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# ON THE CONNECTEDNESS OF THE BRANCH LOCUS OF THE MODULI SPACE OF RIEMANN SURFACES OF LOW GENUS

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Abstract. Let $g$ be an integer $\geq 3$ and let $\mathcal{B}_g = \{X \in \mathcal{M}_g | Aut(X) \neq 1_d\}$, where $\mathcal{M}_g$ denotes the moduli space of compact Riemann surfaces of genus $g$. Using uniformization of Riemann surfaces by Fuchsian groups and the equisymmetric stratification of the branch locus of the moduli space, we prove that the subloci corresponding to Riemann surfaces with automorphism groups isomorphic to cyclic groups of order 2 and 3 belong to the same connected component. We also prove the connectedness of $\mathcal{B}_g$ for $g = 5, 6, 7$ and 8 with the exception of the isolated points given by Kulkarni.

## 1. Introduction

In this article we study the topology of moduli spaces of Riemann surfaces. More concretely we study the connectedness of the branch locus of moduli spaces of Riemann surfaces. The connectedness of subloci of moduli spaces of Riemann surfaces has been widely studied, among others, by [20], [7], [9], [10], [11], [12]. Other subloci of moduli spaces have been studied; see [8] and [16].

Let $g \geq 3$. Then the branch locus $\mathcal{B}_g$ of the moduli space $\mathcal{M}_g$ consists of the surfaces of genus $g$ admitting non-trivial automorphism groups. Two closed Riemann surfaces are called *equisymmetric* if their automorphism groups determine conjugate finite subgroups of the modular group. Harvey [17] alluded to the existence of the equisymmetric stratification of the moduli space. Broughton [2] defined the stratification of $\mathcal{M}_g$ by closed irreducible subvarieties $\overline{\mathcal{M}}_g^{G,\theta}$ with interior $\mathcal{M}_g^{G,\theta}$, if non-empty, as a connected, Zariski dense subvariety in $\overline{\mathcal{M}}_g^{G,\theta}$. Each *equisymmetric stratum* $\mathcal{M}_g^{G,\theta}$ consists of surfaces with full automorphism group conjugated to the finite group $G$ in the modular group, and $\overline{\mathcal{M}}_g^{G,\theta}$ is formed by surfaces such that the automorphism group contains a subgroup of the modular group in the conjugacy class defined by $G$.

In section three we consider the equisymmetric strata corresponding to automorphism groups of order 2 and 3 for the moduli space of Riemann surfaces of an arbitrary genus $g \geq 3$. We show that all the strata $\mathcal{M}_g^{G,\theta}$, where $G$ is a cyclic group of order 2 or 3, belong to the same connected component.

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In section four we consider the moduli spaces of Riemann surfaces of genus $g = 5$, 6, 7, 8 and 9. Each equisymmetric stratum corresponds to a conjugacy class of finite subgroups of the modular group represented as the full group of automorphisms of some Riemann surface of genus $g$. To find the full groups of automorphisms, we use the list of maximal signatures found by Singerman [23]. We shall show, using the equisymmetric stratification, that the branch locus $\mathcal{B}_g$ is connected with exception to the isolated points corresponding to surfaces with full automorphism group of prime order $2g+1$ as found by Kulkarni [20]. The results in section four have been announced in [1].

## 2. Riemann surfaces and Fuchsian groups

A Riemann surface can be realized as the quotient space $\mathcal{D}/\Gamma$ of the hyperbolic plane $\mathcal{D}$, where $\Gamma \in PSL(2,\mathbb{R})$ is a Fuchsian group. If the Fuchsian group $\Gamma$ is isomorphic to an abstract group with presentation

$$\left\langle a_1, b_1, \ldots, a_g, b_g, x_1 \ldots x_k | x_1^{m_1} = \ldots = x_k^{m_k} = \prod x_i \prod [a_i, b_i] = 1 \right\rangle, \tag{1}$$

we say that $\Gamma$ has *signature*

$$s(\Gamma) = (g; m_1, \ldots, m_k). \tag{2}$$

If $s(\Gamma) = (g; -)$, i.e. it has no elliptic generators, we call $\Gamma$ a *surface group*. The relationship between the signatures of a Fuchsian group and subgroups is given in the following theorem of Singerman:

**Theorem 1** ([22]). *Let $\Gamma$ be a Fuchsian group with signature* (2) *and canonical presentation* (1). *Then $\Gamma$ contains a subgroup $\Gamma'$ of index $N$ with signature*

$$s(\Gamma') = (h; m'_{11}, m'_{12}, ..., m'_{1s_1}, ..., m'_{r1}, ..., m'_{rs_r})$$

*if and only if there exists a transitive permutation representation $\theta : \Gamma \to \Sigma_N$ satisfying the following conditions:*

1. *The permutation $\theta(x_i)$ has precisely $s_i$ cycles of lengths less than $m_i$, the lengths of these cycles being $m_i/m'_{i1}, ..., m_i/m'_{is_i}$.*
2. *The Riemann-Hurwitz formula*

$$\mu(\Gamma')/\mu(\Gamma) = N,$$

*where $\mu(\Gamma)$, $\mu(\Gamma')$ are the hyperbolic areas of the surfaces $\mathcal{D}/\Gamma$, $\mathcal{D}/\Gamma'$.*

Given a Riemann surface $X = \mathcal{D}/\Gamma$, with $\Gamma$ a surface Fuchsian group, a finite group $G$ is a group of automorphisms of $X$ if and only if there exists a Fuchsian group $\Delta$ and an epimorphism $\theta : \Delta \to G$ with $ker(\theta) = \Gamma$.

