_1) p_{\text{out},1}}{\left(N_{\text{given}} P_{\text{avg}}^{\text{given}} - N_1 P_1\right)} \right]. \quad (14)$$

The optimization problem in (13) is a mixed-integer non-linear programming (MINLP) problem and one can use MINLP solvers to find the optimal solution [10]. Now we provide two approaches to solve (13). The first approach solves (13) optimally and in the second approach we provide a sub-optimal solution.

B. Solutions to (13)

1) *First Approach:* In the first approach, we solve the optimization problem in (13) using the following method. Since N_1 is an integer variable, for a fixed value of N_1 , one can reduce the two dimensional optimization problem in (13), to a one dimensional sub-problem in the variable P_1 as follows:

$$\min_{P_1} g(P_1) \triangleq \frac{p_{\text{out},1}}{P_1 \left(N_{\text{given}} P_{\text{avg}}^{\text{given}} - N_1 P_1\right)} \quad (15)$$

subject to $0 \leq P_1 \leq P_1^{\text{UB}}$

where P_1^{UB} is the upper bound of P_1 for a fixed value of N_1 , and is given by:

$$P_1^{\text{UB}} = \frac{N_{\text{given}} P_{\text{avg}}^{\text{given}}}{N_1}. \quad (16)$$

One can use constrained non-linear optimization techniques [9] to find the solution to (15). Once the optimal solution P_{1,N_1}^* to (15) is obtained, the corresponding PDP can be computed as

$$p_{\text{drop}}(P_{1,N_1}^*, N_1) = K(N_1) \cdot g(P_{1,N_1}^*) \quad (17)$$

where $K(N_1)$ is a constant and is given by:

$$K(N_1) = \frac{f(P_1, N_1)}{g(P_1)} \quad (18)$$

After solving the sub-problem in (15) for all feasible values of N_1 , we find the optimal solution to (13) by selecting the pair (P_{1,N_1}^*, N_1) which gives the smallest $p_{\text{drop}}(P_{1,N_1}^*, N_1)$ value. The approach is summarized as follows:

Approach 1:

- 1) Initialize $p_{\text{drop}}^{\text{opt}} = \infty$.
- 2) **for** N_1 from 1 to N_{given}
 - a) Solve the sub-problem in (15) to obtain $g(P_{1,N_1}^*)$.
 - b) Compute $p_{\text{drop}}^{\text{current}}$ using (17) and (18).
 - c) **if** $p_{\text{drop}}^{\text{current}} < p_{\text{drop}}^{\text{opt}}$

$$p_{\text{drop}}^{\text{opt}} = p_{\text{drop}}^{\text{current}}.$$

$$P_1^* = P_{1,N_1}^*.$$

$$N_1^* = N_1.$$

else

Increment N_1 .

end (if)

end (for)

- 3) Compute P_2^* from (12) using P_1^* and N_1^* .
-

2) *Second Approach:* The second approach is similar to the first approach except that we provide a sub-optimal closed form solution for the sub-problem in (15). Neglecting the constraint that P_1 is bounded above by P_1^{UB} , following a similar approach as in [5], the stationary points of $g(P_1)$ are given by the following equation:

$$(1 - p_{\text{out},1}) \left(N_{\text{given}} P_{\text{avg}}^{\text{given}} - N_1 P_1\right) (1 - 2^{\delta_1}) = p_{\text{out},1} \left(N_{\text{given}} P_{\text{avg}}^{\text{given}} - 2N_1 P_1\right) P_1 \lambda_1 \quad (19)$$

Now using an approximate Taylor series expansion for $p_{\text{out},1}$ and neglecting the $O\left(\frac{1}{p^3}\right)$ terms, we have

$$p_{\text{out},1} \approx \frac{2^{\delta_1} - 1}{P_1 \lambda_1} - \frac{(2^{\delta_1} - 1)^2}{2P_1^2 \lambda_1^2} \quad (20)$$

Substituting (20) in (19) and simplifying, we obtain the stationary points of $g(P_1)$ in (15) as the roots of the following cubic equation:

$$aP_1^3 + bP_1^2 + cP_1 + d = 0, \quad (21)$$

where $a = 6N_1 z \lambda_1^2$, $b = -4N P_{\text{avg}}^{\text{given}} z \lambda_1^2 - 4N_1 z^2 \lambda_1$, $c = 3N P_{\text{avg}}^{\text{given}} z^2 \lambda_1 + N_1 z^3$, $d = -N z^3 P_{\text{avg}}^{\text{given}}$ and $z = 2^{\delta_1} - 1$. The fact that $d \leq 0$ guarantees that the cubic equation has at least one positive real root. Following similar approach as in [5], we can prove that there is only one positive real root for the cubic equation in (21). Moreover this positive real root is feasible (it satisfies the constraint in (15)) if the following condition is satisfied:

