Robust finite-frequency $H_2$ analysis of uncertain systems

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13th May 2011

Report no.: LiTH-ISY-R-3011
Submitted to Automatica

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Abstract

In many applications, design or analysis is performed over a finite-frequency range of interest. The importance of the $H_2$/robust $H_2$ norm highlights the necessity of computing this norm accordingly. This paper provides different methods for computing upper bounds on the robust finite-frequency $H_2$ norm for systems with structured uncertainties. An application of the robust finite-frequency $H_2$ norm for a comfort analysis problem of an aero-elastic model of an aircraft is also presented.

Keywords: Robust $H_2$ norm, Uncertain systems, Robust control.
Robust finite-frequency $\mathcal{H}_2$ analysis of uncertain systems

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Abstract

In many applications, design or analysis is performed over a finite-frequency range of interest. The importance of the $\mathcal{H}_2$/robust $\mathcal{H}_2$ norm highlights the necessity of computing this norm accordingly. This paper provides different methods for computing upper bounds on the robust finite-frequency $\mathcal{H}_2$ norm for systems with structured uncertainties. An application of the robust finite-frequency $\mathcal{H}_2$ norm for a comfort analysis problem of an aero-elastic model of an aircraft is also presented.

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1 Introduction

The $\mathcal{H}_2$/robust $\mathcal{H}_2$ norm has been one of the pivotal design and analysis criteria in many applications, such as structural dynamics, acoustics, colored noise disturbance rejection, etc., [22], [9], [3]. Due to the importance of the $\mathcal{H}_2$/robust $\mathcal{H}_2$ norm, there has been a substantial amount of research on computation, analysis and design based on these measures, many of which consider the use of Linear Matrix Inequalities (LMIs) and Ricatti equations for this purpose, e.g. [5], [19], [1], [11], [12], [4], [20]. A survey of recent methods in robust $\mathcal{H}_2$ analysis is provided in [14].

Most of the methods presented in the literature consider the whole frequency range for calculating the $\mathcal{H}_2$/robust $\mathcal{H}_2$ norm. However, in some applications it is beneficial to concentrate only on a finite-frequency range of interest and calculate the design/analysis measures accordingly. This can be due to different reasons, e.g. the model is only valid for a specific frequency range or the design is targeted for a specific frequency interval. This motivates the importance of computing the (robust) finite-frequency $\mathcal{H}_2$ norm.

In [6], a method for calculating the finite-frequency $\mathcal{H}_2$ norm for systems without uncertainty is presented, where the key step is to compute the finite-frequency observability Gramian. This is accomplished by first computing the regular observability Gramian and then scaling it by a system dependent matrix.

This paper introduces two methods for calculating an upper bound on the robust finite-frequency $\mathcal{H}_2$ norm for systems with structured uncertainties. The first method combines the notion of finite-frequency Gramians, introduced in [6], with convex optimization tools, [2], commonly used in robust control and calculates the upper bound by solving an underlying optimization problem [10]. The second method, provides a computationally cheaper algorithmic method for calculating the desired upper bound. In contrast to the first approach, the second method performs frequency gridding and breaks the original problem into smaller problems, which are possibly easier to solve. Then it uses the ideas presented in [18] on computing upper bounds on structured singular values, for solving the smaller problems. The results of the smaller problems are then combined to compute the upper bound on the whole desired frequency range, [15].

This paper is structured as follows. First some of the notations used throughout the paper are presented. Section 2 introduces the problem formulation. Mathematical preliminaries are presented in Section 3, which covers
the notion of finite-frequency Gramians and reviews the calculation of upper bounds on the robust $\mathcal{H}_2$ norm. Sections 4 and 5 provide the details of the two methods for calculating upper bounds on the robust finite-frequency $\mathcal{H}_2$ norm. In Section 6 numerical examples are presented. Section 7 provides more insight to the proposed methods by investigating the advantages and disadvantages of them, and finally Section 8 concludes the paper with some final remarks.

1.1 Notation

The notation in this paper is standard. The min and max represent the minimum and maximum of a function or a set, and similarly sup represents the supremum of a function. The symbols $\leq$ and $\preceq$ denote the inequality relation between matrices. A transfer matrix in terms of state-space data is denoted

$$
\begin{bmatrix}
A \\
C
\end{bmatrix} = C(j\omega I - A)^{-1} B + D.
$$

(1)

With $||\cdot||_2$, we denote the Euclidian or 2-norm of a vector or the norm of a matrix induced by the 2-norm. Furthermore $\mathcal{RH}_\infty$ represents real rational functions bounded on Re($s$) $= 0$ including $\infty$. For the sake of brevity of notation, unless necessary, we drop the dependence of functions on frequency.

2 Problem formulation

2.1 $\mathcal{H}_2$ norm of a system

Consider the following system in state space form

$$
\begin{cases}
\dot{x} = Ax + Bu \\
y = Cx
\end{cases}
$$

(2)

and define $G(s)$ as the corresponding transfer function. Then the $\mathcal{H}_2$ norm of the system in (2) is defined as follows

$$
\|G\|_2^2 = \int_{-\infty}^{\infty} \text{Tr} \{G(j\omega)^*G(j\omega)\} \frac{d\omega}{2\pi}.
$$

(3)

This can also be written as

$$
\|G\|_2^2 = \int_0^{\infty} \text{Tr} \left\{ B^T e^{A^T} C^T C e^{A} B \right\} dt \\
= \text{Tr} \left\{ B^T \left( \int_0^{\infty} e^{A^T} C^T C e^{A} dt \right) B \right\} \\
= \text{Tr} \left\{ B^T W_o B \right\},
$$

(4)

where $W_o$ is the observability Gramian of the system.

Similarly the finite-frequency $\mathcal{H}_2$ norm of (2) is defined as

$$
\|G\|_{2,\omega}^2 = \int_{-\omega}^{\omega} \text{Tr} \{G(j\omega)^*G(j\omega)\} \frac{d\omega}{2\pi}.
$$

(5)

2.2 Robust $\mathcal{H}_2$ norm of a system

Consider the uncertain state space system

$$
\begin{cases}
\dot{x} = Ax + B_w w + B_q q \\
p = Cpx + Dpq q \\
z = Cz x + Dzq q \\
q = \Delta p
\end{cases}
$$

(6)

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $z \in \mathbb{R}^l$ and $p, q \in \mathbb{R}^d$. Also $\Delta \in \mathbb{C}^{d \times d}$, which represents the uncertainty present in (6), has the following structure

$$
\Delta = \text{diag} \left[ \delta_1 I_{r_1} \cdots \delta_L I_{r_L} \Delta_{L+1} \cdots \Delta_{L+F} \right],
$$

(7)

where $\delta_i \in \mathbb{R}$ for $i = 1, \cdots, L$, $\Delta_{L+j} \in \mathbb{C}^{m_j \times m_j}$ for $j = 1, \cdots, F$ and $\sum_{i=1}^{L} r_i + \sum_{j=1}^{F} m_j = d$. Also in addition to that and without loss of generality it is assumed that $\Delta \in B_{\Delta}$ where $B_{\Delta}$ is the unit ball for the induced 2-norm. This structure of $\Delta$ can represent both real parametric uncertainties ($\delta_i I_{r_i}$) and un-modeled system dynamics ($\Delta_{L+j}$).