A Fuchsian group $\Gamma$ that is not contained in any other Fuchsian group with finite index is called a *finitely maximal* Fuchsian group (see [23]). To determine if a given finite group is the full automorphism group of some Riemann surface, we need to consider all pairs of signatures $s(\Gamma)$ and $s(\Gamma')$ for Fuchsian groups $\Gamma' \leq \Gamma$. All such pairs were found by Singerman [23]. See also [15].

Let $\Gamma$ be a group with signature (2). The Teichmüller space $T(\Gamma)$ is homeomorphic to a ball of complex dimension $d(\Gamma) = 3g - 3 + r$ (see [21]). Let $M(\Gamma)$ denote the group of outer automorphisms of $\Gamma$. $M(\Gamma)$, which is also called the modular group of $\Gamma$, acts on $T(\Gamma)$ as $[r] \to [r \circ \alpha]$ where $\alpha \in M(\Gamma)$. The moduli space of $\Gamma$ is the quotient space $\mathcal{M}(\Gamma) = T(\Gamma)/M(\Gamma)$. If $\Gamma$ is a surface group of genus $g$, we will denote $\mathcal{M}(\Gamma)$ by $\mathcal{M}_g$. We will study the *branch locus* $\mathcal{B}_g$ of the covering $T_g \to \mathcal{M}_g$;

see [17] and [21]. As an application of Nielsen Realization Theorem one can identify the branch locus of the action of $M(\Gamma)$ as the set $\mathcal{B}_g = \{X \in \mathcal{M}_g | Aut(X) \neq 1_d\}$, for $g \geq 3$. See also [2], [3].

An *(effective and orientable) action* of a finite group $G$ on a Riemann surface $X$ is a representation $\epsilon : G \to \mathrm{Aut}(X)$. Two actions $\epsilon, \epsilon'$ of $G$ on a Riemann surface $X$ are *(weakly) topologically equivalent* if there is an $w \in \mathrm{Aut}(G)$ and an $h \in Hom^+(X)$ such that $\epsilon'(g) = h\epsilon w(g)h^{-1}$; see [2] and [3]. The equisymmetric strata are in correspondence with topological equivalence classes of orientation preserving actions of a finite group $G$ on a surface $X$. See [2]; see also [10].

Let $\Gamma$ be a surface Fuchsian group. Each action of $G$ on the surface $X = \mathcal{D}/\Gamma$ is determined by an epimorphism $\theta : \Delta \to G$ such that $ker(\theta) = \Gamma$. Two epimorphisms $\theta_1$, $\theta_2 : \Delta \to G$ determine two *topologically equivalent actions* of $G$ if and only if there exist automorphisms $\phi \in Aut(\Delta)$ and $w \in Aut(G)$ such that $\theta_2 = w \circ \theta_1 \circ \phi$; see [3]. We classify actions of a finite group using methods found in [3]. Kimura [18] found all actions on surfaces of genus 4, and Kimura and Kuribayashi [19] found all actions on surfaces of genus 5.

Following Broughton [2], let $\mathcal{M}_g^{G,\theta}$ denote the stratum of surfaces with full automorphism group in the conjugacy class of the finite group $G$ in the modular group and let $\overline{\mathcal{M}}_g^{G,\theta}$ denote the set of surfaces such that the automorphisms group contains a subgroup in the class defined by $G$. We have the following theorem by Broughton:

**Theorem 2** ([2])**.** *Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus $g$, and let $G$ be a finite subgroup of the corresponding modular group $M_g$. Then:*

(1) $\overline{\mathcal{M}}_g^{G,\theta}$ *is a closed, irreducible algebraic subvariety of $\mathcal{M}_g$.*

(2) $\mathcal{M}_g^{G,\theta}$*, if it is non-empty, is a smooth, connected, locally closed algebraic subvariety of $\mathcal{M}_g$, Zariski dense in $\overline{\mathcal{M}}_g^{G,\theta}$.*

*Remark* 3. The condition of $\Gamma$ to be a surface Fuchsian group imposes that the order of the image under $\theta$ of an elliptic generator $x_i$ of $\Delta$ is the same as the order of $x_i$ and $\theta(x_1)\theta(x_2)\ldots\theta(x_{r-1}) = \theta(x_r)^{-1}$.

Riemann surfaces and related surfaces with cyclic and abelian groups of automorphisms have been studied recently, e.g. [4], [5], [6] and [13]. For us cyclic groups of automorphisms are of particular interest due to the following lemma.

**Lemma 4** ([10])**.** *The branch locus consists of the union $\mathcal{B}_g = \bigcup \overline{\mathcal{M}}_g^{C_p,\theta}$, where the pair $C_p, \theta$ runs over all classes of actions of cyclic groups $C_p$ of prime order $p$.*

*Proof.* Every group $G$ contains a subgroup of prime order $p$ where $p$ is a divisor of $|G|$. This subgroup is isomorphic to $C_p$; thus $\mathcal{M}_g^{G,\alpha} \subset \overline{\mathcal{M}}_g^{C_p,\theta}$ for some actions $\alpha$ of $G$ and $\theta$ of $C_p$, where $\alpha|_{C_p} = \theta$. $\square$

Kulkarni [20] determined conditions for the existence of isolated points stated in the following theorem:

**Theorem 5** ([20])**.** *The number of isolated points in $\mathcal{B}_g$ is 1 if $g = 2$, $\lfloor (g-2)/3 \rfloor$ if $q = 2g + 1$ is a prime $> 7$ and 0 otherwise.*

## 3. Strata corresponding to cyclic groups of order 2 or 3

We will show that all the strata given by actions of cyclic groups of order 2 or 3 belong to the same connected component by finding the appropriate surface epimorphisms $\theta : \Delta \to G$ where $G = C_2 \times C_2$, $C_6$ or $D_3$.