$$2P_1^{\text{UB}} \geq \frac{2^{\delta_1} - 1}{\lambda_1} \quad (22)$$

When the condition in (22) is satisfied, the optimal P_{1,N_1}^* for the sub-problem in (15) is given by the real positive root of (21). On the contrary, if (22) is not satisfied, then we have

$$2P_1^{\text{UB}} < \frac{2^{\delta_1} - 1}{\lambda_1} \implies -\frac{2^{\delta_1} - 1}{P_1^{\text{UB}} \lambda_1} < -2$$

which implies that $p_{\text{out},1}$ in (7) is close 1. In this case, we can choose

$$P_{1,N_1}^* = \frac{P_1^{\text{UB}}}{2}, \quad (23)$$

which is the value of P_1 that maximizes the denominator of $g(P_1)$ in (15). The second approach is summarized below:

Approach 2:

- 1) Initialize $p_{\text{drop}}^{\text{opt}} = \infty$.
- 2) **for** N_1 from 1 to N_{given}
 - a) Solve for P_{1,N_1}^* as follows:
 - if** (22) is satisfied
 - Obtain P_{1,N_1}^* as the positive real root of (21).

```

else
  Obtain  $P_{1,N_1}^*$  using (23).
end (if)
b) Compute  $p_{\text{drop}}^{\text{current}}$  using (15), (17) and (18).
c) if  $p_{\text{drop}}^{\text{current}} < p_{\text{drop}}^{\text{opt}}$ 
   $p_{\text{drop}}^{\text{opt}} = p_{\text{drop}}^{\text{current}}$ .
   $P_1^* = P_{1,N_1}^*$ .
   $N_1^* = N_1$ .
else
  Increment  $N_1$ .
end (if)
end (for)
3) Compute  $P_2^*$  from (12) using  $P_1^*$  and  $N_1^*$ .

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IV. NUMERICAL RESULTS

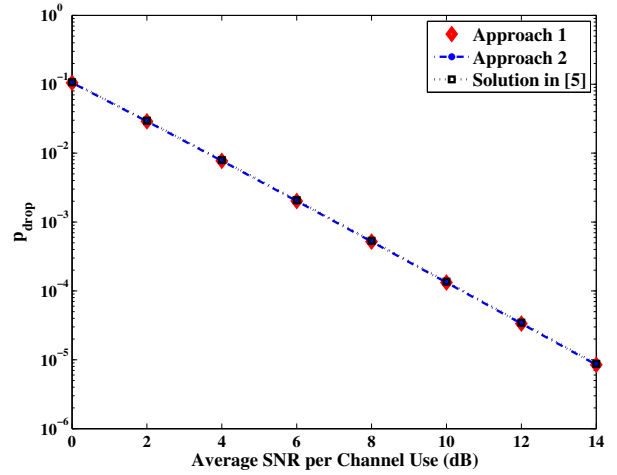
In this section, we present numerical results comparing the performance of both approaches presented in Section III-B for solving the optimization problem in (9). Fig. 4 shows the p_{drop} comparison for different spectral efficiency values. For comparison purposes, we have also plotted the result with only optimal power allocation of [5] obtained by assuming $N_1 = N_2 = \frac{N}{2}$. In Fig. 4(a) with $N_I = 400$ bits and $N = 1000$ channel uses, i.e., using more number of channel uses to transmit a small number of information bits, the solution provided by the optimal resource allocation (by both Approach 1 and Approach 2) has similar performance as that of optimal power allocation by assuming $N_1 = N_2 = \frac{N}{2}$ [5]. Intuitively this implies that for low values of $\frac{N_I}{N}$, in order to minimize p_{drop} , the optimal way to allocate channel uses is to use $N_1 = N_2 = \frac{N}{2}$, this has also been confirmed from the optimal N_1 value obtained from both the proposed approaches. We can also see from the figure that Approach 2 has practically the same performance as that of Approach 1.

Fig. 4(b) shows the result for a higher value of $\frac{N_I}{N}$, i.e., using a smaller number of total channel uses to transmit a larger number of information bits. As we can see, in this case, the optimal resource allocation (number of channel uses and power) has better performance than that of just using optimal power allocation assuming equal channel uses. At high average SNR values, the optimal allocation of channel uses for both the approaches is observed as $N_1 \approx 1.5N_2$. The gain for optimal resource allocation over only optimal power allocation is about 0.6 dB. In terms of complexity, Approach 2 involves finding a positive real root of a cubic equation compared to exact one dimensional minimization in Approach 1, for all possible values of N_1 .³ However the solution in [5] involves the finding a root of a cubic equation only for $N_1 = N_{\text{given}}/2$.

