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Pairs of projections on a Hilbert space: properties and generalized invertibility

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*To Aleksa, my own Oneiros,
for passing through the gates of horn and ivory with me*

Abstract

This thesis is concerned with the problem of characterizing sums, differences, and products of two projections on a separable Hilbert space. Other objective is characterizing the Moore-Penrose and the Drazin inverse for pairs of operators. We use reasoning similar to one presented in the famous P. Halmos' two projection theorem: using matrix representation of two orthogonal projection depending on the relations between their ranges and null-spaces gives us simpler form of their matrices and allows us to involve matrix theory in solving problems. We extend research to idempotents, generalized and hypergeneralized projections and their combinations.

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Linköping, March 12, 2012

Sonja Radosavljević

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Introduction

In this thesis, we want to examine some properties of sums, differences, and products of two operators on Hilbert space. We start with the orthogonal projections, i.e., operators such that $P = P^2 = P^*$, but we also extend research to idempotent operator P (which satisfy $P = P^2$), generalized projections (for which $P^* = P^2$) and hypergeneralized projections (satisfying $P^\dagger = P^2$, where P^\dagger is the Moore-Penrose inverse of operator P). The method comes from P. Halmos' two projection theorem, (see [16]), stating that for two orthogonal projections P and Q on a finite dimensional Hilbert space H there exists suitable matrix representation in accordance to the relations between their ranges and null-spaces. We generalize this approach to the wider classes of operators and also to infinite dimensional Hilbert spaces.

As we know, every Hilbert space can be represented as a direct sum of two orthogonal subspaces, i.e.,

$$H = L \oplus L^\perp,$$

where L is arbitrary closed subspace of H . If P is an orthogonal projection and $\mathcal{R}(P) = \{Px : x \in H\}$ its range and $\mathcal{N}(P) = \{x \in H : Px = 0\}$ its null-space, then from $\mathcal{R}(P)^\perp = \mathcal{N}(P^*)$ and $\mathcal{R}(P^*)^\perp = \mathcal{N}(P)$ we get

$$H = \mathcal{R}(P) \oplus \mathcal{N}(P).$$

The matrix representation of P then becomes

$$P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(P) \\ \mathcal{N}(P) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(P) \\ \mathcal{N}(P) \end{bmatrix}, \quad (1)$$

and if we use the fact that $P_1 : \mathcal{R}(P) \rightarrow \mathcal{R}(P)$, $P_2 : \mathcal{N}(P) \rightarrow \mathcal{R}(P)$, $P_3 : \mathcal{R}(P) \rightarrow \mathcal{N}(P)$ and $P_4 : \mathcal{N}(P) \rightarrow \mathcal{N}(P)$, we conclude that $P_1 = I_{\mathcal{R}(P)}$ and $P_2 = P_3 = P_4 = 0$. If we denote $\mathcal{R}(P) = L$, then

$$P = \begin{bmatrix} I_L & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \rightarrow \begin{bmatrix} L \\ L^\perp \end{bmatrix}. \quad (2)$$

We can see now that representation for Q under the same decomposition of the space is

$$Q = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \rightarrow \begin{bmatrix} L \\ L^\perp \end{bmatrix}, \quad (3)$$

where A and D are self-adjoint operators.

What Halmos theorem says, and what we presented here in the simplified form, is that if there are two orthogonal projections on Hilbert space, one of them can be used to generate “coordinates”. Expressing everything in the terms of the orthogonal projection P gives us wanted coordinates and we are usually left with straightforward computations. (Similar relations can be seen between analytical and Euclidean geometry: coordinate system can make things easier.) We are now able to further discuss properties of the operators A , B and D as well as properties of the sums, differences, and products of the projections P and Q by studying their matrices. An advantage of using this method lies in the fact that the projection P is in its simplest form in (2). Any other representation of P would have a more complicated form and would lead to a more complicated form of other operators in which P appears.

The second and perhaps more direct influence to our work comes from D. Djordjević and J. Koliha, (see [11], [12]), who gave matrix representation of a closed range operator $A \in \mathcal{L}(H)$ on infinite dimensional Hilbert space H depending on the different decomposition of the space. For the proof of the following lemma see [12].

Lemma 0.1

Let $A \in \mathcal{L}(H)$ be a closed range operator. Then the operator A has the following three matrix representations with respect to the orthogonal sums of subspaces $H = \mathcal{R}(A) \oplus \mathcal{N}(A^) = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$:*

(a) *We have*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where A_1 is invertible.

(b) *We have*

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $B = A_1A_1^ + A_2A_2^*$ maps $\mathcal{R}(A)$ onto itself and $B > 0$.*

(c) *Alternatively,*

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix},$$

*where $B = A_1^*A_1 + A_2^*A_2$ maps $\mathcal{R}(A^*)$ onto itself and $B > 0$.*

The operators A_i are different in all three cases.

