On the Moore-Penrose and the Drazin inverse of two projections on Hilbert space

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Abstract

For two given orthogonal, generalized or hypergeneralized projections $P$ and $Q$ on Hilbert space $H$, we gave their matrix representation. We also gave canonical forms of the Moore-Penrose and the Drazin inverses of their product, difference and sum. In addition, it is showed when these operators are EP and some simple correlations between mentioned operators are established.

1 Introduction

Motivation for writing this paper came from publications of Deng and Wei ([5], [6]) and Baksalary and Trenkler, ([1], [2]). Namely, Deng and Wei studied Drazin invertibility for product, difference and sum of idempotents and Baksalary and Trenkler used matrix representation of the Moore-Penrose inverse of product, difference and sum of orthogonal projections. Our main goal is to give canonical form of the Moore-Penrose and the Drazin inverse for product, difference and sum of two orthogonal, generalized or hypergeneralized projections on an arbitrary Hilbert space. Using the canonical forms, we can examine when the Moore-Penrose and the Drazin inverse exist. Also, we can describe the relation between inverses (if any), estimate the Drazin index and establish necessary and sufficient conditions under which these operators are EP. Although some of the results are the same or similar to results in mentioned papers, our results are different since for the starting operators we used generalized and hypergeneralized projections and not only orthogonal projections. There is also a partial difference in the sense of examined properties.

Throughout the paper, $H$ will stand for Hilbert space and $\mathcal{L}(H)$ will stand for set of all bounded linear operators on space $H$. The symbols $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $A^*$ will denote range, null space and adjoint operator of operator $A \in \mathcal{L}(H)$.

Operator $P \in \mathcal{L}(H)$ is idempotent if $P = P^2$ and it is an orthogonal projection if $P = P^2 = P^*.$

Generalized and hypergeneralized projections were introduced in [7] by Groβ and Trenkler.
Definition 1.1. Operator $G \in \mathcal{L}(H)$ is

(a) a generalized projection if $G^2 = G^*$,

(b) a hypergeneralized projection if $G^2 = G^\dagger$.

Set of all generalized projection on $H$ is denoted by $\mathcal{GP}(H)$ and set of all hypergeneralized projection is denoted by $\mathcal{HGP}(H)$.

Here $A^\dagger$ is the Moore-Penrose inverse of $A \in \mathcal{L}(H)$ i.e. the unique solution to the equations

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$ 

Notice that $A^\dagger$ exists if and only if $\mathcal{R}(A)$ is closed. Then $AA^\dagger$ is orthogonal projection onto $\mathcal{R}(A)$ parallel to $\mathcal{N}(A^*)$, and $A^\dagger A$ is orthogonal projection onto $\mathcal{R}(A^*)$ parallel to $\mathcal{N}(A)$. Consequently, $I - AA^\dagger$ is orthogonal projection onto $\mathcal{N}(A^*)$ and $I - A^\dagger A$ is orthogonal projection onto $\mathcal{N}(A)$. An essential property of any $P \in \mathcal{L}(H)$ is that $P$ is an orthogonal projection if and only if it is expressible as $AA^\dagger$, for some $A \in \mathcal{L}(H)$.

For $A \in \mathcal{L}(H)$, an element $B \in \mathcal{L}(H)$ is the Drazin inverse of $A$ if the following hold:

$$BAB = B, \quad BA = AB, \quad A^{n+1}B = A^n,$$

for some non-negative integer $n$. The smallest such $n$ is called the Drazin index of $A$. By $A^D$ we denote Drazin inverse of $A$ and by $ind(A)$ we denote Drazin index of $A$.

If such $n$ does not exist, $ind(A) = \infty$ and operator $A$ is generalized Drazin invertible. Its invers is denoted by $A^d$.

Operator $A$ is invertible if and only if $ind(A) = 0$.

If $ind(A) \leq 1$, $A$ is group invertible and $A^D$ is group inverse, usually denoted by $A^\#$.

Notice that if the Drazin inverse exists, it is unique. Operator $A \in \mathcal{L}(H)$ is Drazin invertible if and only if $asc(A) < \infty$ and $dsc(A) < \infty$, where $asc(A)$ is the minimal integer such that $\mathcal{N}(A^{n+1}) = \mathcal{N}(A^n)$ and $dsc(A)$ is the minimal integer such that $\mathcal{R}(A^{n+1}) = \mathcal{R}(A^n)$. In this case, $ind(A) = asc(A) = dsc(A) = n$.

Recall that if $\mathcal{R}(A^n)$ is closed for some integer $n$, then $asc(A) = dsc(A) < \infty$.

Operator $A \in \mathcal{L}(H)$ is EP if $AA^\dagger = A^\dagger A$, or, in the other words, if $A^\dagger = A^D = A^\#$. There are many characterization of EP operators. In this paper, we use results from Djordjević and Koliha, ([4]).

In what follows, $\overline{A}$ will stand for $I - A$ and $P_A$ will stand for $AA^\dagger$.