The possible actions of $C_2$ on surfaces of genus $g$ are determined by the signatures $(i; 2, \overset{2g+2-4i}{\ldots}, 2)$, $i = 0, \ldots, \lfloor \frac{g+1}{2} \rfloor$. Each signature gives one action, yielding the stratum $\mathcal{M}_g^{C_2,i}$.

**Theorem 6.** *Let $g \geq 3$. Then the strata $\mathcal{M}_g^{C_2,i}$, $i = 0, \ldots, \lfloor \frac{g+1}{2} \rfloor$, belong to the same connected component. In particular, $\overline{\mathcal{M}}_g^{C_2,i} \cap \overline{\mathcal{M}}_g^{C_2,\lfloor \frac{g+1}{2} \rfloor} \neq \emptyset$.*

*Proof.* Consider groups of automorphisms isomorphic to $C_2 \times C_2 = \langle a, b | a^2 = b^2 = (ab)^2 = 1 \rangle$. By the Riemann-Hurwitz formula we find that a surface kernel epimorphism $\theta : \Delta \to C_2 \times C_2$ exists if $s(\Delta) = (\gamma; 2, \overset{g-4\gamma+3}{\ldots}, 2)$. Now let $\gamma = 0$ so that $s(\Delta) = (0; 2, \overset{g+3}{\ldots}, 2)$ We have two cases depending as $g$ is odd or even.

**(1)** $g$ odd. Define $\theta_j : \Delta \to C_2 \times C_2$, $0 \leq j \leq \lfloor \frac{g-1}{4} \rfloor$, by $\theta_j(x_i) = a$, $1 \leq i \leq g+1-2j$, $\theta_j(x_i) = b$, $g+2-2j \leq i \leq g+3$. By applying Theorem 1 we get subgroups with signatures $s(\theta_0^{-1} \langle a \rangle) = (j; 2, \overset{2g+2-4j}{\ldots}, 2)$, $s(\theta_0^{-1} \langle b \rangle) = (\frac{g-1}{2} - j; 2, \overset{4+4j}{\ldots}, 2)$ and $s(\theta_0^{-1} \langle ab \rangle) = (\frac{g+1}{2}; -)$. Thus

$$\overline{\mathcal{M}}_g^{C_2,j} \cap \overline{\mathcal{M}}_g^{C_2,\frac{g-1}{2}-j} \cap \overline{\mathcal{M}}_g^{C_2,\frac{g+1}{2}} \neq \emptyset.$$

Therefore, $\overline{\mathcal{M}}_g^{C_2,j} \cap \overline{\mathcal{M}}_g^{C_2,\frac{g+1}{2}} \neq \emptyset$, as desired.

**(2)** $g$ even. Define epimorphisms $\theta_j : \Delta \to C_2 \times C_2$ by $\theta_j(x_i) = a$, $1 \leq i \leq g+1-2j$, $\theta_j(x_i) = ab$, $g+2-2j \leq i \leq g+2$ and $\theta_j(x_{g+3}) = b$. By Theorem 1 the epimorphisms induce subgroups with signature $(j; 2, \overset{2g+2-4j}{\ldots}, 2)$, $(\frac{g}{2} - j; 2, \overset{2+4j}{\ldots}, 2)$ and $(\frac{g}{2}; 2, 2)$. Thus

$$\overline{\mathcal{M}}_g^{C_2,j} \cap \overline{\mathcal{M}}_g^{C_2,\frac{g}{2}-j} \cap \overline{\mathcal{M}}_g^{C_2,\frac{g}{2}} \neq \emptyset.$$

Therefore, $\overline{\mathcal{M}}_g^{C_2,j} \cap \overline{\mathcal{M}}_g^{C_2,\frac{g}{2}} \neq \emptyset$, and thus $\overline{\mathcal{M}}_g^{C_2,j} \cap \overline{\mathcal{M}}_g^{C_2,\lfloor \frac{g+1}{2} \rfloor} \neq \emptyset$. □

The actions of $C_3 = \langle a | a^3 = 1 \rangle$ on surfaces of genus $g$ are induced by signatures $(\gamma; 3, \overset{g+2-3\gamma}{\ldots}, 3)$. An epimorphism $\theta : \Delta(\gamma; 3, \overset{g+2-3\gamma}{\ldots}, 3) \to C_3$ is equivalent to one of the following epimorphisms:

$$\theta_{j,k} : \begin{cases} x_i \to a, & 1 \leq i \leq g+2-3\gamma-3j-k \\ x_i \to a^2, & g+3-3\gamma-3j-k \leq i \leq g+2-3\gamma \end{cases}, \; g+2 \equiv -k \; mod \; 3.$$

The actions $\theta_{j,k} : \Delta(\gamma; 3, \overset{g+2-3\gamma}{\ldots}, 3) \to C_3$ induce the strata $\mathcal{M}_g^{C_3,\gamma j}$.