The transfer matrix for the uncertain system in (6) is defined as below, see Figure 1,
The following definition of this transfer matrix will also be used later in the upcoming sections

$$M(j\omega) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} A & B_q & B_w \\ C_p & D_{pq} & 0 \\ C_z & D_{rz} & 0 \end{bmatrix},$$  \label{eq: transfer_matrix}

where $M \in \mathbb{C}^{(d+l) \times (d+m)}$, $M_{11} \in \mathbb{C}^{d \times d}$, $M_{12} \in \mathbb{C}^{d \times m}$, $M_{21} \in \mathbb{C}^{l \times d}$ and $M_{22} \in \mathbb{C}^{l \times m}$.

The finite-frequency observability Gramian is defined as

$$W_o(\omega) = \int_{-\infty}^{\infty} H(j\omega)^* C^T CH(j\omega) \frac{d\omega}{2\pi}.$$ \label{eq: observability_gramian}

To continue with the study of the robust finite-frequency $\mathcal{H}_2$ problem, next the notion of finite-frequency observability Gramian is introduced, as proposed in [6]. The finite-frequency observability Gramian is defined as

$$W_o(\bar{\omega}) = \int_{-\bar{\omega}}^{\bar{\omega}} H(j\omega)^* C^T CH(j\omega) \frac{d\omega}{2\pi}.$$ \label{eq: observability_gramian_upper_bound}

The next lemma provides a way to compute $W_o(\bar{\omega})$ in terms of the observability Gramian, $W_o$.

**Lemma 1** The finite-frequency observability Gramian can be computed as

$$W_o(\bar{\omega}) = L(A, \bar{\omega})^* W_o + W_o L(A, \bar{\omega}),$$ \label{eq: observability_gramian_upper_bound2}

where $W_o$ is defined by (14) or equivalently (15) and

$$L(A, \bar{\omega}) = \int_{-\bar{\omega}}^{\bar{\omega}} H(j\omega) \frac{d\omega}{2\pi} = \frac{j}{2\pi} \ln[(A + j\bar{\omega}I)(A - j\bar{\omega}I)^{-1}].$$ \label{eq: observability_gramian_upper_bound3}

**PROOF.** See [6, page 100]. □

### 3.2 An upper bound on the robust $\mathcal{H}_2$ norm

Let $X$ represent Hermitian, block diagonal positive definite matrices that commute with $\Delta$, i.e. every $X \in \mathcal{X}$ has the following structure

$$X = \begin{bmatrix} X_1 & \cdots & X_L \\ x_{L+1} & \cdots & x_{L+P} \end{bmatrix}.$$ \label{eq: general_matrix_structure}

This paper proposes methods for calculating upper bounds on (13).
where the blocks in $X$ have compatible dimensions with their corresponding blocks in $\Delta$. The following condition plays a central role throughout this section.

**Condition 1** Consider the system in (6). There exists $X(\omega) \in \mathbb{X}$ where $\mathbb{X} \subseteq \mathbb{R}^{d \times d}$, Hermitian $Y(\omega) \in \mathbb{R}^{m \times m}$ and $\epsilon > 0$ such that

$$M(j\omega)^\ast \begin{bmatrix} X(\omega) & 0 \\ 0 & I \end{bmatrix} M(j\omega) - \begin{bmatrix} X(\omega) & 0 \\ 0 & Y(\omega) \end{bmatrix} \preceq \begin{bmatrix} -\epsilon I & 0 \\ 0 & 0 \end{bmatrix}. \tag{20}$$

The set of operators $\mathbb{X}$ are often called scaling matrices. In many cases it is customary to use constant scaling matrices to make the problem easier to handle. However the results achieved based on constant scaling matrices can be conservative. One of the ways to reduce the conservativeness and keep the computational complexity reasonable is to use special classes of dynamic scaling matrices. This will be investigated in more detail in Section 3.2.2.

Next, two methods for computing upper bounds on robust $H_2$ norm of systems with structured uncertainties are presented. The first method explicitly defines $Y(\omega)$ in Condition 1 and uses $Y(\omega)$ to construct the upper bound on the robust $H_2$ norm of the system. This method will be referred to as explicit upper bound calculation. The second method calculates the upper bound through computing the observability Gramian via solving a set of LMIs. This method is referred to as Gramian based upper bound calculation.

### 3.2.1 Explicit upper bound calculation

Consider Condition 1. This condition can be restated as follows

**Lemma 2** If there exists $X(\omega) \in \mathbb{X}$ such that

$$M_{11}^* X(\omega) M_{11} + M_{21}^* M_{21} - X(\omega) \prec 0, \tag{21}$$

then Condition 1 is satisfied if and only if there exists $Y(\omega) = Y(\omega)^\ast$ such that,

$$M_{12}^* X(\omega) M_{12} + M_{22}^* M_{22} - (M_{12}^* X(\omega) M_{11} + M_{22}^* M_{21}) \times (M_{11}^* X(\omega) M_{11} + M_{21}^* M_{21} - X(\omega))^{-1} \times (M_{12}^* X(\omega) M_{11} + M_{22}^* M_{21})^\ast \preceq Y(\omega). \tag{22}$$

**PROOF.** See Appendix A.

Using Condition 1 and Lemma 2, the following theorem provides upper bounds on the gain of the system for all frequencies and will be used to provide an upper bound on the robust $H_2$ norm for systems with structured uncertainty.

**Theorem 1** If there exists $X(\omega) \in \mathbb{X}$ such that (21) is satisfied $\forall \omega$ and we define $Y(\omega)$ as below

$$Y(\omega) = M_{12}^* X(\omega) M_{12} + M_{22}^* M_{22} - (M_{12}^* X(\omega) M_{11} + M_{22}^* M_{21}) \times (M_{11}^* X(\omega) M_{11} + M_{21}^* M_{21} - X(\omega))^{-1} \times (M_{12}^* X(\omega) M_{11} + M_{22}^* M_{21})^\ast, \tag{23}$$

then $\Delta \ast M(j\omega)^\ast (\Delta \ast M)(j\omega) \preceq Y(\omega) \ \forall \omega$.

**PROOF.** See Appendix B.

**Corollary 1** If there exists $X(\omega) \in \mathbb{X}$ and a frequency interval centered at $\omega_i$, $I(\omega_i) = [\omega_i + \omega_{\min}, \omega_i + \omega_{\max}]$, such that

$$M_{11}^* X M_{11} + M_{21}^* M_{21} - X \prec 0 \quad \forall \omega \in I(\omega_i), \tag{24}$$

and we consider $Y(\omega)$ as defined in (23) for the mentioned frequency interval then

$$\int_{\omega \in I(\omega_i)} \text{Tr} \{Y(\omega)\} \frac{d\omega}{2\pi} \leq \int_{\omega \in I(\omega_i)} \text{Tr} \{(\Delta \ast M)(\Delta \ast M)^\ast\} \frac{d\omega}{2\pi}, \tag{25}$$

for all $\Delta \in B_\Delta$, and specifically if $I(\omega_i)$ covers all frequencies

$$\sup_{\Delta \in B_\Delta} \|\Delta \ast M\|^2 \leq \int_{-\infty}^{\infty} \text{Tr} \{Y(\omega)\} \frac{d\omega}{2\pi}. \tag{26}$$