## 2 Auxiliary results

Let $P, Q \in \mathcal{L}(H)$ be orthogonal projectons and $\mathcal{R}(P) = L$. Since $H = \mathcal{R}(P) \oplus \mathcal{R}(P)^\perp = L \oplus L^\perp$, we have the following representaton of projections $P, \overline{P}, Q, \overline{Q} \in \mathcal{L}(H)$.
$\mathcal{L}(H)$ with respect to the decomposition of space:

\[
P = \begin{bmatrix} \phantom{I}P_1 \ 0 \ 0 \\
0 \ 0 \ 0
\end{bmatrix} = \begin{bmatrix} I_L \ 0 \ 0 \\
0 \ L \ 0
\end{bmatrix} \rightarrow \begin{bmatrix} L \ 0 \\
L \ \perp
\end{bmatrix},
\]
with $A \in \mathcal{L}(L)$ and $D \in \mathcal{L}(L^\perp)$ being Hermitian and non-negative.

Next two theorems are known for matrices on $\mathbb{C}^n$, see [2].

**Theorem 2.1.** Let $Q \in \mathcal{L}(H)$ be represented as in (3). Then the following holds:

(a) $A = A^2 + BB^*$, or, equivalently, $A\overline{A} = BB^*$,

(b) $B = AB + BD$, or, equivalently, $B^* = B^*A + DB^*$,

(c) $D = D^2 + B^*B$, or, equivalently, $D\overline{D} = B^*B$.

**Proof.** Since $Q = Q^2$, we obtain

\[
\begin{bmatrix} A & B \\
B^* & D
\end{bmatrix} \begin{bmatrix} A & B \\
B^* & D
\end{bmatrix} = \begin{bmatrix} A^2 + BB^* & AB + BD \\
B^*A + DB^* & B^*B + D^2
\end{bmatrix} = \begin{bmatrix} A & B \\
B^* & D
\end{bmatrix}
\]

implying that $A = A^2 + BB^*$, $B = AB + BD$ and $D = D^2 + B^*B$. 

**Theorem 2.2.** Let $Q \in \mathcal{L}(H)$ be represented as in (3). Then:

(a) $\mathcal{R}(B) \subseteq \mathcal{R}(A)$,

(b) $\mathcal{R}(B) \subseteq \mathcal{R}(\overline{A})$,

(c) $\mathcal{R}(B^*) \subseteq \mathcal{R}(D)$,

(d) $\mathcal{R}(B^*) \subseteq \mathcal{R}(\overline{D})$,

(e) $A^\dagger B = BD^\dagger$,

(f) $\overline{A} B = BD^\dagger$,

(g) $A$ is a contraction,

(h) $D$ is a contraction,

(i) $A - BD^\dagger B^* = I_L - \overline{\overline{A}} \overline{A}$. 

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Proof. (a) Since $A = A^2 + BB^*$, we have

$$\mathcal{R}(A) = \mathcal{R}(A^2 + BB^*) = \mathcal{R}(AA^* + BB^*).$$

To prove that $\mathcal{R}(AA^* + BB^*) = \mathcal{R}(A) + \mathcal{R}(B)$, observe operator matrix

$$M = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}.$$  

For any $x \in \mathcal{R}(MM^*)$, there exist $y \in H$ such that $x = MM^*y = M(M^*y)$ and $x \in \mathcal{R}(M)$. On the other hand, for $x \in \mathcal{R}(M)$, there is $y \in H$ and $x = My$. Besides, $MM^*x = MM^*My = My = x$ and $MM^* = MM^*(MM^*)^\dagger = P_{\mathcal{R}(MM^*)}$ implying $x \in \mathcal{R}(MM^*)$. Hence, $\mathcal{R}(M) = \mathcal{R}(MM^*)$ and

$$\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(M) = \mathcal{R}(MM^*) = \mathcal{R}(AA^* + BB^*)$$

and we have

$$\mathcal{R}(A) = \mathcal{R}(A) + \mathcal{R}(B)$$

implying $\mathcal{R}(B) \subseteq \mathcal{R}(A)$.

(b) Since $A = I - \overline{A}$, from Theorem 2.1 (a), we get $\overline{A} = \overline{A^2 + BB^*}$. The rest of the proof is analogous to the point (a) of this theorem.

(c), (d) Similarly.

(e) Since $B = AB + BD$, we have $A^\dagger B = A^\dagger (AB + BD) = A^\dagger AB + A^\dagger BD$ and using the facts that $A^\dagger A = P_{\mathcal{R}(A^\dagger)}$ and $\mathcal{R}(B) \subseteq \mathcal{R}(A^\dagger)$, we get $A^\dagger AB = B$ and $A^\dagger B = B + A^\dagger BD$, or, equivalently $B = A^\dagger B D$. Postmultiplying this equation by $D^\dagger$ and using item (d) of this Theorem, in its equivalent form $B D D^\dagger = B$, we obtain (e).

(f) Analogously to the previous proof.

(g) Since $A = A^*$, from Theorem 2.1 (a), we have that

$$I_L - AA^* = I_L - (A - BB^*) = \overline{A} + BB^*,$$

and the right hand side is nonnegative as a sum of two nonnegative operators implying that $A$ is a contraction.

(h) This part of the proof is dual to the part (g).

(i) From Theorem 2.1 (a), item (f) of this Theorem and the fact that hermitian operator $A$ commutes with its MP-inverse, it follows that

$$BD^\dagger B^* = \overline{A}^\dagger BB^* = \overline{A}^\dagger AA = \overline{A}^\dagger(I - A) \overline{A} = \overline{A}^\dagger \overline{A} - \overline{A}^\dagger \overline{A} \overline{A} = \overline{A}^\dagger \overline{A} - \overline{A}$$

by taking into account that $\overline{A} \overline{A} = \overline{A}^\dagger \overline{A}$. Now we get

$$A - BD^\dagger B^* = I - \overline{A}^\dagger \overline{A},$$

establishing the condition. \qed
Following the results of Groβ and Trenkler for matrices, we will formulate a few theorems for generalized and hypergeneralized projections on arbitrary Hilbert space. We start with the result which is very similar to Theorem (1) in [7].