**Theorem 7.** *Let $g \geq 4$. Then for each stratum $\mathcal{M}_g^{C_3,\gamma j}$ there exists a stratum $\mathcal{M}_g^{C_2,i}$ such that $\overline{\mathcal{M}}_g^{C_3,\gamma j} \cap \overline{\mathcal{M}}_g^{C_2,i} \neq \emptyset$.*

*Proof.* We will look at epimorphisms $\phi : \Delta' \to C_6 = \langle b | b^6 = 1 \rangle$ for groups $\Delta'$ with signature defined below. We begin with the epimorphisms induced by $(0; 3, \overset{g+2}{\ldots}, 3)$.

**(1)** $g$ odd. Observe that $g+2 \equiv 0,1,2 \ mod\ 3$ implies $g+1 \equiv 2,0,4 \ mod\ 6$ respectively. Let $\Delta'$ have signature $(0;2,3,\overset{\frac{g+1}{2}}{\ldots},3,6)$ and $\phi_{j,k}$ be defined as

$$\phi_{2n,0}:\begin{cases} x_1 \to b^3 \\ x_i \to b^2, \quad 2 \le i \le \dfrac{g+1}{2}+1-3n \\ x_i \to b^4, \quad \dfrac{g+1}{2}+2-3n \le i \le \dfrac{g+1}{2}+1 \\ x_{\frac{g+1}{2}+2} \to b \end{cases}$$

$$\phi_{2n+1,0}:\begin{cases} x_1 \to b^3 \\ x_i \to b^2, \quad 2 \le i \le \dfrac{g+1}{2}-3n \\ x_i \to b^4, \quad \dfrac{g+1}{2}+1-3n \le i \le \dfrac{g+1}{2}+1 \\ x_{\frac{g+1}{2}+2} \to b^5 \end{cases}$$

$$\phi_{2n,1}:\begin{cases} x_1 \to b^3 \\ x_i \to b^2, \quad 2 \le i \le \dfrac{g+1}{2}+1-3n \\ x_i \to b^4, \quad \dfrac{g+1}{2}+2-3n \le i \le \dfrac{g+1}{2}+1 \\ x_{\frac{g+1}{2}+2} \to b^5 \end{cases}$$

$$\phi_{2n+1,1}:\begin{cases} x_1 \to b^3 \\ x_i \to b^2, \quad 2 \le i \le \dfrac{g-1}{2}-3n \\ x_i \to b^4, \quad \dfrac{g+1}{2}-3n \le i \le \dfrac{g+1}{2}+1 \\ x_{\frac{g+1}{2}+2} \to b \end{cases}$$

$$\phi_{2n,2}:\begin{cases} x_1 \to b^3 \\ x_i \to b^2, \quad 2 \le i \le \dfrac{g+1}{2}-3n \\ x_i \to b^4, \quad \dfrac{g+1}{2}+1-3n \le i \le \dfrac{g+1}{2}+1 \\ x_{\frac{g+1}{2}+2} \to b \end{cases}$$

$$\phi_{2n+1,2}:\begin{cases} x_1 \to b^3 \\ x_i \to b^2, \quad 2 \le i \le \dfrac{g-1}{2}-3n \\ x_i \to b^4, \quad \dfrac{g+1}{2}-3n \le i \le \dfrac{g+1}{2}+1 \\ x_{\frac{g+1}{2}+2} \to b^5 \end{cases}$$

It is easy to see that $\theta_{j,k}$ extends to $\phi_{j,k}$ and that by Theorem 1

$$s(\phi_{j,k}^{-1}\langle b^3\rangle) = (\frac{g-1}{2};2,2,2,2).$$

**(2)** $g$ even. Again $g+2 \equiv 0,1,2 \ mod\ 3$ implies $g+2 \equiv 0,4,2 \ mod\ 6$ respectively. Let $\Delta'$ have signature $(0;3,\overset{\frac{g}{2}}{\ldots},3,6,6)$ and $\phi_{j,k}$ be defined as

$$\phi_{2n,0}: \begin{cases} x_i \to b^2, & 1 \leq i \leq \frac{g}{2} - 3n \\ x_i \to b^4, & \frac{g}{2} + 1 - 3n \leq i \leq \frac{g}{2} \\ x_{\frac{g}{2}+1} \to b \\ x_{\frac{g}{2}+2} \to b \end{cases}$$

$$\phi_{2n+1,0}: \begin{cases} x_i \to b^2, & 1 \leq i \leq \frac{g}{2} - 1 - 3n \\ x_i \to b^4, & \frac{g}{2} - 3n \leq i \leq \frac{g}{2} \\ x_{\frac{g}{2}+1} \to b \\ x_{\frac{g}{2}+2} \to b^5 \end{cases}$$

$$\phi_{2n,1}: \begin{cases} x_i \to b^2, & 1 \leq i \leq \frac{g}{2} - 3n \\ x_i \to b^4, & \frac{g}{2} + 1 - 3n \leq i \leq \frac{g}{2} \\ x_{\frac{g}{2}+1} \to b \\ x_{\frac{g}{2}+2} \to b^5 \end{cases}$$

$$\phi_{2n+1,1}: \begin{cases} x_i \to b^2, & 1 \leq i \leq \frac{g}{2} - 1 - 3n \\ x_i \to b^4, & \frac{g}{2} - 3n \leq i \leq \frac{g}{2} \\ x_{\frac{g}{2}+1} \to b^5 \\ x_{\frac{g}{2}+2} \to b^5 \end{cases}$$

$$\phi_{2n,2}: \begin{cases} x_i \to b^2, & 1 \leq i \leq \frac{g}{2} - 1 - 3n \\ x_i \to b^4, & \frac{g}{2} - 3n \leq i \leq \frac{g}{2} \\ x_{\frac{g}{2}+1} \to b \\ x_{\frac{g}{2}+2} \to b \end{cases}$$

$$\phi_{2n+1,2}: \begin{cases} x_i \to b^2, & 1 \leq i \leq \frac{g}{2} - 2 - 3n \\ x_i \to b^4, & \frac{g}{2} - 1 - 3n \leq i \leq \frac{g}{2} \\ x_{\frac{g}{2}+1} \to b \\ x_{\frac{g}{2}+2} \to b^5 \end{cases}$$