### 3.2.2 Gramian-based upper bound calculation

In this section a class of dynamic scaling matrices with the following structure will be considered

$$X(\omega) = \psi(j\omega) X \psi(j\omega)^\ast = \left[C_\psi(j\omega I - A_\psi)^{-1} I\right]^\ast \left[C_\psi(j\omega I - A_\psi)^{-1} I\right], \tag{27}$$

where $A_\psi \in \mathbb{R}^{n_\psi \times n_\psi}$ and $C_\psi \in \mathbb{R}^{d \times n_\psi}$ are fixed matrices with appropriate dimensions such that $A_\psi$ is stable and $(C_\psi, A_\psi)$ is observable. Also note that $X \in \mathbb{R}^{d \times d}$.
\[ \mathbb{R}^{(d+n_\psi)\times(d+n_\psi)} \] is a free basis for the parameters such that \( \mathcal{X}(s) \in \mathbb{X} \). As shown in [7] using this class of scaling matrices, Condition 1 can be rewritten as follows

**Lemma 3** Consider the partitioning \( M = [M_1 \ M_2] \), defined in (9), for the transfer matrix of system in (6). By replacing \( \mathcal{X}(\omega) \) with \( \mathcal{X}(\omega)^{-1} \) in (20), it can be restated as

\[
\begin{bmatrix}
M_1(j\omega)\mathcal{X}(\omega)M_1(j\omega)^* - \begin{bmatrix} \mathcal{X}(\omega) & 0 \\ 0 & I \end{bmatrix} M_2(j\omega) \\
M_2(j\omega)^* - Y(\omega)
\end{bmatrix} \leq 0.
\] (28)

**PROOF.** See [7, Lemma 1]. \( \Box \)

The upper left block of (28) can be expressed, up to its sign, as

\[
\begin{align*}
C_{11} := & \begin{bmatrix} \mathcal{X}(\omega) & 0 \\ 0 & I \end{bmatrix} - M_1(j\omega)\mathcal{X}(\omega)M_1(j\omega)^* \\
= & \begin{bmatrix} \psi & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \psi^* & M_{11} \psi \\ M_{21} \psi & 0 \end{bmatrix} X \begin{bmatrix} \psi \psi^* \\ M_{11} \psi \psi^* \\ M_{21} \psi \psi^* \end{bmatrix} \\
= & \begin{bmatrix} M_{11} \psi \psi^* & 0 \\ M_{21} \psi \psi^* & 0 \end{bmatrix} \begin{bmatrix} -X & 0 \\ 0 & -X \end{bmatrix} \begin{bmatrix} M_{11} \psi \psi^* & 0 \\ M_{21} \psi \psi^* & 0 \end{bmatrix}.
\end{align*}
\] (29)

By introducing the following transfer matrix

\[
\begin{align*}
\hat{C}(j\omega I - \hat{A})^{-1} \hat{B}_q + \hat{D} & = \begin{bmatrix} M_{11} \psi \psi^* \\ M_{21} \psi \psi^* \end{bmatrix},
\end{align*}
\] (30)

and setting \( \Gamma = \begin{bmatrix} 0 & I \end{bmatrix}^T \), (29) can be reformulated as

\[
\begin{align*}
C_{11} = & \left( \begin{bmatrix} \hat{C}(j\omega I - \hat{A})^{-1} I \\ \hat{D} \end{bmatrix} \begin{bmatrix} \hat{B}_q \\ 0 \end{bmatrix} \right) \times \\
& \left( \begin{bmatrix} \hat{C}(j\omega I - \hat{A})^{-1} I \\ \hat{D} \end{bmatrix} \begin{bmatrix} \hat{B}_q \\ 0 \end{bmatrix} \right)^*,
\end{align*}
\] (31)

where

\[
\hat{A} = \begin{bmatrix} A & B_q C_\psi & 0 \\ 0 & A_\psi & 0 \\ 0 & 0 & A_\psi \end{bmatrix}, \quad \hat{B}_q = \begin{bmatrix} 0 & B_q & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
\hat{C} = \begin{bmatrix} C & DC_\psi \\ 0 & C_\psi \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix},
\] (32)

where \( \hat{A} \in \mathbb{R}^{n_\psi \times n_\psi}, \hat{B}_q \in \mathbb{R}^{n_\psi \times n'}, \hat{C} \in \mathbb{R}^{(l+d)\times n_\psi} \) and \( \hat{D} \in \mathbb{R}^{l+d \times d} \), with \( n = 2n_\psi + n' \) and \( d = 2n_\psi + 2d \).

Let \( \Pi(X, \hat{B}_q, \hat{D}) \) be an affine function of \( X \) defined as below

\[
\Pi(X, \hat{B}_q, \hat{D}) = \begin{bmatrix} \hat{B}_q \\ \hat{D} \end{bmatrix} \begin{bmatrix} -X & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} \hat{B}_q^* \\ \hat{D} \end{bmatrix}^T
\]

\[
= \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{I2} & \Pi_{22} \end{bmatrix},
\] (33)

where \( \Pi_{11} \in \mathbb{R}^{n_\psi \times n_\psi}, \Pi_{12} \in \mathbb{R}^{n_\psi \times (l+d)} \) and \( \Pi_{22} \in \mathbb{R}^{l+d \times l+d} \).

The following theorem taken from [11] can be used to calculate upper bounds on the robust \( H_2 \) norm.

**Theorem 2** If there exist matrix \( X \) such that \( \mathcal{X}(\omega) \) in (27) satisfies \( \mathcal{X}(\omega) \in \mathbb{X} \), and Hermitian matrices \( P_-, P_+ \in \mathbb{R}^{n_\psi \times n_\psi}, Q \in \mathbb{R}^{n_\psi \times n_\psi}, W_\psi \in \mathbb{R}^{n_\psi \times n_\psi} \), such that

\[
\begin{bmatrix} P_- & Q \gg 0, \\ A_\psi Q + QA_\psi^T & QC_\psi^T \\ C_\psi Q & 0 \end{bmatrix} - X \prec 0,
\]

\[
\begin{bmatrix} \hat{A}P_+ + P_- \hat{A}^T & P_- \hat{C}^T \\ \hat{A}P_+ + P_+ \hat{A}^T & P_+ \hat{C}^T \end{bmatrix} - \Pi(X, \hat{B}_q, \hat{D}) \prec 0,
\]

\[
\begin{bmatrix} \hat{W}_\psi & \hat{I} \\ \hat{I} & P_+ - P_- \end{bmatrix} \prec 0,
\]

\[
\text{Tr} \left( \begin{bmatrix} \hat{B}_q^T & 0 \end{bmatrix} \hat{W}_\psi \begin{bmatrix} \hat{B}_q \\ 0 \end{bmatrix} \right) < \gamma^2,
\]

then \( \mathcal{X}(\omega) \) satisfies (28) and the system \( (\Delta * M) \) defined in (11) has robust \( H_2 \) norm less than \( \gamma^2 \).
Theorem 2 includes the problem with constant scaling matrices as a special case. Let

$$\dot{A} = A, \dot{B}_q = [B_q 0], \dot{C} = C, \dot{D} = \begin{bmatrix} D & [I_d 0] \end{bmatrix}. \quad (35)$$

Then the following Corollary is a restatement of Theorem 2 for constant scaling matrices, i.e. $\mathcal{X}(\omega) = X$.