**Theorem 2.3.** Let $G \in \mathcal{L}(H)$ be a generalized projection. Then $G$ is a closed range operator and $G^3$ is an orthogonal projection on $\mathcal{R}(G)$. Moreover, $H$ has decomposition

$$H = \mathcal{R}(G) \oplus \mathcal{N}(G)$$

and $G$ has the following matrix representation

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix},$$

where restriction $G_1 = G|_{\mathcal{R}(G)}$ is unitary on $\mathcal{R}(G)$.

**Proof.** If $G$ is a generalized projection, then $G^4 = (G^2)^2 = (G^*)^2 = (G^2)^* = G$. From $GG^* = G^4 = G$ follows that $G$ is a partial isometry implying that

$$G^3 = GG^* = P_{\mathcal{R}(G)},$$

$$G^3 = G^*G = P_{\mathcal{N}(G)^\perp}.$$

Thus, $G^3$ is an orthogonal projection onto $\mathcal{R}(G) = \mathcal{N}(G)^\perp = \mathcal{R}(G^*)$. Consequently, $\mathcal{R}(G)$ is a closed subset in $H$ as a range of an orthogonal projection on a Hilbert space. From Lemma (1.2) in [4] we get the following decomposition of the space

$$H = \mathcal{R}(G^*) \oplus \mathcal{N}(G) = \mathcal{R}(G) \oplus \mathcal{N}(G).$$

Now, $G$ has the following matrix representation in accordance with this decomposition:

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix},$$

where $G_1^2 = G_1^\dagger$, $G_1^1 = G_1$ and $G_1G_1^\dagger = G_1^*G_1 = G_1^3 = I_{\mathcal{R}(G)}$.

**Theorem 2.4.** Let $G, H \in \mathcal{GP}(H)$ and $H = \mathcal{R}(G) \oplus \mathcal{N}(G)$. Then $G$ and $H$ has the following representation with respect to decomposition of the space:

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix},$$

$$H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix},$$

where

$$H_1^* = H_1^2 + H_2H_3,$$

$$H_2^* = H_3H_1 + H_4H_3,$$

$$H_3^* = H_1H_2 + H_2H_4,$$

$$H_4^* = H_3H_2 + H_4^2.$$
Furthermore, $H_2 = 0$ if and only if $H_3 = 0$.

**Proof.** Let $H = \mathcal{R}(G) \oplus \mathcal{N}(G)$. Then representation of $G$ follows from Theorem (1) in [7] and let $H$ has representation

$$H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}.$$ 

Then, from

$$H^2 = \begin{bmatrix} H_1^2 + H_2H_3 & H_1H_2 + H_2H_4 \\ H_3H_1 + H_4H_3 & H_3H_2 + H_4^2 \end{bmatrix} = \begin{bmatrix} H_1^* & H_2^* \\ H_3^* & H_4^* \end{bmatrix} = H^*,$$

conclusion follows directly.

If $H_2 = 0$, then $H_3^* = H_1H_2 + H_2H_4 = 0$ and $H_3 = 0$. Analogously, $H_3 = 0$ implies $H_2 = 0$. \hfill \Box

**Theorem 2.5.** Let $G \in \mathcal{L}(H)$ be a hypergeneralized projection. Then $G$ is a closed range operator and $H$ has decomposition

$$H = \mathcal{R}(G) \oplus \mathcal{N}(G).$$

Also, $G$ has the following matrix representation with the respect to decomposition of the space

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix},$$

where restriction $G_1 = G|_{\mathcal{R}(G)}$ satisfies $G_1^2 = I_{\mathcal{R}(G)}$.

**Proof.** If $G$ is a hypergeneralized projector, then $G$ and $G^\dagger$ commute and $G$ is EP. Using Lemma (1.2) in [4], we get the following decomposition of the space $H = \mathcal{R}(G) \oplus \mathcal{N}(G)$ and $G$ has the required representation. \hfill \Box

### 3 The Moore-Penrose and the Drazin inverse of two orthogonal projections

We start this section with theorem which gives matrix representation of the Moore-Penrose inverse of product, difference and sum of orthogonal projections.

**Theorem 3.1.** Let orthogonal projections $P, Q \in \mathcal{L}(H)$ be represented as in (1) and (2). Then the Moore-Penrose inverse of $PQ$, $P - Q$ and $P + Q$ exists and the following holds:

(a) $(PQ)^\dagger = \begin{bmatrix} AA^\dagger & 0 \\ B^\dagger A & 0 \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \rightarrow \begin{bmatrix} L \\ L^\perp \end{bmatrix}$ and

$$\mathcal{R}(PQ) = \mathcal{R}(A)$$
(b) \((P - Q)^\dagger = \begin{bmatrix}
  \bar{A}A^\dagger & -BD^\dagger \\
  -B^*A^\dagger & -DD^\dagger
\end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \rightarrow \begin{bmatrix} L \\ L^\perp \end{bmatrix} \) and
\[ \mathcal{R}(P - Q) = \mathcal{R}(\bar{A}) \oplus \mathcal{R}(D) \]