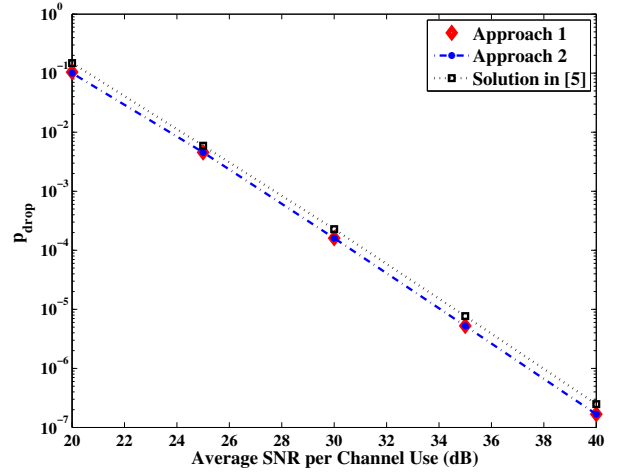
V. CONCLUSIONS AND DISCUSSION

In general, the closed-form expression provided for the PDP of an IR-HARQ scheme with limited number of retransmissions may not be directly useful for solving the resource allocation problem. Approximate expressions to PDP are required

³This is the worst-case scenario. When channel statistics are changing slowly, one can search only over a subset of all possible N_1 values by using knowledge about previous solutions.



(a) With $N_I = 400$ and $N = 1000$.



(b) With $N_I = 4000$ and $N = 1000$.

Figure 4. p_{drop} comparison using both the approaches presented in Section III-B. For comparison purposes we also presented the result from [5], which assumes an equal number of channel uses per transmission and uses optimal power allocation. The parameters for the simulation are $P_{\text{avg}}^{\text{given}} = 10$ dB. The average SNR is varied by varying $\lambda_1 = \lambda_2 = \lambda$.

to solve these optimization problems. For the special case of two transmissions, and low spectral efficiency values, it is optimal to allocate equal number of channel uses for both ARQ rounds. For high target spectral efficiency values, the equal number of channel uses has a small performance degradation compared to optimal allocation.

APPENDIX A INCOMPLETE FOX'S H-FUNCTION

The generalized upper incomplete Fox's H-function is defined as [8]:

$$\begin{aligned}
\mathcal{H}_{p,q}^{x,y}(r) &\triangleq \mathcal{H}_{p,q}^{x,y} \left(r \left| \begin{array}{c} (a_1, \gamma_1, A_1), \dots, (a_p, \gamma_p, A_p) \\ (b_1, \eta_1, B_1), \dots, (b_q, \eta_q, B_q) \end{array} \right. \right) \\
&= \frac{1}{2\pi\sqrt{-1}} \oint_C M_{p,q}^{x,y}[s] r^{-s} ds \quad (24)
\end{aligned}$$

where x, y, p and q are integers with $0 \leq x \leq p$ and $0 \leq y \leq q$. $a_i, b_i \in \mathbb{C}$ and $\gamma_i, \eta_j, A_i, B_j \in \mathbb{R}^+$ for $1 \leq i \leq p$ and

$1 \leq j \leq q$. $M_{p,q}^{x,y}[s]$ is the Mellin transform of $\mathcal{H}_{p,q}^{x,y}(r)$, and is defined as:

$$M_{p,q}^{x,y}[s] = \frac{\prod_{i=1}^y \Gamma(1 - a_i - \gamma_i s, A_i) \prod_{j=1}^x \Gamma(b_j + \beta_j s, B_j)}{\prod_{i=y+1}^p \Gamma(a_i + \gamma_i s, A_i) \prod_{j=x+1}^q \Gamma(1 - b_j - \beta_j s, B_j)} \quad (25)$$

where $\Gamma(\cdot, \cdot)$ is the upper incomplete Gamma function. The contour of integration C in (24) is chosen such that the integral corresponds to Mellin-Barnes integral [11].

APPENDIX B PROOF OF PROPOSITION 1

From (2) and (3), we have