**Corollary 2** If there exist matrix $X \in \mathbb{X}$ and symmetric matrices $P_- , P_+ , Z \in \mathbb{R}^{n \times n}$ such that

$$\begin{cases}
P_-, X > 0, \\
\begin{bmatrix}
\hat{A}P_+ + P_- \hat{A}^T & P_- \hat{C}^T \\
\hat{C}P_+ & 0
\end{bmatrix} - \Pi(X, \hat{B}_q, \hat{D}) < 0,
\end{cases}$$

$$\begin{cases}
\hat{A}P_+ + P_- \hat{A}^T & P_+ \hat{C}^T \\
\hat{C}P_+ & 0
\end{cases} - \Pi(X, \hat{B}_q, \hat{D}) < 0, \quad (36)
$$

$$\begin{bmatrix} Z & I \\
I & P_+ - P_- \end{bmatrix} > 0,$$

$$\text{Tr} \left( B_w^T Z B_w \right) < \gamma^2.$$  

then $\mathcal{X}(\omega) = X$ satisfies (28) and the system $(\Delta \ast M)$ defined in (11) has robust $\mathcal{H}_2$ norm less than $\gamma^2$.

**PROOF.** See [11]. \qed

Aside from upper bounds on the robust $\mathcal{H}_2$ norm of the system, Theorem 2 also provides additional information that will be used in the upcoming sections. These additional information are highlighted in the following Lemma.

**Lemma 4** Let $P_-, P_+, X, Q$ and $\tilde{W}_o$ satisfy (34), and $C_{11}$ be defined as in (29). Then $C_{11} > 0$ with spectral factor $\tilde{N}$ such that $\tilde{N}, \tilde{N}^{-1} \in \mathcal{R} \mathcal{H}_\infty$, i.e. $C_{11} = \tilde{N} \tilde{N}^*$. Also let the scaled $M$ be defined as

$$\hat{M} = \begin{bmatrix} \mathcal{X}(\omega)^{1/2} & 0 \\
0 & \mathcal{X}(\omega)^{-1/2} \end{bmatrix} \begin{bmatrix} \mathcal{X}(\omega)^{1/2} & 0 \\
0 & \mathcal{X}(\omega)^{-1/2} \end{bmatrix}, \quad (37)$$

and be partitioned as $\hat{M} = [\hat{M}_1 \hat{M}_2]$. Then $\|\tilde{N}^{-1} \hat{M}_2\|^2 < \gamma^2$. A state space realization for $\tilde{N}^{-1} \hat{M}_2$ is given by

$$\tilde{N}^{-1} \hat{M}_2 = \begin{bmatrix} \hat{A} - (\Pi_{12} - P_- \hat{C}^T) \Pi_{22}^{-1} \hat{C} & \hat{B}_w \\
\Pi_{22}^{-1} \hat{C} & 0 \end{bmatrix}. \quad (38)$$

Moreover $W_o$ is the observability Gramian of $\tilde{N}^{-1} \hat{M}_2$.

**PROOF.** See [11]. \qed

4 Gramian-based upper bound on the robust finite-frequency $\mathcal{H}_2$ norm

In this section the first method for calculating an upper bound on the robust finite-frequency $\mathcal{H}_2$ norm of system in (6) is presented. The following theorem combines the ideas presented in Section 3.1, regarding the finite-frequency observability Gramians, with the results of Section 3.2.2, and computes the upper bound on the robust finite-frequency $\mathcal{H}_2$ norm for (6). Hereafter this method is referred to as Method 1.

**Theorem 3** Let $P_-, P_+, X, Q$ and $\tilde{W}_o$ be a solution to (34), then

$$\sup_{\Delta \in B_\Delta} \|\Delta \ast M\|^2 \leq \text{Tr} \left( \begin{bmatrix} B_w \\
0 \end{bmatrix} \left( L(\hat{A}, \omega)^* \tilde{W}_o + \tilde{W}_o L(\hat{A}, \omega) \right) \begin{bmatrix} B_w \\
0 \end{bmatrix} \right) \quad (39)$$

where $L(\hat{A}, \omega)$ is defined in (18) and $\hat{A} = \tilde{A} - (\Pi_{12} - P_- \hat{C}^T) \Pi_{22}^{-1} \hat{C}$.

**PROOF.** See Appendix C.

As was mentioned in Section 3.2, by using dynamic scaling matrices and increasing the order of these scaling matrices, it is possible to reduce the conservativeness of the results. In order to further reduce the conservativeness of the bounds and improve the numerical properties of the optimization problems, it is useful to perform uncertainty partitioning. In this approach, for each of the uncertainty partitions, the upper bound on the robust finite-frequency $\mathcal{H}_2$ norm of the system is computed and the maximum of these bounds is considered as the final result.
5 Frequency gridding based upper bound on the robust finite-frequency $H_2$ norm

In this section the second method to compute upper bounds on the robust finite-frequency $H_2$ norm is presented.

The following corollary to Theorem 1 plays a central role in the proposed algorithm.

**Corollary 3** Let $\mathcal{I}(\omega_i)$ for $i = 1, \ldots, m$ be disjoint frequency intervals such that $\bigcup_{i=1}^{m} \mathcal{I}(\omega_i) = [-\bar{\omega}, \bar{\omega}]$. Also let the constant matrices $X_i$ for $i = 1, \ldots, m$ be the scaling matrices for which $M_{11}^*X_iM_{11} + M_{21}^*M_{21} - X_i < 0$ $\forall \omega \in \mathcal{I}(\omega_i)$. Then, it holds that

$$\sup_{\Delta \in \mathcal{B}_\Delta} \| \Delta \ast M \|_{2,\bar{\omega}}^2 \leq \sup_{\Delta \in \mathcal{B}_\Delta} \sum_{i=1}^{m} \int_{\omega \in \mathcal{I}(\omega_i)} \Tr \{ (\Delta \ast M)^*(\Delta \ast M) \} \frac{d\omega}{2\pi} \leq \sum_{i=1}^{m} \int_{\omega \in \mathcal{I}(\omega_i)} \Tr \{ Y_i(\omega) \} \frac{d\omega}{2\pi},$$

(40)

where $Y_i(\omega)$ is defined as in (23), with $X(\omega) = X_i$.

Corollary 3 provides a sketch for calculating upper bounds on the robust finite-frequency $H_2$ norm via frequency gridding. However calculating a suitable scaling matrix $X$ for $\mathcal{I}(\omega_i)$ requires checking $M_{11}^*X_iM_{11} + M_{21}^*M_{21} - X_i < 0$ for an infinite number of frequencies in $\mathcal{I}(\omega_i)$. Next a method is proposed to solve this issue. Consider the following two LMI's

$$M_{11}(j\omega)^*X(\omega)M_{11}(j\omega) + M_{21}(j\omega)^*M_{21}(j\omega) - X(\omega) < 0, \quad \text{(41)}$$

$$\begin{bmatrix} M_{11}(j\omega) & 0 \\ M_{21}(j\omega) & 0 \end{bmatrix}^* \tilde{X}(\omega) \begin{bmatrix} M_{11}(j\omega) & 0 \\ M_{21}(j\omega) & 0 \end{bmatrix} - \tilde{X}(\omega) < 0. \quad \text{(42)}$$

Then $\tilde{X}_i = \begin{bmatrix} X_i & 0 \\ 0 & I \end{bmatrix} \in \mathbb{R}^{(d+l) \times (d+l)}$ satisfies (42) for $\omega = \omega_i$, if and only if $X_i$ satisfies (41) for the same frequency.