(c) \((P + Q)^\dagger = \begin{bmatrix}
  \frac{1}{2}(I + \bar{A}\bar{A}^\dagger) & -BD^\dagger \\
  -D^\dagger B^* & 2D^\dagger - DD^\dagger
\end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \rightarrow \begin{bmatrix} L \\ L^\perp \end{bmatrix} \) and
\[ \mathcal{R}(P + Q) = L \oplus \mathcal{R}(D). \]

**Proof.** (a) Using representations (1) and (3) for orthogonal projections \(P, Q \in \mathcal{L}(H)\), the well known Harte-Mbekhta formula \((PQ)^\dagger = (PQ)^* (PQ(PQ)^*)^\dagger\) and Theorem 2.1(a), we obtain
\[ (PQ)^\dagger = \begin{bmatrix} A & 0 \\ B^* & 0 \end{bmatrix} \begin{bmatrix} A^2 + BB^* & 0 \\ 0 & 0 \end{bmatrix}^\dagger = \begin{bmatrix} AA^\dagger & 0 \\ B^*A^\dagger & 0 \end{bmatrix}. \]

From \(PQ(PQ)^\dagger = P_{\mathcal{R}(PQ)}\), we obtain
\[ (PQ)(PQ)^\dagger = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} AA^\dagger & 0 \\ B^*A^\dagger & 0 \end{bmatrix} = \begin{bmatrix} AA^\dagger & 0 \\ 0 & 0 \end{bmatrix}, \]
or, in the other words, \(\mathcal{R}(PQ) = \mathcal{R}(A)\).

(b) Similarly to part (a), we can calculate the Moore-Penrose inverse of \(P - Q\) as follows
\[ (P - Q)^\dagger = (P - Q)^* ((P - Q)(P - Q)^*)^\dagger \]
\[ = \begin{bmatrix} \bar{A} & -B \\ -B^* & -D \end{bmatrix} \begin{bmatrix} \bar{A}^2 + BB^* & -\bar{A}B + BD \\ -B^*\bar{A} + DB^* & B^*B + D^2 \end{bmatrix} \]
\[ = \begin{bmatrix} \bar{A} & -B \\ -B^* & -D \end{bmatrix} \begin{bmatrix} \bar{A}^\dagger & 0 \\ 0 & D^\dagger \end{bmatrix} \]
\[ = \begin{bmatrix} \bar{A}\bar{A}^\dagger & -BD^\dagger \\
  -B^*\bar{A}^\dagger & -DD^\dagger
\end{bmatrix}. \]

For the range of \(P - Q\) we have
\[ P_{\mathcal{R}(P - Q)} = (P - Q)(P - Q)^\dagger \]
\[ = \begin{bmatrix} \bar{A}\bar{A}\bar{A}^\dagger + BD^\dagger B^* & -\bar{A}BD^\dagger + BDD^\dagger \\ -B^*\bar{A}\bar{A}^\dagger + DD^\dagger B^* & B^*BD^\dagger + DDD^\dagger \end{bmatrix} \]
\[ = \begin{bmatrix} \bar{A}\bar{A}^\dagger & 0 \\ 0 & DD^\dagger \end{bmatrix}. \]
implying
\[ \mathcal{R}(P - Q) = \mathcal{R}(A) \oplus \mathcal{R}(D). \]

(c) The Moore-Penrose inverse of \( P + Q \) has the following representation with the respect to decomposition of the space:
\[
(P + Q)\dagger = \begin{bmatrix}
X_1 & X_2 \\
X_3 & X_4
\end{bmatrix} : \begin{bmatrix}
L \\
L^\perp
\end{bmatrix} \rightarrow \begin{bmatrix}
L \\
L^\perp
\end{bmatrix}.
\]

In order to calculate \((P + Q)\dagger\), we will use Moore-Penrose equations. From the first Moore-Penrose equation, \((P + Q)(P + Q)\dagger = P + Q\), we have
\[
((I + A)X_1 + BX_3)(I + A) + ((I + A)X_2 + BX_4)B^* = I + A,
\]
\[
((I + A)X_1 + BX_3)B + ((I + A)X_2 + BX_4)D = B,
\]
\[
(B^*X_1 + DX_3)(I + A) + (B^*X_2 + DX_4)B^* = B^*,
\]
\[
(B^*X_1 + DX_3)B + (B^*X_2 + DX_4)D = D.
\]

The second Moore-Penrose equation, \((P + Q)\dagger(P + Q)(P + Q)\dagger = (P + Q)\dagger\), implies
\[
(X_1(I + A) + X_2B^*)X_1 + (X_1B + X_2D)X_3 = X_1,
\]
\[
(X_1(I + A) + X_2B^*)X_2 + (X_1B + X_2D)X_4 = X_2,
\]
\[
(X_3(I + A) + X_4B^*)X_1 + (X_3B + X_4D)X_3 = X_3,
\]
\[
(X_3(I + A) + X_4B^*)X_2 + (X_3B + X_4D)X_4 = X_4,
\]

while the third and fourth Moore-Penrose equations, \(((P + Q)(P + Q)\dagger)^* = (P + Q)(P + Q)\dagger\) and \(((P + Q)\dagger(P + Q))^* = (P + Q)\dagger(P + Q)\), give \(X_3 = X_3^*\). Further calculations show that
\[
(I + A)X_1 + BX_3^* = I_L,
\]
\[
(I + A)X_2 + BX_4 = 0,
\]
\[
B^*X_1 + DX_3^* = 0,
\]
\[
B^*X_2 + DX_4 = DD^\dagger.
\]

According to Theorem 2.2 (b), (c), from \(B^*X_1 + DX_3^* = 0\) we get \(D^\dagger B^* X_1 + X_2 = 0\), or equivalently, \(X_2^* = -D^\dagger B^* X_1\).