It is easy to see that $\theta_{j,k}$ extends to $\phi_{j,k}$. $s(\phi_{j,k}^{-1}\langle b^3 \rangle) = (\frac{g}{2};2,2)$ and we have $\overline{\mathcal{M}}_g^{C_3,0j} \cap \overline{\mathcal{M}}_g^{C_2,\lfloor \frac{g}{2} \rfloor} \neq \emptyset$.

**(3)** Now assume that $\Delta$ has signature $(\gamma; 3, \overset{g+2-3\gamma}{\ldots}, 3)$, $0 < \gamma < \frac{g+2}{3}$. Also note that $g+2-3\gamma \equiv g+2 \ mod\ 3$. Thus, if $g+2-3\gamma$ is odd, we can consider $\Delta'$ with signature $(0; 2, \overset{2\gamma+1}{\ldots}, 2, 3, \overset{\frac{g+1-3\gamma}{2}}{\ldots}, 3, 6)$. Since $3(1+2\gamma) \equiv 3 \ mod\ 6$ and $\frac{g+1-3\gamma}{2} \equiv 0,\ 2 \ or\ 4 \ mod\ 6$, $\theta : \Delta \to C_3$ extends to an epimorphism $\phi : \Delta' \to C_6$ as above. Similarly, if $g+2-3\gamma$ is even, we consider the signature $(0; 2, \overset{2\gamma}{\ldots}, 2, 3, \overset{\frac{g-3\gamma}{2}}{\ldots}, 3, 6, 6)$ and an epimorphism $\phi : \Delta' \to C_6$ as above. Thus we see that $\overline{\mathcal{M}}_g^{C_3,\gamma j} \cap \overline{\mathcal{M}}_g^{C_2,\lfloor\frac{g-3\gamma}{2}\rfloor} \neq \emptyset$.

**(4)** Finally we need to consider groups $\Delta$ with signature $(\frac{g+2}{3}; -)$. In this case there exist a group $\Delta'$ with signature $(0; 2, \overset{2(\frac{g+2}{3}+1)}{\ldots}, 2)$ and an epimorphism $\phi : \Delta' \to D_3 = \left\langle s, t | s^2 = t^2 = (st)^3 = 1 \right\rangle$ defined by

$$\phi : \begin{cases} x_i \to s, & 1 \leq i \leq \dfrac{2(g+2)}{3}, \\ x_i \to t, & \dfrac{2(g+2)}{3} + 1 \leq i \leq \dfrac{2(g+2)}{3} + 2 \end{cases}$$

and $s(\phi^{-1}\left\langle s \right\rangle) = (\frac{g-1}{3}; 2, \overset{\frac{2(g+2)}{3}+2}{\ldots}, 2)$. Thus $\overline{\mathcal{M}}_g^{C_3,\frac{g+2}{3}} \cap \overline{\mathcal{M}}_g^{C_2,\frac{g-1}{3}} \neq \emptyset$. $\square$

## 4. On the connectedness of the branch locus of the moduli space of Riemann surfaces of low genus

It is well known that the branch loci of $\mathcal{M}_2$, with the exception of one isolated point given by a pentagonal curve, and $\mathcal{M}_3$ are connected; see also [1]. Costa and Izquierdo [10] showed that $\mathcal{B}_4$ is connected. Kulkarni [20] found for which genera the branch locus $\mathcal{B}_g$ contains isolated points, and Costa and Izquierdo [14] listed the genera of which $\mathcal{B}_g$ contains isolated strata of dimension one.

Here we shall show that the branch loci of $\mathcal{M}_5$ and $\mathcal{M}_6$ are connected with the exception of one isolated point in each, the branch locus of $\mathcal{M}_7$ is connected, and the branch locus of $\mathcal{M}_8$ is connected with the exception of 2 isolated points. Theorems 6 and 7 prove the connectedness of the strata of surfaces with automorphisms of order 2 and 3; thus we will only regard automorphisms of higher order. By Lemma 4 we know that the branch locus is the union of equisymmetric strata determined by actions of cyclic groups of prime order. For each genus we will consider prime orders satisfying the Riemann-Hurwitz formula (Theorem 1).

**Proposition 8.** *The branch locus $\mathcal{B}_5$ of $\mathcal{M}_5$ is the union*

$$\overline{\mathcal{M}}_5^{C_2,0} \cup \overline{\mathcal{M}}_5^{C_2,1} \cup \overline{\mathcal{M}}_5^{C_2,2} \cup \overline{\mathcal{M}}_5^{C_2,3} \cup \overline{\mathcal{M}}_5^{C_3,0} \cup \overline{\mathcal{M}}_5^{C_3,1} \cup \overline{\mathcal{M}}_5^{C_{11},01}.$$

*Proof.* **(1)** $\mathcal{M}_5^{C_2,0}$, $\mathcal{M}_5^{C_2,1}$, $\mathcal{M}_5^{C_2,2}$ and $\mathcal{M}_5^{C_2,3}$ correspond to epimorphisms $\theta : \Delta \to C_2$ with signatures $s(\Delta_0) = (0; 2, \overset{12}{\ldots}, 2)$, $s(\Delta_1) = (1; 2, \overset{8}{\ldots}, 2)$, $s(\Delta_2) = (2; 2, 2, 2, 2)$ and $s(\Delta_3) = (3; -)$ respectively.