The following theorem taken from [18], solves the issue of infinite dimensionality of the problem in Corollary 3 by providing a way to extend the validity of a scaling matrix that satisfies $M_{11}^*X_iM_{11} + M_{21}^*M_{21} - X_i < 0$ for a single frequency, e.g. $\omega = \omega_i$, to a frequency interval, $\mathcal{I}(\omega_i)$.

**Theorem 4** Let $\tilde{M} = \begin{bmatrix} M_{11} & 0 \\ M_{21} & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, and let

$$D = \tilde{X}_i^\circ, \quad \text{where } \tilde{X}_i \text{ satisfies the LMI in (42). Define}$$

$$G = A_X - B_X D_X^{-1} C_X,$$

(43)

where

$$A_X = \begin{bmatrix} A_G & 0 \\ -C_G^* C_G & -A_G^* \end{bmatrix}, \quad B_X = \begin{bmatrix} -B_G \\ C_G D_G \end{bmatrix}, \quad C_X = \begin{bmatrix} D_G^* C_G & B_G^* \end{bmatrix}, \quad D_X = I - D_G D_G$$

(44)

in which

$$G = \begin{bmatrix} A_G & B_G \\ C_G & D_G \end{bmatrix} = \begin{bmatrix} A - j\omega_i I & \tilde{B} D^{-1} \\ D \tilde{C} & D \tilde{D} D^{-1} \end{bmatrix},$$

(45)

and define $\omega_{\text{low}}$ and $\omega_{\text{high}}$ as

$$\omega_{\text{low}} = \begin{cases} -\omega_i, & \text{if } jG \text{ has no positive real eigenvalue} \\ \max\{ \lambda \in \mathbb{R}_+ : \det(\lambda I + jG) = 0 \}, & \text{otherwise} \end{cases}$$

(46)

$$\omega_{\text{high}} = \begin{cases} \infty, & \text{if } jG \text{ has no negative real eigenvalue} \\ \min\{ \lambda \in \mathbb{R}_+ : \det(\lambda I + jG) = 0 \}, & \text{otherwise} \end{cases}$$

(47)

Then $\tilde{X}_i$ satisfies (42) $\forall \omega \in \mathcal{I}(\omega_i) = [\omega_i + \omega_{\text{low}}, \omega_i + \omega_{\text{high}}]$.

**PROOF.** See Appendix D.

Using Corollary 3 and Theorem 2, the following algorithm can be used for calculating an upper bound on the robust finite-frequency $H_2$ norm. This algorithmic method is referred to as Method 2.
(I) Divide the frequency interval of interest into a desired number of disjoint partitions, \( \mathcal{I}(\omega_i) \), where \( \omega_i \) is the center of the respective partition.

(II) For each of the partitions, compute \( X_i \) such that it satisfies (41) for \( \omega = \omega_i \). In case there exist a partition for which there exists no feasible solution, the system is not robustly stable and this method cannot be applied to this system.

(III) Construct \( X_i \) from the achieved \( X_i \) in (II).

(IV) Using Theorem 4 calculate the valid frequency range for the mentioned LMI s in (II). If the achieved frequency range does not cover the respective frequency partition, i.e., \( \mathcal{I}(\omega_i) \not\subseteq \mathcal{I}(\omega) \), go back to (I) and choose a finer partitioning for the frequency interval of interest.

(V) Define \( Y_i(\omega) \) using (23) with \( X(\omega) = X_i \).

(VI) Use numerical integration to calculate
\[
\int_{\omega \in \mathcal{I}(\omega)} \text{Tr} \{ Y_i(\omega) \} \, d\omega.
\]

(VII) By Corollary 3, calculate the upper bound by summing up the integrals computed in (VI).

The second step of Algorithm 1, requires computation of constant scaling matrices that satisfy (41) for \( \omega = \omega_i \) for each of the partitions. This can be accomplished through different approaches. However, considering the expression in (40) and the importance of \( \text{Tr} \{ Y_i(\omega) \} \) in the quality (closeness to the actual value) of the proposed upper bound on the robust finite-frequency \( \mathcal{H}_2 \) norm in (40), it seems intuitive to calculate the scaling matrices while aiming at minimizing \( \text{Tr} \{ Y_i(\omega) \} \). The following two approaches utilize this in the process of computing suitable scaling matrices.

**Approach 1** Compute \( X_i \) in Step (II) of Algorithm 1 as the solution of the following optimization problem

\[
\min_{X_i, Y_i} \text{Tr} \{ Y_i \}
\]
subject to
\[
(20) \quad \text{with} \quad \omega = \omega_i.
\] (48)

**Remark 1** The idea of frequency gridding was also presented in [13], where the authors consider the \( \mathcal{H}_2 \) performance problem for discrete time systems. In that paper, an optimization problem similar to (48) for frequencies \( 0 = \omega_0 \ldots \omega_N = 2\pi \) is formulated and then the integral \( \int_{\omega = 0}^{\omega = 2\pi} \text{trace}(Y(\omega)) \, d\omega \) is approximated by the following Riemann sum expression

\[
\frac{1}{2\pi} \sum_{i=1}^{N} \text{trace}(Y_i(\omega_i - \omega_{i-1})),
\]
where \( 0 = \omega_0 \ldots \omega_N = 2\pi \).

However, this approach does not necessarily provide a guaranteed upper bound on the robust \( \mathcal{H}_2 \) norm of the system.

For any \( X_i \) satisfying the LMI in (41) for \( \omega = \omega_i \) let

\[
f(\alpha) = \text{Tr} \{ M_{12}^* \alpha X_i M_{12} + M_{22}^* M_{22} - (M_{12}^* \alpha X_i M_{11}^* + M_{22}^* M_{22}) \times (M_{11}^* \alpha X_i M_{11}^* + M_{21}^* M_{21} - \alpha X_i)^{-1} \times (M_{12}^* \alpha X_i M_{11}^* + M_{22}^* M_{22})^* \}.
\] (50)

This function is convex with respect to \( \alpha \). Next, following the same objectives as in Approach 1, an alternative method for calculating suitable scaling matrices is introduced.

**Approach 2** Compute \( X_i \) in Step (II) of Algorithm 1 using the following sequential method

(I) Find \( X_i \) such that it satisfies the LMI in (41) for \( \omega = \omega_i \).

(II) Minimize \( f(\alpha) \), in (50), with the achieved \( X_i \) with respect to all \( \alpha \) such that \( \alpha X_i \) still satisfies the LMI in (41) for \( \omega = \omega_i \).

Denote \( \alpha^* \) as the minimizing \( \alpha \). Then \( \alpha^* X_i \) will be used within the remaining steps of Algorithm 1. In order to assure that \( \alpha^* X_i \) satisfies (41) the search for \( \alpha \) should be subject to the constraint \( \alpha > \alpha_{\min} \), where

\[
\alpha_{\min} = \min \left\{ \text{eig} \left[ \begin{bmatrix} \Lambda^{\frac{-1}{2}} & 0 \\ 0 & I \end{bmatrix} U(-M_{11}^* X_i M_{11} + X_i U^*) \begin{bmatrix} \Lambda^{\frac{-1}{2}} & 0 \\ 0 & I \end{bmatrix} \right] \right\},
\]

in which \( U \), a unitary matrix, and \( \Lambda \), are defined by the singular value decomposition \( M_{21}^* M_{21} = U^* \begin{bmatrix} \Lambda^{\frac{-1}{2}} & 0 \\ 0 & I \end{bmatrix} U \).