Here we again use the fact that $\mathcal{R}(A)^\perp = \mathcal{N}(A^*)$ and $\mathcal{R}(A^*)^\perp = \mathcal{N}(A)$. Hence,

$$H = \mathcal{R}(A) \oplus \mathcal{N}(A^*) = \mathcal{R}(A^*) \oplus \mathcal{N}(A).$$

Like in the case with orthogonal projections, here is obtained simpler form of operator A with two generalizations: operator is in the infinite dimensional settings and it is closed range operator, not necessarily orthogonal projection. Three different forms are the result of different decompositions of the space.

Based on the previous lemma and results that deal with two operators and the suitable decomposition of the underlying Hilbert space, we now shift to specific operator classes. Since the literature on two projection theory is quite vast (see [7] for introduction to the topic), our idea is to extend research to pairs of idempotents, generalized and hypergeneralized projections and their combinations. J. Gross and G. Trenkler in [15] and J. K. Baksalary, O. M. Baksalary, X. Liu and G. Trenkler in [2], [3], [4] examined some properties of such operators on $C^{m \times n}$. We are concerned with the same classes of operators. However, the method from [12] and [11] gives us the opportunity to use matrix representation of these operators on an infinite dimensional Hilbert space and to involve matrix theory in solving problems coming from operator theory. Note that EP operators are those satisfying $\mathcal{R}(A) = \mathcal{R}(A^*)$. Denote by $\mathcal{OP}(H)$, $\mathcal{GP}(H)$, $\mathcal{HGP}(H)$, and $\mathcal{EP}(H)$ classes of orthogonal, generalized and hypergeneralized projections and EP operators, respectively. Then,

$$\mathcal{OP}(H) \subset \mathcal{GP}(H) \subset \mathcal{HGP}(H) \subset \mathcal{EP}(H)$$

and generalization is justified.

Generalized and hypergeneralized projections do not have all the properties of orthogonal projections, so it is important to know what properties they have and under what conditions their product, sum and difference keep them. Among many properties that an operator can have, we were especially interested in generalized invertibility.

Let $A \in \mathcal{L}(H)$ and $b \in H$. Consider the equation

$$Ax = b.$$

If A is invertible, then $A^{-1}b$ is the unique solution to the equation. If A is not invertible and $b \in \mathcal{R}(A)$, then there exists solution (possible more than one). If $b \notin \mathcal{R}(A)$, then there are no solutions. However, it is possible to obtain generalized solutions by using the Moore-Penrose inverse of A (see [1], [8], [13] for more details and proofs), denoted by A^\dagger and defined as the unique solution to the equations

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$

It turns out that the least square solution of the linear equation $Ax = b$ is $x_0 = A^\dagger b$, i.e.,

$$\|Ax_0 - b\| = \min_x \|Ax - b\|.$$

Moreover, if M is the set of all least square solutions of the linear equation $Ax = b$, then x_0 is the minimal norm least square solution, i.e.,

$$\|x_0\| = \min_{x \in M} \|x\|.$$

Depending on applications, sometimes we can give up the equation solving properties of the Moore-penrose inverse in exchange for commutativity or some spectral property

that the ordinary inverse possess. Thus we come to Drazin inverse: for $A \in \mathcal{L}(H)$, A^D is the Drazin inverse if

$$A^D A A^D = A^D, \quad A^D A = A A^D, \quad A^{n+1} A^D = A^n,$$

for some non-negative integer n . The smallest such n is called the Drazin index of A .

Several authors have results on Drazin inverse of the sum and difference of idempotents (see [9], [10]). Using an algebraic approach, we are able to either simplify existing proofs or to give some new characterizations of the Moore-Penrose and the Drazin inverse of sums, differences and products of projections, generalized and hypergeneralized projections. Similarly to Lemma (0.1), we find canonical forms of the Moore-Penrose inverse of a closed range operator A depending on decomposition of the space. So, if $A \in \mathcal{L}(H)$ is a closed range operator and $H = \mathcal{R}(A^*) \oplus \mathcal{N}(A) = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$, then we have three different representations of operator and its inverse:

(a)

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}, \quad A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

(b)

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}, \quad A^\dagger = \begin{bmatrix} A_1^* B^{-1} & 0 \\ A_2^* B^{-1} & 0 \end{bmatrix},$$

where $B = A_1 A_1^* + A_2 A_2^*$ maps $\mathcal{R}(A)$ onto itself and $B > 0$.

(c)

$$A = \begin{bmatrix} A_3 & 0 \\ A_4 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}, \quad A^\dagger = \begin{bmatrix} C^{-1} A_3^* & C^{-1} A_4^* \\ 0 & 0 \end{bmatrix},$$

where $C = A_3^* A_3 + A_4^* A_4$ maps $\mathcal{R}(A^*)$ onto itself and $C > 0$.

We can use these representations for solving various problems. The algebraic character of the method provides transparency of the proofs. As an illustration of the method, we state the following theorem.