$$\begin{aligned} p_{\text{out},l} &= \Pr \left[\left(\sum_{m=1}^l N_m \log(1 + P_m \alpha_m) \right) \leq N_I \right] \\ &= \Pr \left[\log \left(\prod_{m=1}^l (1 + P_m \alpha_m)^{\frac{1}{\delta_m}} \right) \leq 1 \right] \\ &= \Pr \left[\log \left(\prod_{m=1}^l (X_m)^{\frac{1}{\delta_m}} \right) \leq 1 \right] \quad (26) \\ &= \Pr \left[\log \left(\prod_{m=1}^l Z_m \right) \leq 1 \right] = \Pr[Y \leq 1] = F_Y(1) \end{aligned}$$

where X_m for $1 \leq m \leq l$ is a shifted exponential random variable with $p_{X_m}(x_m) = \frac{1}{P_m \lambda_m} \exp\left(-\frac{(x_m-1)}{P_m \lambda_m}\right)$, $x_m \geq 1$. $F_Y(y) = \Pr[Y \leq y]$ is cumulative distribution function (CDF) of Y . Using Mellin transform property from [11], we can write the Mellin transform of $W = \prod_{m=1}^l Z_m$ as:

$$M_W \{p_W(w)\}(s) = \prod_{m=1}^l M_{Z_m} \{p_{Z_m}(z_m)\}(s), \quad (27)$$

and the Mellin transform of Z_m , $1 \leq m \leq l$ is defined as:

$$\begin{aligned} M_{Z_m} \{p_{Z_m}(z_m)\}(s) &= \int_0^\infty x_m^{\frac{(s-1)}{\delta_m}} p_{X_m}(x_m) dx_m \\ &= \frac{\exp\left(\frac{1}{P_m \lambda_m}\right)}{P_m \lambda_m} \times \\ &\int_1^\infty x_m^{\frac{(s-1)}{\delta_m}} \exp\left(-\frac{x_m}{P_m \lambda_m}\right) dx_m \quad (28) \\ &\stackrel{(a)}{=} \exp\left(\frac{1}{P_m \lambda_m}\right) \times \\ &\left(\frac{1}{P_m \lambda_m}\right)^{\frac{1-s}{\delta_m}} \Gamma\left(\frac{1-s}{\delta_m} + 1, \frac{1}{P_m \lambda_m}\right) \end{aligned}$$

in (a), we have used the result from [12, pp. 340]. The region of convergence for the result in (28) is $\mathcal{R}\{s\} > 0$. Now using (27) and (28), we have

$$M_W \{p_W(w)\}(s) = K_l \cdot \left[(M_l)^{1-s} \prod_{m=1}^l \Gamma\left(\frac{1-s}{\delta_m} + 1, \frac{1}{P_m \lambda_m}\right) \right], \quad (29)$$

where K_l and M_l are defined in Proposition 1. The probability density function (PDF) of $W = \prod_{m=1}^l Z_m$ is given by the Mellin-Barnes integral as:

$$\begin{aligned} p_W(w) &= \frac{1}{2\pi\sqrt{-1}} \oint_{C:\mathcal{R}\{s\}>0} M_W \{p_W(w)\}(s) w^{-s} ds \\ &= \frac{M_l K_l}{2\pi\sqrt{-1}} \oint_{C:\mathcal{R}\{s\}>0} \left[\prod_{m=1}^l \left(\frac{w}{(P_m \lambda_m)^{\frac{1}{\delta_m}}} \right)^{-s} \right. \\ &\quad \left. \Gamma\left(\frac{1-s}{\delta_m} + 1, \frac{1}{P_m \lambda_m}\right) \right] ds \quad (30) \\ &= K_l M_l \times \\ &\quad \mathcal{H}_{0,l}^{l,0} \left[\frac{w}{M_l} \middle| \left(1 - \frac{1}{\delta_1}, \frac{1}{\delta_1}, \frac{1}{P_1 \lambda_1}\right), \dots, \left(1 - \frac{1}{\delta_l}, \frac{1}{\delta_l}, \frac{1}{P_l \lambda_l}\right) \right] \end{aligned}$$

We can write the PDF of $Y = \log W$ (from [8, A.10]) as :

$$p_Y(y) = K_l \cdot \ln 2 \cdot \mathcal{H}_{0,l}^{l,0} \left[\frac{2^y}{M_l} \middle| \left(1, \frac{1}{\delta_1}, \frac{1}{P_1 \lambda_1}\right), \dots, \left(1, \frac{1}{\delta_l}, \frac{1}{P_l \lambda_l}\right) \right] \quad (31)$$

and the CDF of Y can be expressed (from [8, A.14]) as:

$$F_Y(y) = 1 - K_l \times \mathcal{H}_{1,l+1}^{l+1,0} \left[\frac{2^y}{M_l} \middle| (0, 1, 0), \left(1, \frac{1}{\delta_1}, \frac{1}{P_1 \lambda_1}\right), \dots, \left(1, \frac{1}{\delta_l}, \frac{1}{P_l \lambda_l}\right) \right] \quad (32)$$

from (26) and (32), we can obtain (5).

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