It is important to note that for some problems it might be required to perform many iterations between the first and the fourth steps of Algorithm 1. One of the ways
to alleviate this issue and even calculate better upper bounds, is to modify the proposed approaches by augmenting new constraints for other frequencies from the partition under investigation. In this case the cost function can also be modified accordingly. As an example, Approach 1 can be modified as follows

\[
\begin{aligned}
\min_{X_i, Y_i} & \quad \text{Tr} \{ Y_i \} \\
\text{subj. to} & \quad M(j\omega)^* \begin{bmatrix} X_i & 0 \\ 0 & I \end{bmatrix} M(j\omega) - \begin{bmatrix} X_i & 0 \\ 0 & Y_i \end{bmatrix} \preceq 0 \\
\text{for } & \quad \omega = \omega_j \in I(\omega_i), j = 1, \ldots, N_i,
\end{aligned}
\]

or alternatively as

\[
\begin{aligned}
\min_{X_i, Y_i, j=1,\ldots,N_i} & \quad \sum_{j=1}^{N_i} \text{Tr} \{ Y_i^j \} \\
\text{subj. to} & \quad M(j\omega)^* \begin{bmatrix} X_i & 0 \\ 0 & I \end{bmatrix} M(j\omega) - \begin{bmatrix} X_i & 0 \\ 0 & Y_i^j \end{bmatrix} \preceq 0 \\
\text{for } & \quad \omega = \omega_j \in I(\omega_i), j = 1, \ldots, N_i.
\end{aligned}
\]

Similar to Method 1, uncertainty partitioning improves the quality of the calculated upper bound on this method too.

**Remark 2** Note that although the calculated value for the upper bound using Algorithm 1 has a decreasing trend with respect to the number of partitions, this trend is not necessarily monotonically decreasing. This is due to the fact that the calculated upper bound not only is dependent on the number of partitions but also on the quality of the calculated scaling matrices and how they affect the numerical integration procedure.

6 Numerical examples

In this section the proposed methods are tested on theoretical and practical examples. The chosen theoretical example can be solved analytically, i.e. the robust $H_2$ norm for this example can be computed via routine calculations. The achieved results for this example are reported in Section 6.1.

As a practical example, an application to the comfort analysis problem for a civil aircraft model is discussed. Due to its more complex uncertainty structure, this example is computationally more challenging. Section 6.2 presents the analysis results for this example. It should be pointed out that all the computations, for both examples, are conducted using the Yalmip toolbox [8] with the SDPT3 solver [21]. The platform used for the simulations uses a Dual Core AMD Opteron$^{TM}$ Processor 270 as the CPU and 4 GB of RAM.

6.1 Theoretical Example

Consider the uncertain system in (6) with the following system matrices

\[
A = \begin{bmatrix} -2.5 & 0 & 0 & -50 & 0 \\ 0 & -1 & 0.5 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -100 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_q = \begin{bmatrix} 0.25 & -0.5 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},
\]

\[
B_w = \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \quad C = \begin{bmatrix} C_p \\ C_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\]

\[
D = \begin{bmatrix} D_{pq} \\ D_{zq} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}.
\]

In this example $\Delta(\delta) = \delta I_2$ with $-1 \leq \delta \leq 1$. This system is known to have robust $H_2$ norm, as defined in (12), equal to 1.5311 which is attained for $\delta = 0.25$.

Figure 2 illustrates the gain plots of the system for different values of the uncertain parameter. The aim is to calculate the robust finite-frequency $H_2$ norm of the system and avoid the peak occurring at 100 rad/s. This is motivated by Figure 3 which presents the calculated finite-frequency $H_2$ norm of the system in (54), with respect to different values for the uncertain parameter and frequency bounds. As can be seen from this figure and the jump at $\tilde{\omega} = 100$ rad/s, the contribution of this peak to the robust finite-frequency $H_2$ norm cannot be neglected. In order to avoid this peak, the frequency bound that has been considered for this example is $\tilde{\omega} = 50$ rad/s. The actual value for the robust finite-frequency $H_2$ norm for (54) with this frequency bound is 0.8919.

Method 1, presented in Section 4, utilizes the following class of dynamic scaling matrices
ψ(s) = \left[ \frac{(s-p)^{n_{\psi} - 1}}{(s-p)^{n_{\psi}}} I_2 \left( \frac{(s-p)^{n_{\psi} - 2}}{(s-p)^{n_{\psi}}} I_2 \ldots \frac{1}{(s-p)^{n_{\psi}} I_2} I_2 \right) \right] (55)

with \( p = 150 \), and via Theorem 3 it calculates the upper bound on the robust finite-frequency \( \mathcal{H}_2 \) norm of the system. For this particular example dynamic scaling matrices with order higher than 3 do not produce any better upper bounds, so only scaling matrices up to order 3 are considered.

Method 2, presented in Section 5, has been applied to this example with Approaches 1 and 2. The number of frequency partitions is increased until either the performance matches the performance of Method 1 or the improvement in the computed upper bound is not discernible anymore.

Figure 4 illustrates the achieved upper bounds on different frequency bounds, \( \bar{\omega} \). The curve marked with the solid line reports the actual values for the robust finite-frequency \( \mathcal{H}_2 \) norm of the system. The dashed lines present the achieved upper bounds using Method 1. As can be seen from the figure as the order of the dynamic scaling matrices increases the computed upper bound becomes tighter. Note that the upper bounds computed using scaling matrices with \( n_{\psi} \geq 1 \) are practically indistinguishable. The bounds presented with the dashed-dotted lines are results achieved by applying Method 2 to this example. As can be seen from Figure 4, Method 2 with Approach 1 can produce better upper bounds than the second approach and can match the performance of Method 1 with 40 partitions. Figure 6 illustrates the calculated upper bounds on the systems gains, \( Y(\omega) \), for different frequencies. Table 1 presents a summary of the achieved results.

So far the presented results are achieved without any uncertainty partitioning. In order to illustrate the effect of uncertainty partitioning on the performance of the proposed methods, Method 1 and Method 2 with Approach 1 are applied to this example with uncertainty partitioning. Figures 6 and 7 present the achieved upper bounds on robust finite-frequency \( \mathcal{H}_2 \) norm of the system with \( \bar{\omega} = 50 \text{ rad/s} \) using Methods 1 and 2, respectively. These figures illustrate the upper bound with respect to number of uncertainty partitions and order of dynamic scaling matrices, for Method 1, and number of frequency partitions, for Method 2.
Fig. 5. Magnitudes of $\|\Delta M\|^2_2$ for different uncertainty values (solid lines), and the calculated upper bound on each frequency point. The dashed and dashed-dotted lines represent the achieved upper bounds using Methods 1 and 2 for different orders of scaling matrix and numbers of frequency partition numbers, respectively.

grid points, for Method 2. As can be seen from the figures and considering the actual robust finite-frequency $H_2$ norm of the system, the computed upper bounds using both methods are extremely tight. A summary of the results from this analysis is presented in Tables 2 and 3.

As can be observed from Tables 2 and 3, although both methods produce equally tight upper bounds, Method 1 achieves this goal with lower computation time.

Table 1
Numerical results for the theoretical example.