From \((I + A)X_1 + BX_3^* = I_L\) and Theorem 2.2 (i), we get \((2I - A\overline{A})X_1 = I_L\)
i.e. \(X_1 = (2I - A\overline{A})^{-1} = \frac{1}{2}(I + A\overline{A})\). Theorem 2.1 (c) and \(B^*X_2 + DX_4 = DD^\dagger\)
imply \(-B^* BD^\dagger + DX_4 = DD^\dagger\). Finally, we have \(X_2 = -BD^\dagger, X_3 = -D^\dagger B^*, \)
\(X_4 = 2D^\dagger - DD^\dagger\) and
\[
(P + Q)\dagger = \begin{bmatrix}
\frac{1}{2}(I + \overline{A}\overline{A}) & -BD^\dagger \\
-B^* D^\dagger & 2D^\dagger - DD^\dagger
\end{bmatrix}.
\]
Like in part (b),
\[
P_{\mathcal{R}(P+Q)} = (P + Q)(P + Q)^\dagger
\]
\[
= \begin{bmatrix}
\frac{1}{2}(I + A)(I + AA^\dagger) - BD^\dagger B^* & -(I + A)BD^\dagger + 2BD^\dagger - BDD^\dagger \\
\frac{1}{2}B^*(I + AA^\dagger) - DD^\dagger B^* & -B^* BD^\dagger + 2DD^\dagger - DDD^\dagger 
\end{bmatrix}
\]
which proves that
\[
\mathcal{R}(P + Q) = L \oplus \mathcal{R}(D).
\]

To prove the existence of the Moore-Penrose inverse of \(PQ\), \(P - Q\) and \(P + Q\), it is sufficient to prove that these operators have closed range. Since \(Q\) is an orthogonal projection, \(\mathcal{R}(Q)\) is closed subset of \(H\). Also,
\[
\mathcal{R}(Q) = Q(H) = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} \begin{bmatrix} L \\ L^\perp \end{bmatrix} = \begin{bmatrix} \mathcal{R}(A) + \mathcal{R}(B) \\ \mathcal{R}(B^*) + \mathcal{R}(D) \end{bmatrix} = \mathcal{R}(A) + \mathcal{R}(D),
\]
because items (a), (c) of Theorem 2.2 state that \(\mathcal{R}(B) \subseteq \mathcal{R}(A)\) and \(\mathcal{R}(B^*) \subseteq \mathcal{R}(D)\). This implies that \(\mathcal{R}(A)\) and \(\mathcal{R}(D)\) are closed subsets of \(L\) and \(L^\perp\) respectively. If \(\mathcal{R}(A)\) is closed, then for every sequence \((x_n) \subseteq L\), \(x_n \to x\) and \(Ax_n \to y\) imply \(x \in L\) and \(Ax = y\). Now, \((I - A)x_n \to x - y\) and \(x - y \in L\), \((I - A)x = x - y\) which proves that \(\mathcal{R}(I - A)\) is closed. Consequently, \(\mathcal{R}(PQ)\), \(\mathcal{R}(I - A)\) and \(\mathcal{R}(A + I)\) are closed which completes the proof. \(\square\)

Similar to Theorem 3.1 in [6], we have the following result.

**Theorem 3.2.** Let orthogonal projections \(P, Q \in \mathcal{L}(H)\) be represented as in (1) and (3). Then the Drazin inverses of \(PQ\), \(P - Q\) and \(P + Q\) exist, \(P - Q\) and \(P + Q\) are EP operators and the following holds:

(a) \((PQ)^D = \begin{bmatrix} A^D & (A^D)^2B \\
0 & 0 \end{bmatrix}\) and \(\text{ind}(PQ) \leq \text{ind}(A) + 1\),

(b) \((P - Q)^D = (P - Q)^\dagger\) and \(\text{ind}(P - Q) \leq 1\),

(c) \((P + Q)^D = (P + Q)^\dagger\) and \(\text{ind}(P + Q) \leq 1\).

**Proof.** (a) Theorem 3.1 proves that \(\mathcal{R}(PQ)\) is closed subset of \(H\). Thus, the Drazin inverse for this operators exists. According to representations (1) and (3) of projections \(P, Q\), their product \(PQ\) and the Drazin inverse \((PQ)^D\) can be written in the following way:
\[
PQ = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \quad (PQ)^D = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \to \begin{bmatrix} L \\ L^\perp \end{bmatrix}.
\]

Equations that describe Drazin inverse are
\[
(PQ)^D PQ(PQ)^D = \begin{bmatrix} X_1AX_1 & X_1AX_2 \\ X_3AX_1 & X_3AX_2 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} = (PQ)^D,
\]

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\[(PQ)^D PQ = \begin{bmatrix} X_1A & X_1B \\ X_3A & X_3B \end{bmatrix} = \begin{bmatrix} AX_1 & AX_2 \\ 0 & 0 \end{bmatrix} = PQ(PQ)^D,\]

\[(PQ)^{n+1}(PQ)^D = \begin{bmatrix} A^{n+1}X_1 & A^{n+1}X_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A^n & A^{n-1}B \\ 0 & 0 \end{bmatrix} = (PQ)^n.\]