**(2)** The strata $\mathcal{M}_5^{C_3,0}$ and $\mathcal{M}_5^{C_3,1}$ correspond to epimorphisms $\theta : \Delta \to C_3$ where $s(\Delta_0) = (0; 3, \overset{7}{\ldots}, 3)$ and $s(\Delta_1) = (1; 3, 3, 3, 3)$ respectively. Note that by the construction in the proof of Theorem 7 there exists only one class of epimorphisms of each type.

**(3)** $\overline{\mathcal{M}}_5^{C_5,1}$ is induced by non-maximal epimorphisms $\theta : \Delta \to C_5$, $s(\Delta) = (1;5,5)$. They extend to surface kernel epimorphism $\phi : \Delta' \to D_5 = \langle a, s | a^5 = s^2 = (sa)^2 = 1\rangle$, $s(\Delta') = (0;2,2,2,2,5)$, defined by $\phi(x_i) = s,\ i = 1,2,3$, and $\phi(x_4) = sa$. We see that $s(\phi^{-1}\langle a\rangle) = (1;5,5)$ and $s(\phi^{-1}\langle s\rangle) = (2;2,2,2,2)$. Thus $\overline{\mathcal{M}}_5^{C_5,1} \equiv \overline{\mathcal{M}}_5^{D_5,\theta} \subset \overline{\mathcal{M}}_5^{C_2,2}$.

**(4)** Signature $(0;11,11,11)$. There are two classes of actions of $C_{11}$ with representatives $\theta_1 : \Delta \to C_{11}$, defined by $\theta_1(x_1) = a$, $\theta_1(x_2) = a^2$ and $\theta_1(x_3) = a^{-3}$, and $\theta_2 : \Delta \to C_{11}$, defined by $\theta_2(x_1) = a$, $\theta_2(x_2) = a$ and $\theta_2(x_3) = a^{-2}$. Now $\theta_2$ extends to $\phi : \Delta(0;2,11,22) \to C_{22}$, defined by $\phi(x_1) = b^{11}$ and $\phi(x_2) = b^{10}$. By Theorem 1 $\phi^{-1}\left\langle b^2\right\rangle$ is a group with signature $(0;11,11,11)$, and the images of the elliptic generators by $\phi$ (with the isomorphism $b^2 \to a$) are $a$, $a$ and $a^{-2}$. $\overline{\mathcal{M}}_5^{C_{11},02} \equiv \overline{\mathcal{M}}_5^{C_{22}}$. By Theorem 1 $s(\phi_2^{-1}\left\langle b^{11}\right\rangle) = (0;2,\overset{12}{\ldots},2)$, thus $\overline{\mathcal{M}}_5^{C_{22}} \subset \overline{\mathcal{M}}_5^{C_2,0}$. The epimorphism $\theta_1$ yields a maximal action of $C_{11}$ in $\mathcal{M}_5$ producing an isolated point $\overline{\mathcal{M}}_5^{C_{11},01}$. □

**Theorem 9.** *The branch locus of $\mathcal{M}_5$ is connected with the exception of one isolated point.*

*Proof.* It follows from Theorem 6 and Theorem 7, together with the results in the proof of Proposition 8. □

**Theorem 10.** *The branch locus of $\mathcal{M}_6$ is connected with the exception of one isolated point.*

*Proof.* **(1)** By Theorem 6 and Theorem 7 the strata corresponding to the actions of $C_2$ and $C_3$ belong to the same connected component of $\mathcal{B}_6$.

**(2)** $s(\Delta) = (0;5,5,5,5,5)$. There are three classes of epimorphisms $\theta(\Delta) \to C_5$ defined by $\theta_1(x_i) = a,\ i = 1,\ldots,5$, $\theta_2(x_i) = a,\ i = 1,\ldots,3$, $\theta_2(x_4) = a^3$ and $\theta_3(x_i) = a,\ i = 1,2$, $\theta_3(x_i) = a^2,\ i = 3,4$. $\theta_1$ and $\theta_3$ extends to epimorphisms $\phi_1, \phi_3 : \Delta(0;2,5,5,10) \to C_{10}$ defined by $\phi_1(x_i) = b^2,\ i = 2,3$, and $\phi_3(x_2) = b^2$, $\phi_3(x_3) = b^4$. $\theta_2$ extends to an epimorphism $\phi_2 : \Delta(0;5,15,15) \to C_{15}$ defined by $\phi_2(x_1) = b^3$, $\phi_2(x_2) = b^4$ and $\phi_2(x_3) = b^8$. By Theorem 1 $\phi_1^{-1}\left\langle b^5\right\rangle = (2;2,\overset{6}{\ldots},2)$, $\phi_2^{-1}\left\langle b^5\right\rangle = (2;3,3)$ and $\phi_3^{-1}\left\langle b^5\right\rangle = (2;2,\overset{6}{\ldots},2)$, i.e. $\overline{\mathcal{M}}_6^{C_5,0i} \cap \overline{\mathcal{M}}_6^{C_2,2} \neq \emptyset$ for $i = 1,3$ and $\overline{\mathcal{M}}_6^{C_5,02} \cap \overline{\mathcal{M}}_6^{C_3,2} \neq \emptyset$.