<table>
<thead>
<tr>
<th>Method</th>
<th>Estimated Upper bound</th>
<th>Elapsed Time[sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>M.1, $n_{\omega} = 0$</td>
<td>1.2609</td>
<td>11</td>
</tr>
<tr>
<td>M.1, $n_{\omega} = 1$</td>
<td>1.1972</td>
<td>10</td>
</tr>
<tr>
<td>M.1, $n_{\omega} = 2$</td>
<td>1.1944</td>
<td>12</td>
</tr>
<tr>
<td>M.1, $n_{\omega} = 3$</td>
<td>1.1911</td>
<td>13</td>
</tr>
<tr>
<td>M.2, App.1, $n_{par} = 40$</td>
<td>1.189</td>
<td>44</td>
</tr>
<tr>
<td>M.2, App.1, $n_{par} = 200$</td>
<td>1.186</td>
<td>144</td>
</tr>
<tr>
<td>M.2, App.2, $n_{par} = 200$</td>
<td>1.3184</td>
<td>552</td>
</tr>
</tbody>
</table>

Table 2
Numerical results for the theoretical example Using Method 1.

<table>
<thead>
<tr>
<th>$n_{\omega}$</th>
<th>No. Uncer. Par.</th>
<th>Estimated Upper bound</th>
<th>Elapsed Time[sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>1.1944</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>0.8928</td>
<td>434</td>
</tr>
</tbody>
</table>

Table 3
Numerical results for the theoretical example Using Method 2.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>1</td>
<td>1.1945</td>
<td>30</td>
</tr>
<tr>
<td>20</td>
<td>20</td>
<td>0.8924</td>
<td>532</td>
</tr>
</tbody>
</table>

6.2 Comfort Analysis Application

The problem considered in this section involves a model of a civil aircraft, including both rigid and flexible modes. This model is to be used to evaluate the effect of wind turbulence on different points of the aircraft, and is re-
ferred to as comfort analysis. This problem can be reformulated as an $H_2$ performance analysis problem for an extended system, including the model of the aircraft, a Von Karman filter, modeling the wind spectrum, and an output filter, accounting for the turbulence field, [16]. In the provided aircraft model the uncertain parameter $\delta$ corresponds to the level of fullness of the fuel tanks and it is normalized to vary within the range $[-1, 1]$. The overall extended system is presented in LFT form, as in (6), with $n = 21$ states and an uncertainty block size of $d = 14$.

The provided aircraft model is valid for frequencies up to 15 rad/s and beyond that does not have any physical meaning [17]. This motivates performing finite-frequency $H_2$ performance analysis, limited to this frequency range.

Figure 8, illustrates the gain plots of the system under consideration as a function of frequency. Different curves in this figure correspond to different uncertainty values. As can be seen from the figure, the frequency bound at 15 rad/s is necessary to avoid the peak at approximately 20 rad/s which is outside the validity range of the model.

The methods considered for performing comfort analysis are Methods 1 and 2 with the use of constant scaling matrices and Approach 1, respectively. Tables 4 and 5 summarize the achieved results using Methods 1 and 2, respectively. As can be seen from the tables, both methods perform equally accurate in estimating the robust finite-frequency $H_2$ norm of the system. However, in contrast to the example in Section 6.1, Method 2 is faster in calculating the upper bound with equal accuracy.

Similar to Section 6.1, it is possible to improve the computed upper bounds via uncertainty partitioning. This can be observed from Tables 4 and 5.

### 7 Discussion and General remarks

This section highlights the advantages and disadvantages of the proposed methods and provides insight on how to improve the performance of the methods considering the characteristics of the problem at hand.

#### 7.1 The observability Gramian based method

This method considers the frequency interval of interest as a whole and calculates an upper bound on the robust finite-frequency $H_2$ norm of the system in one shot or one iteration by solving an SDP. However the dimension of this optimization problem grows rapidly with the number of states and/or size of the uncertainty block. This limits the capabilities of this method in handling medium or large sized problems, i.e. analysis of systems with high number of states or large uncertainty blocks.

The most apparent possibility to improve the accuracy of the computed upper bound using this method is to increase the order of the dynamic scaling matrices. This comes at the cost of rapidly increasing the number of optimization variables in the underlying SDP and affects the computational tractability of the method.

Another way of improving the computed upper bound is to perform uncertainty partitioning, which proved to be effective through the examples presented in Section 6. However, this improvement comes at the cost of a much higher computational burden, see Table 4.

#### 7.2 The frequency gridding based method

This method starts with an initial partitioning of the desired frequency interval and calculates the upper bound on the robust finite-frequency $H_2$ norm by solving the corresponding SDP for each of the partitions.

### Table 4
Numerical results for the theoretical example Using Method 1.

<table>
<thead>
<tr>
<th>$n_\psi$</th>
<th>No. Uncer. Par.</th>
<th>Estimated Upper bound</th>
<th>Elapsed Time[h]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>50</td>
<td>1.2434</td>
<td>8.62</td>
</tr>
<tr>
<td>0</td>
<td>450</td>
<td>0.7970</td>
<td>59.24</td>
</tr>
</tbody>
</table>

### Table 5
Numerical results for the theoretical example Using Method 2.

<table>
<thead>
<tr>
<th>No. freq. Grids</th>
<th>No. Uncer. Par.</th>
<th>Estimated Upper bound</th>
<th>Elapsed Time[h]</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>1</td>
<td>1.2382</td>
<td>0.5611</td>
</tr>
<tr>
<td>80</td>
<td>10</td>
<td>0.7911</td>
<td>4.25</td>
</tr>
</tbody>
</table>
The size of the underlying SDPs in this method is smaller than the previous method and is mainly dependent on the size of the uncertainty block. Consequently, this method can handle larger problems. However for large problems, the algorithm might require some iterations between steps IV and I of the algorithm, to be able to produce consistent results. Another issue with this method is the requirement to perform numerical integration on a rational function in step VI of the algorithm. This can become slightly problematic for high order systems.

There are two main ways to improve the computed upper bounds using this method, namely increasing the number of partitions, and augmenting the SDP for each partition with more constraints for other frequency points in the partition and/or adding more variables to the SDPs corresponding to the partitions. This proved to scale better considering the computation time, as compared to Method 1 see Table 5.

8 Conclusion

This paper has provided two methods for calculating upper bounds on the robust finite-frequency $H_2$ norm. Through the paper different guidelines for improving the performance of the proposed methods have been presented and their effectiveness has been illustrated using both a theoretical and a practical example.

The proposed methods consider different formulations for calculating a consistent upper bound on the robust finite-frequency $H_2$ norm. Due to this, although both methods can produce equally tight upper bounds, they have different computational properties. Method 1 is more suitable for small-sized problems and produce results faster than the second method for this type of problems. On the other hand, Method 2 can handle larger problems and produce results more rapidly for this class of problems.

Acknowledgements

The authors wish to thank involved personnel form AIRBUS, Clément Roos and Carsten Döll from ONERA and Simon Hecker and Andras Varga from DLR for providing the model of the civil aircraft used in Section 6.2.