Thus, from the first equation we have

\[X_1AX_1 = X_1, \quad X_1AX_2 = X_2, \quad X_3AX_1 = X_3, \quad X_3AX_2 = X_4,\]

from the second equation

\[X_1A = AX_1, \quad AX_2 = X_1B, \quad X_3A = 0, \quad X_3B = 0,\]

and the third equation implies

\[A^{n+1}X_1 = A^n, \quad A^{n+1}X_2 = A^{n-1}B.\]

It is easy to conclude that \(X_1 = A^D, X_3 = 0, X_4 = 0.\) Equations \(X_1AX_2 = X_2\) and \(AX_2 = X_1B\) give \(X_1^2B = X_2.\) Finally,

\[(PQ)^D = \begin{bmatrix} A^D & (A^D)^2B \\ 0 & 0 \end{bmatrix}.\]

To estimate the Drazin index of \(PQ,\) suppose that \(\text{ind}(A) = n.\) Then

\[(PQ)^{n+2}(PQ)^D = \begin{bmatrix} A^{n+2} & A^{n+1}B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A^D & (A^D)^2B \\ 0 & 0 \end{bmatrix}
= \begin{bmatrix} A^{n+1} & A^{n+1}A^DB \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A^n & A^nB \\ 0 & 0 \end{bmatrix} = (PQ)^{n+1}\]

implying that \(\text{ind}(PQ) \leq \text{ind}(A) + 1.\)

(b) Since \((P - Q)(P - Q)^* = (P - Q)^*(P - Q)\) and \(\mathcal{R}(P - Q) = \mathcal{R}(\mathcal{A}) \oplus \mathcal{R}(D)\) is closed , \(P - Q\) is EP operator as normal operator with closed range and \((P - Q)^\dagger = (P - Q)^D.\) Besides,

\[(P - Q)^2(P - Q)^D = (P - Q)(P - Q)^\dagger(P - Q) = P - Q\]

and \(\text{ind}(P - Q) \leq 1.\)

(c) Similarly to (b), \(P + Q\) is EP operator and \((P + Q)^\dagger = (P + Q)^D,\)
\(\text{ind}(P + Q) \leq 1.\)

\[\text{Theorem 3.3. Let orthogonal projections } P, Q \in \mathcal{L}(H) \text{ be represented as in (1) and (3). Then the following holds:}\]

(a) If \(PQ = QP\) or \(PQP = PQ,\) then

\[(P + Q)^D = \begin{bmatrix} I_L - \frac{1}{2}A & 0 \\ 0 & D \end{bmatrix}, \quad (P - Q)^D = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & -D \end{bmatrix}.\]
(b) If \(PQP = P\), then
\[
(P + Q)^D = \begin{bmatrix} \frac{1}{2}I_L & 0 \\ 0 & D \end{bmatrix}, \quad (P - Q)^D = \begin{bmatrix} 0 & 0 \\ 0 & -D \end{bmatrix}.
\]

(c) If \(PQP = Q\), then
\[
(P + Q)^D = \begin{bmatrix} I_L - \frac{1}{2}A & 0 \\ 0 & 0 \end{bmatrix}, \quad (P - Q)^D = \begin{bmatrix} \bar{A} & 0 \\ 0 & 0 \end{bmatrix} = P - Q.
\]

(d) If \(PQP = 0\), then
\[
(P + Q)^D = \begin{bmatrix} I_L & 0 \\ 0 & D \end{bmatrix} = P + Q, \quad (P - Q)^D = \begin{bmatrix} I_L & 0 \\ 0 & -D \end{bmatrix} = P - Q.
\]

Proof. Let
\[
(P + Q)^D = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \to \begin{bmatrix} L \\ L^\perp \end{bmatrix}.
\]

(a) If
\[
PQ = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ B^* & 0 \end{bmatrix} = QP
\]
or
\[
PQP = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} = PQ,
\]
then \(B = B^* = 0\), \(I_L + A\) is invertible and \((I_L + A)^{-1} = I_L - \frac{1}{2}A\) and according to Theorem 2.1 (c), \(D = D^2\). Thus, we can write
\[
Q = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \quad P + Q = \begin{bmatrix} I_L + A & 0 \\ 0 & D \end{bmatrix}, \quad (P + Q)^n = \begin{bmatrix} (I_L + A)^n & 0 \\ 0 & D \end{bmatrix}.
\]

Verifying the equation
\[
(P + Q)^2(P + Q)^D = \begin{bmatrix} (I_L + A)^2X_1 & (I_L + A)^2X_2 \\ DX_3 & DX_4 \end{bmatrix}
\]
\[
= \begin{bmatrix} I_L + A & 0 \\ 0 & D \end{bmatrix} = P + Q
\]
we get
\[
X_2 = X_3 = 0, \quad DX_4 = D.
\]
The other two equations, \((P + Q)^D(P + Q)(P + Q)^D = (P + Q)^D\) and \((P + Q)^D(P + Q) = (P + Q)(P + Q)^D\), give
\[
X_4DX_4 = X_4, \quad X_4D = DX_4
\]
i.e. \(X_4 = D\). Thus,
\[
(P + Q)^D = \begin{bmatrix} I_L - \frac{1}{2}A & 0 \\ 0 & D \end{bmatrix}.
\]
Formula

\[(P - Q)^D = \begin{bmatrix} \bar{A} & 0 \\ 0 & -D \end{bmatrix}\]

follows form Theorem 3.2 (b) and the fact that \(A = A^2\) implies \(\bar{A}^D = \bar{A}^2 = \bar{A}\).