**(3)** $s(\Delta) = (0;7,7,7,7)$. There are the following four classes of epimorphisms $\theta : \Delta \to C_7$ defined by $\theta_1(x_i) = a,\ i = 1,2,3$, $\theta_2(x_i) = a,\ i = 1,2$, $\theta_2(x_3) = a^{-1}$, $\theta_3(x_i) = a,\ i = 1,2$, $\theta_3(x_3) = a^2$ and $\theta_4(x_1) = a$, $\theta_4(x_2) = a^2$, $\theta_4(x_3) = a^5$. $\theta_1$ is induced by $\phi_1 : \Delta(0;3,7,21) \to C_{21}$, defined by $\phi_1(x_1) = b^7$, $\phi_1(x_2) = b^3$. We find by Theorem 1 that $\phi_1^{-1}\left\langle b^7\right\rangle = (0;3,\overset{8}{\ldots},3)$ and $\overline{\mathcal{M}}_6^{C_7,01} \cap \overline{\mathcal{M}}_6^{C_3,01} \neq \emptyset$. $\theta_2$ extends to an epimorphism $\phi_2 : \Delta(0;2,2,7,7) \to C_{14}$ defined by $\phi_2(x_i) = b^7,\ i = 1,2$, and $\phi_2(x_3) = b^2$. $\theta_4$ extends to an epimorphism $\phi_4 : \Delta(0;2,2,7,7) \to D_7 = \left\langle a, s | a^7 = s^2 = (sa)^2 = 1\right\rangle$ defined by $\phi_4(x_1) = s$, $\phi_4(x_2) = sa$ and $\phi_4(x_3) = a$. Finally assume $s(\Delta) = (0;7,14,14)$ and let the epimorphism $\phi_3 : \Delta \to C_{14}$ be defined by $\phi_3(x_1) = b^2$ and $\phi_3(x_2) = b^3$. Then $\phi_3$ induces $\theta_3$. By Theorem 1 it follows that $\phi_2^{-1}\left\langle b^7\right\rangle = (0;2,\overset{14}{\ldots},2)$, $\phi_3^{-1}\left\langle b^7\right\rangle = (3;2,2)$ and $\phi_4^{-1}\langle s\rangle = (3;2,2)$. $\overline{\mathcal{M}}_6^{C_7,02} \cap \overline{\mathcal{M}}_6^{C_2,0} \neq \emptyset$, $\overline{\mathcal{M}}_6^{C_7,03} \cap \overline{\mathcal{M}}_6^{C_2,3} \neq \emptyset$ and $\overline{\mathcal{M}}_6^{C_7,04} \cap \overline{\mathcal{M}}_6^{C_2,3} \neq \emptyset$.

**(4)** $s(\Delta) = (0; 13, 13, 13)$. We have three possible epimorphisms $\theta : \Delta \to C_{13}$ which are defined by $\theta_1(x_i) = a$, $i = 1, 2$, $\theta_2(x_1) = a$, $\theta_2(x_2) = a^2$ and $\theta_3(x_1) = a$, $\theta_2(x_2) = a^3$. $\theta_1$ extends to $\phi_1 : \Delta(0; 2, 13, 26) \to C_{26}$, defined by $\phi_1(x_1) = b^{13}$, $\phi_1(x_2) = b_2$ and $\phi_1^{-1}\langle b^{13}\rangle = (0; 2, \overset{14}{\ldots}, 2)$. $\theta_3$ extends to an epimorphism $\phi_3$ : $\Delta(0; 3, 3, 13) \to C_{13} \rtimes C_3 = C_{13} \rtimes C_3 = \langle a, b | a^{13} = b^3 = bab^2a^{10} = 1\rangle$, defined by $\phi_3(x_1) = b$, $\phi_1(x_2) = b^2a^{12}$. $\phi_3^{-1}\langle b\rangle = (2; 3, 3)$. By Theorem 5 we know that $\theta_2$ yields a maximal action of $C_{13}$ in $\mathcal{M}_6$ producing an isolated point of $\mathcal{B}_6$ (see [20]). □

**Theorem 11.** *The branch locus of $\mathcal{M}_7$ is connected.*

*Proof.* **(1)** By Theorem 6 and Theorem 7 the strata corresponding to the actions of $C_2$ and $C_3$ belong to the same connected component of $\mathcal{B}_7$.

**(2)** $s(\Delta) = (1; 5, 5, 5)$. There is only one class of epimorphisms $\theta : \Delta \to C_5$, and it is defined by $\theta(x_i) = a$, $i = 1, 2$, $\theta(\alpha) = 1$. This class extends to $\phi$ : $\Delta'(0; 2, 10, 10, 10) \to C_{10}$, defined by $\phi(x_1) = b^5$, $\phi(x_i) = b$, $i = 2, 3$. By Theorem 1 $s(\phi^{-1}\langle b^5\rangle) = (2; 2, \overset{8}{\ldots}, 2)$. Therefore, $\mathcal{M}_7^{C_5,1} \cap \mathcal{M}_7^{C_2,2} \neq \emptyset$.