References


A Proof of Lemma 2

Let

\[\begin{align*}
C_{11} &= M_{11}^* X(\omega) M_{11} + M_{21}^* M_{21} - X(\omega), \\
C_{12} &= M_{11}^* X(\omega) M_{12} + M_{21}^* M_{22}, \\
C_{21} &= M_{12}^* X(\omega) M_{11} + M_{22}^* M_{21}, \\
C_{22} &= M_{12}^* X(\omega) M_{12} + M_{22}^* M_{22}. 
\end{align*}\]  

(A.1)

Then the left hand side of Condition 1 can be written as

\[
M^*(j\omega) \begin{bmatrix} X(\omega) & 0 \\ 0 & I \end{bmatrix} M(j\omega) - \begin{bmatrix} X(\omega) & 0 \\ 0 & Y(\omega) \end{bmatrix} = \\
\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} - Y(\omega) \end{bmatrix}.
\]

(A.2)

Now if we assume that there exists \(X(\omega) \in \mathbb{K}\) such that \(C_{11} \prec 0\), then Lemma 2 is the direct outcome of Schur’s lemma. □

B Proof of Theorem 1

If the assumptions of the theorem are satisfied, then by Lemma 2, Condition 1 is valid, i.e. (20) holds. Define

\[
\hat{M} = \begin{bmatrix} X(\omega)^{\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} M \begin{bmatrix} X(\omega)^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix}.
\]

(B.1)

Then (20) can be rewritten as

\[
\hat{M}^* \hat{M} - \begin{bmatrix} I & 0 \\ 0 & Y(\omega) \end{bmatrix} \preceq \begin{bmatrix} -\epsilon I & 0 \\ 0 & 0 \end{bmatrix}. 
\]

(B.2)

As a result

\[
\hat{M}^* \hat{M} \preceq \begin{bmatrix} I & 0 \\ 0 & Y(\omega) \end{bmatrix}.
\]

(B.3)

Define \(\bar{q}(j\omega) = X(\omega)^{\frac{1}{2}} q(j\omega)\) and \(\bar{p}(j\omega) = X(\omega)^{\frac{1}{2}} p(j\omega)\).

By pre and post multiplying both sides of (B.3) by

\[
\begin{bmatrix} \bar{q}(j\omega)^* \\
\bar{p}(j\omega)^* \end{bmatrix}
\]

and

\[
\begin{bmatrix} \bar{q}(j\omega) \\
\bar{p}(j\omega) \end{bmatrix},
\]

respectively, we have

\[
|z(j\omega)|^2 + |\bar{p}(j\omega)|^2 \leq |\bar{q}(j\omega)|^2 + w(j\omega)^* Y(\omega) w(j\omega). 
\]

(B.4)

For all frequencies \(\Delta\) commutes with \(X(\omega)^{-\frac{1}{2}}\), and hence \(\bar{q} = X^{\frac{1}{2}} q = X^{\frac{1}{2}} \Delta X^{-\frac{1}{2}} \Delta p = \Delta \bar{p}\). Considering the fact that \(\Delta \in B\Delta\), it now follows from (11) and (B.4) that

\[
|z(j\omega)|^2 = w(j\omega)^* (\Delta * M)(j\omega)^* (\Delta * M)(j\omega) w(j\omega) 
\]

\[
\leq w(j\omega)^* Y(\omega) w(j\omega), 
\]

(B.5)

which completes the proof. □

C Proof of Theorem 3

Let \(P_-, P_+, X, Q\) and \(\tilde{W}_o\) satisfy (34). Define

\[
\hat{Y} = (\tilde{N}^{-1} \tilde{M}_2)^* (\tilde{N}^{-1} \tilde{M}_2) = \tilde{M}_2^2 C_{11}^{-1} \tilde{M}_2 \succeq 0. 
\]

(C.1)

where \(\tilde{N}\) and \(\tilde{M}_2\) are defined in Lemma 4. From (C.1) \(C_{11} \succ 0\). If we set \(Y = M_{11}^2 C_{11}^{-1} \tilde{M}_2\), by Schur’s lemma it follows that

\[
\begin{bmatrix} -C_{11} & \tilde{M}_2 \\ \tilde{M}_2^* & -Y \end{bmatrix} \succeq 0. 
\]

(C.2)

By replacing \(X(\omega)\) with \(X(\omega)^{-1}\) in (B.1) and using Lemma 3, (C.2) is equivalent to (B.2). In other words

\[
M(j\omega)^* \begin{bmatrix} X(\omega)^{-1} & 0 \\ 0 & I \end{bmatrix} M(j\omega) - \begin{bmatrix} X(\omega)^{-1} & 0 \\ 0 & Y(\omega) \end{bmatrix} \preceq \begin{bmatrix} -\epsilon I & 0 \\ 0 & 0 \end{bmatrix}. 
\]

(C.3)

By (C.3) and the same arguments as in the proof of Theorem 1, \((\Delta * M)(j\omega)^* (\Delta * M)(j\omega) \preceq Y(\omega) \forall \omega, \forall \Delta \in B\Delta.\) As a result by using lemmas 1 and 4, (39) follows.

□

D Proof of Theorem 4

Consider the LMI in (42) with \(\bar{X}(\omega) = \tilde{X}_I\). This LMI can be rewritten as
\( X_i^{-\frac{1}{2}} \bar{M}^* \tilde{X}_i^\frac{1}{2} \tilde{X}_i^\frac{1}{2} \bar{M} X_i^{-\frac{1}{2}} - I < 0. \) \tag{D.1}

Let \( G(j\omega) = \tilde{X}_i^\frac{1}{2} \bar{M}(j(\omega + \omega_i)) \tilde{X}_i^{-\frac{1}{2}} \). It now follows that
\[
G = \begin{bmatrix}
A_G & B_G \\
C_G & D_G
\end{bmatrix}.
\]

In this theorem we are looking for the largest frequency interval, for which the LMI in (D.1) is valid. On the boundary of this interval \( I - G(j\omega)^* G(j\omega) \) becomes singular, i.e. \( \det(I - G(j\omega)^* G(j\omega)) = 0 \).

By (44) and (45), \( I - G(j\omega)^* G(j\omega) = \begin{bmatrix}
A_X & B_X \\
C_X & D_X
\end{bmatrix} \). Using Sylvester’s determinant lemma and some simple matrix manipulation we have

\[
\det(I - G(j\omega)^* G(j\omega)) = 0 \iff \\
\det(I + D_X^{-\frac{1}{2}} C_X(j\omega I - A_X)^{-1} B_X D_X^{-\frac{1}{2}}) = 0 \iff \\
\det(I + (j\omega I - A_X)^{-1} B_X D_X^{-1} C_X). \tag{D.2}
\]

By using the matrix determinant lemma and the definition of \( G \) it is also straightforward to establish equivalence between the following expressions

\[
\det(I + (j\omega I - A_X)^{-1} B_X D_X^{-1} C_X) \iff \\
\det(j\omega I - (A_X - B_X D_X^{-1} C_X)) = 0 \iff \det(\omega I + j\bar{G}) = 0, \tag{D.3}
\]

which completes the proof. \( \square \)
Robust finite-frequency $\mathcal{H}_2$ analysis of uncertain systems

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In many applications, design or analysis is performed over a finite-frequency range of interest. The importance of the $\mathcal{H}_2$/robust $\mathcal{H}_2$ norm highlights the necessity of computing this norm accordingly. This paper provides different methods for computing upper bounds on the robust finite-frequency $\mathcal{H}_2$ norm for systems with structured uncertainties. An application of the robust finite-frequency $\mathcal{H}_2$ norm for a comfort analysis problem of an aero-elastic model of an aircraft is also presented.