(b) If \(PQP = P\), then \(A = I_L\) and Theorem 2.1 implies \(B = B^* = 0\). Then,

\[Q = \begin{bmatrix} I_L & 0 \\ 0 & D \end{bmatrix}\]

and from part (a) of this Theorem we conclude

\[(P + Q)^D = \begin{bmatrix} \frac{1}{2}I_L & 0 \\ 0 & D \end{bmatrix}, \quad (P - Q)^D = \begin{bmatrix} 0 & 0 \\ 0 & -D \end{bmatrix} .\]

(c) From \(PQP = Q\) we get \(B = B^* = D = 0\) and \(A = A^2\). Now, \(I_L + A\) is invertible and

\[(P + Q)^D = (P + Q)^{-1} = \begin{bmatrix} (I_L + A)^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_L - \frac{1}{2}A & 0 \\ 0 & 0 \end{bmatrix} \]

and

\[(P - Q)^D = \begin{bmatrix} \bar{A} & 0 \\ 0 & 0 \end{bmatrix} .\]

(d) If \(PQP = 0\), then \(A = 0\) and since \(\mathcal{R}(B) \subseteq \mathcal{R}(A)\), we conclude \(B = B^* = 0\). In this case,

\[Q = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}, \quad P + Q = \begin{bmatrix} I_L & 0 \\ 0 & D \end{bmatrix} \]

implying

\[(P + Q)^D = P + Q = \begin{bmatrix} I_L & 0 \\ 0 & D \end{bmatrix}, \quad (P - Q)^D = P - Q = \begin{bmatrix} I_L & 0 \\ 0 & -D \end{bmatrix}. \]

\[\square\]

**Theorem 3.4.** Let orthogonal projections \(P, Q \in \mathcal{L}(H)\) be represented as in (1) and (3). Then

\[(PQ)^D = (QP)^\dagger (QP)^\dagger (QP)^\dagger.\]

Moreover, if \(PQ = QP\), then \(PQ\) is EP operator and

\[(PQ)^D = (PQ)^\dagger, \quad \text{ind}(PQ) \leq 1.\]

**Proof.** Corollary 5.2 in [8] states that \((PQ)^\dagger\) is idempotent for every orthogonal projections \(P\) and \(Q\). Thus we can write

\[(PQ)^\dagger = \begin{bmatrix} I & 0 \\ K & 0 \end{bmatrix} \]

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and
\[ PQ = (PQ)^{††} = \begin{bmatrix} (I + K^*K)^{-1} & (I + K^*K)^{-1}K^* \\ 0 & 0 \end{bmatrix}. \]

Denote by \( A = (I + K^*K)^{-1} \) and \( B = (I + K^*K)^{-1}K^* = AK^* \). Then
\[ PQ = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \]

and according to Theorem 3.3 (a),
\[ (PQ)^D = \begin{bmatrix} A^D & (A^D)^2B \\ 0 & 0 \end{bmatrix} \]
\[ = \begin{bmatrix} I + K^*K & (I + K^*K)^2(I + K^*K)^{-1}K^* \\ 0 & 0 \end{bmatrix} \]
\[ = \begin{bmatrix} I + K^*K & (I + K^*K)K^* \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I + K^*K & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I \\ K^* \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} I \\ K^* \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \]
\[ = (QP)^†((PQ)^†)^†,...(QP)^†. \]

If \( P \) and \( Q \) commute, then \( PQ \) is normal operator with closed range which means it is EP operator \((PQ)^† = (PQ)^D\).

\[ (PQ)^D = ((PQ)^*)^† = (QP)^†, \]

where we used \(((PQ)^*)^† = ((PQ)^†)^† = (QP)^†\).

\[ \text{Proof.} \quad \text{From Theorems 2.3, 2.4 and 2.5, we see that } R(G) = R(G_1) \text{ is closed and pair of generalized or hypergeneralized projections has matrix form} \]
\[ G = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}. \]

\[ \text{4 The Moore-Penrose and the Drazin inverse of generalized and hypergeneralized projections} \]

Some of the results obtained in the previous section we can extend to generalized and hypergeneralized projections.

**Theorem 4.1.** Let \( G, H \in \mathcal{L}(H) \) be two generalized or hypergeneralized projections. Then the Moore-Penrose inverse of \( GH \) exists and has the following matrix representation
\[ (GH)^† = \begin{bmatrix} (G_1H_1)^*D^{-1} & 0 \\ (G_1H_2)^*D^{-1} & 0 \end{bmatrix}, \]

where \( D = G_1H_1(G_1H_1)^* + G_1H_2(G_1H_2)^* > 0 \) is invertible.

**Proof.** From Theorems 2.3, 2.4 and 2.5, we see that \( R(G) = R(G_1) \) is closed and pair of generalized or hypergeneralized projections has matrix form
\[ G = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}. \]
Then
\[ GH = \begin{bmatrix} G_1H_1 & G_1H_2 \\ 0 & 0 \end{bmatrix} \]
and analogously to the proof of Theorem 3.1 (a), we obtain mentioned matrix form. Since \( R(GH) = R(G_1) \) is closed, the Moore-Penrose \((GH)^\dagger\) exists.