**(3)** $s(\Delta) = (1; 7, 7)$. Let the surface kernel epimorphism $\phi : \Delta(0; 2, 2, 2, 2, 7) \to D_7 = \langle a, s | a^7 = s^2 = (sa)^2 = 1, \rangle$ be defined by $\theta(x_i) = s$, $i = 1, 2, 3$, and $\theta(x_4) = sa$. We see that $s(\theta^{-1}\langle a\rangle) = (1; 7, 7)$ and $s(\theta^{-1}\langle s\rangle) = (3; 2, 2, 2, 2)$. Therefore, $\overline{\mathcal{M}}_7^{C_7,1} \equiv \overline{\mathcal{M}}_7^{D_7,\theta} \subset \overline{\mathcal{M}}_7^{C_2,3}$. □

**Theorem 12.** *The branch locus of $\mathcal{M}_8$ is connected with the exception of two isolated points.*

*Proof.* **(1)** By Theorem 6 and Theorem 7 the strata corresponding to the actions of $C_2$ and $C_3$ belong to the same connected component of $\mathcal{B}_8$.

**(2)** $s(\Delta) = (0; 5, 5, 5, 5, 5, 5)$. There exist five classes of epimorphisms $\theta : \Delta \to C_5$ defined by $\theta_1(x_i) = a$, $i = 1, 2, 3, 4$, $\theta_1(x_5) = a^3$, $\theta_2(x_i) = a$, $i = 1, 2, 3, 4$, $\theta_2(x_5) = a^2$, $\theta_3(x_i) = a$, $i = 1, 2, 3$, $\theta_3(x_i) = a^4$, $i = 4, 5, 6$, $\theta_4(x_i) = a$, $i = 1, 2, 3$, $\theta_4(x_i) = a^2$, $i = 4, 5$, and $\theta_5(x_i) = a$, $i = 1, 2$, $\theta_5(x_i) = a^4$, $i = 3, 4$, $\theta_5(x_5) = a^2$. Each $\theta_i$ extends to an epimorphism $\phi_i : \Delta'(0; 5, 5, 10, 10) \to C_{10}$ defined by $\phi_1(x_i) = b^2$, $i = 1, 2$, $\phi_1(x_i) = b^3$, $i = 3, 4$, $\phi_2(x_i) = b^2$, $i = 1, 2$, $\phi_2(x_3) = b^7$, $\phi_3(x_1) = b^2$, $\phi_3(x_2) = b^8$, $\phi_3(x_3) = b$, $\phi_4(x_1) = b^2$, $\phi_4(x_2) = b^4$, $\phi_4(x_3) = b$ and $\phi_5(x_1) = b^2$, $\phi_5(x_2) = b^8$, $\phi_3(x_3) = b^3$. By Theorem 1 $s(\phi_i^{-1}\langle b^5\rangle) = (4; 2, 2)$, $i = 1, \ldots, 5$. Thus $\overline{\mathcal{M}}_8^{C_5} \cap \overline{\mathcal{M}}_8^{C_2,4}$.

**(3)** $s(\Delta) = (2; -)$. The single class of epimorphisms is non-maximal and extends to $\phi : \Delta'(0; 2, 2, 2, 2, 2, 2) \to D_7 = \langle s, t | s^2 = t^2 = (st)^7 = 1\rangle$ defined by $\phi(x_i) = s$, $i = 1, 2, 3, 4$, $\phi(x_i) = t$, $i = 5, 6$. $s(\phi^{-1}\langle s\rangle) = (3; 2, \overset{6}{\ldots}, 2)$ and $\overline{\mathcal{M}}_8^{C_7} \equiv \overline{\mathcal{M}}_8^{D_7} \subset \overline{\mathcal{M}}_8^{C_2,3}$.

**(4)** $s(\Delta) = (0; 17, 17, 17)$. There are three classes of epimorphisms $\theta : \Delta \to C_{17}$. One non-maximal defined by $\theta_1(x_i) = a$, $i = 1, 2$, extending to $\phi : \Delta'(2, 17, 34) \to C_{34}$ which is defined by $\phi(x_1) = b^{17}$ and $\phi(x_2) = b^2$, and $\overline{\mathcal{M}}_8^{C_{17},01} \equiv \overline{\mathcal{M}}_8^{C_{34}} \subset \overline{\mathcal{M}}_8^{C_2,0}$. The other two classes are maximal and produce one isolated point each (see [20]). □

*Remark* 13. The branch locus of $\mathcal{M}_9$ contains two isolated strata of dimension 2. Indeed, consider the signature $s(\Delta) = (0; 7, 7, 7, 7, 7)$ and epimorphisms $\theta_1, \theta_2 : \Delta \to C_7$, defined by $\theta_1(x_i) = a$, $i = 1, 2$, $\theta_1(x_3) = a^3$, $\theta_1(x_4) = a^4$ and $\theta_2(x_i) = a$, $i = 1, 2$, $\theta_2(x_3) = a^2$, $\theta_2(x_4) = a^4$. The only possibilities to extend an epimorphism $\theta : \Delta \to C_7$ are to epimorphisms $\phi_1 : \Lambda(0; 2, 7, 7, 14) \to C_{14}$ or $\phi_2 : \Lambda(0; 7, 21, 21) \to C_{21}$. However, if $\phi_1(x_2) = b^{2m}$ and $\phi_1(x_3) = b^{2n}$, then $\phi_1$ induces a class of epimorphisms $\bar{\theta}_1 : \Delta \to C_7$ defined by $\bar{\theta}_1(x_i) = a^m$, $i = 1, 2$, and $\theta_1(x_i) = a^n$, $i = 3, 4$. Similarly if $\phi_2(x_1) = b^{3m}$, then $\phi_2$ induce a class of epimorphisms $\bar{\theta}_2 : \Delta \to C_7$, defined by $\bar{\theta}_2(x_i) = a^m$, $i = 1, 2, 3$. Clearly $\theta_1$ and $\theta_2$ are in neither of these classes, thus producing isolated strata of dimension 2.

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