**Theorem 4.2.** Let \( G, H \in \mathcal{L}(H) \) be two generalized or hypergeneralized projections. Then the Drazin inverse of \( GH \) exists and has the following matrix representation
\[
(GH)^D = \begin{bmatrix} (G_1H_1)^D & ((G_1H_1)^D)^2G_1H_2 \\ 0 & 0 \end{bmatrix}.
\]

**Proof.** Similarly to the proof of Theorem 3.2 (a) and using Theorem 2.5.

**Theorem 4.3.** Let \( G, H \in \mathcal{L}(H) \) be two generalized projections.

(a) If \( GH = HG \), then \( GH \) is EP and
\[
(GH)^\dagger = (GH)^D = (GH)^* = (GH)^2 = (GH)^{-1},
\]
\[
(GH)^\dagger = \begin{bmatrix} (G_1H_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}.
\]

(b) If \( GH = HG = 0 \), then \( G + H \) is EP and
\[
(G + H)^\dagger = (G + H)^D = (G + H)^* = (G + H)^2 = (G + H)^{-1},
\]
\[
(G + H)^\dagger = \begin{bmatrix} G_1^{-1} & 0 \\ 0 & H_1^{-1} \end{bmatrix}.
\]

(c) If \( GH = HG = H^* \), then \( G - H \) is EP and
\[
(G - H)^\dagger = (G - H)^D = (G - H)^* = (G - H)^2 = (G - H)^{-1},
\]
\[
(G - H)^\dagger = \begin{bmatrix} (G_1 - H_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}.
\]

**Proof.** (a) If \( G, H \in \mathcal{L}(H) \) are two commuting generalized projections, then from
\[
(GH)^* = H^*G^* = H^2G^2 = (HG)^2 = (GH)^2
\]
we conclude that \( GH \) is also generalized projection, and therefore EP operator. Checking the Moore-Penrose equations for \((GH)^2\), we see that they hold. From the uniqueness of the Moore-Penrose inverse follows \((GH)^2 = (GH)^\dagger\) and
\[
(GH)^\dagger = (GH)^D = (GH)^2
\]
From $GH(GH)^\dagger = P_{R(GH)}$, using matrix form of $GH$, we get $G_1 H_1 (G_1 H_1)^\dagger = I$, or equivalently, $(G_1 H_1)^\dagger = (G_1 H_1)^{-1}$. Finally,

$$(GH)^\dagger = (GH)^D = (GH)^* = (GH)^2 = (GH)^{-1}.$$  

(b) If $GH = HG = 0$, then $(G + H)^2 = G^2 + H^2 = G^* + H^* = (G + H)^*$ and $G + H$ is a generalized projection. The rest of the proof is similar to part (a).

(c) If $GH = HG = H^*$, then $(G - H)^2 = G^2 - H^2 = G^* - H^* = (G - H)^*$ and the rest of the proof is similar to part (a).

Matrix representations are easily obtained by using canonical forms of $G$ and $H$ given in Theorem 2.4.

Theorem 4.4. Let $G, H \in \mathcal{L}(H)$ be two hypergeneralized projections.

(a) If $GH = HG$, then $GH$ is EP and

$$(GH)^\dagger = (GH)^D = (GH)^2 = (GH)^{-1},$$

$$(G_1 H_1)^{-1} 0 \quad 0 \quad 0 \quad 0.$$

(b) If $GH = HG = 0$, then $G + H$ is EP and

$$(G + H)^\dagger = (G + H)^D = (G + H)^2 = (G + H)^{-1},$$

$$(G_1 H_1)^{-1} 0 \quad 0 \quad 0 \quad 0.$$

(c) If $GH = HG = H^*$, then $G - H$ is EP and

$$(G - H)^\dagger = (G - H)^D = (G - H)^2 = (G - H)^{-1},$$

$$(G_1 - H_1)^{-1} 0 \quad 0 \quad 0.$$

Proof. (a) $GH$ is EP operator and $(GH)^4 = GH$, so it is a hypergeneralized projection. Since $(GH)^2 = (GH)^\dagger$, operator $GH$ commutes with its Moore-Penrose inverse and $(GH)^\dagger = (GH)^D$. From

$$GH(GH)^\dagger = \left[ \begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right],$$

follows $(GH)^\dagger = (GH)^{-1}$. Thus,

$$(GH)^\dagger = (GH)^D = (GH)^2 = (GH)^{-1}.$$  

(b) If $GH = HG = 0$, then $(G + H)^2 = (G + H)^\dagger$ and $H_1 = H_2 = H_3 = 0$ implies

$$G + H = \left[ \begin{array}{cc} G_1 & 0 \\ 0 & H_4 \end{array} \right], \quad (G + H)^\dagger = \left[ \begin{array}{cc} G_1^\dagger & 0 \\ 0 & H_4^\dagger \end{array} \right].$$
From \((G+H)(G+H)^\dagger = P_{R(G+H)} = P_{R(G)} + P_{R(H)}\) and
\[
(G+H)(G+H)^\dagger = \begin{bmatrix}
G_1G_1^\dagger & 0 \\
0 & H_4H_4^\dagger
\end{bmatrix} = \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}
\]
we conclude that \(G_1^\dagger = G_1^{-1}, \ H_4^\dagger = H_4^{-1}\) and \((G+H)^\dagger = (G+H)^{-1}\).

(c) Similarly to (b). \(\square\)

References


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