Theorem 0.1

Let $K \in \mathcal{L}(H)$. Then K is idempotent if and only if it is expressible in the form $K = (PQ)^\dagger$ for some orthogonal projections $P, Q \in \mathcal{L}(H)$. Moreover, $K = QKP$.

Proof: If K is idempotent, $\mathcal{R}(K) = L$ and $H = L \oplus L^\perp$, then it has the form:

$$K = \begin{bmatrix} K_1 & 0 \\ K_2 & 0 \end{bmatrix},$$

where $K_1^2 = K_1$ and K_2 is arbitrary. It is easy to see that

$$P = \begin{bmatrix} K_1^\dagger K_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} K_1 K_1^\dagger & 0 \\ 0 & 0 \end{bmatrix}$$

are wanted orthogonal projections.

Conversely, suppose that we have the representations (2) and (3) for the orthogonal projections P and Q . From formula $(PQ)^\dagger = (PQ)^*(PQ(PQ)^*)^\dagger$, we obtain

$$(PQ)^\dagger = \begin{bmatrix} AA^\dagger & 0 \\ B^*A^\dagger & 0 \end{bmatrix}$$

and a direct verification shows that $(PQ)^\dagger$ is an idempotent. □

Summary of papers

Two papers are included in thesis. Below is a short summary for each of the papers.

Paper A: On pairs of generalized and hypergeneralized projections on a Hilbert space

In this paper, we characterize generalized and hypergeneralized projections (bounded linear operators which satisfy conditions $A^2 = A^*$ and $A^2 = A^\dagger$). We give their matrix representations and examine under what conditions the product, difference and sum of these operators are operators of the same class.

Paper B: On the Moore-Penrose and the Drazin inverse of two projections on a Hilbert space

For two given orthogonal, generalized or hypergeneralized projections P and Q on a Hilbert space H , we give their matrix representation. We also give canonical forms of the Moore-Penrose and the Drazin inverses of their product, difference, and sum. In addition, we provide results showing when these operators are EP operators and some simple relationships between the mentioned operators are established.

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Paper A

On pairs of generalized and hypergeneralized projections on a Hilbert space

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On pairs of generalized and hypergeneralized projections on a Hilbert space

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Abstract

In this paper, we characterize generalized and hypergeneralized projections (bounded linear operators which satisfy conditions $A^2 = A^*$ and $A^2 = A^\dagger$). We give their matrix representations and examine under what conditions the product, difference and sum of these operators are operators of the same class.

Keywords: Generalized projections, hypergeneralized projections.

1 Introduction

Let H be a separable Hilbert space and $\mathcal{L}(H)$ be a space of all bounded linear operators on H . The symbols $\mathcal{R}(A)$, $\mathcal{N}(A)$ and A^* denote range, null space and adjoint operator of operator $A \in \mathcal{L}(H)$. Operator $A \in \mathcal{L}(H)$ is a projection (idempotent) if $A^2 = A$, while it is an orthogonal projection if $A^* = A = A^2$. Operator is hermitian (self adjoint) if $A = A^*$, normal if $AA^* = A^*A$ and unitary if $AA^* = A^*A = I$. All these operators have been extensively studied and there are plenty of characterizations both of these operators and their linear combinations ([5]).

The Moore-Penrose inverse of $A \in \mathcal{L}(H)$, denoted by A^\dagger , is the unique solution to the equations

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$

Notice that A^\dagger exists if and only if $\mathcal{R}(A)$ is closed. Then AA^\dagger is the orthogonal projection onto $\mathcal{R}(A)$ parallel to $\mathcal{N}(A^*)$, and $A^\dagger A$ is the orthogonal projection onto $\mathcal{R}(A^*)$ parallel to $\mathcal{N}(A)$. Consequently, $I - AA^\dagger$ is the orthogonal projection onto $\mathcal{N}(A^*)$ and $I - A^\dagger A$ is the orthogonal projection onto $\mathcal{N}(A)$.

For $A \in \mathcal{L}(H)$, an element $B \in \mathcal{L}(H)$ is the Drazin inverse of A , if the following hold:

$$BAB = B, \quad BA = AB, \quad A^{n+1}B = A^n,$$

for some non-negative integer n . The smallest such n is called the Drazin index of A . By A^D we denote Drazin inverse of A and by $ind(A)$ we denote Drazin index of A .

If such n does not exist, $ind(A) = \infty$ and operator A is generalized Drazin invertible. Its invers is denoted by A^d .

Operator A is invertible if and only if $ind(A) = 0$.

If $ind(A) \leq 1$, operator A is group invertible and A^D is its group inverse, usually denoted by $A^\#$.

Notice that if the Drazin inverse exists, it is unique. Drazin inverse exists if $\mathcal{R}(A^n)$ is closed for some non-negative integer n .

Operator $A \in \mathcal{L}(H)$ is a partial isometry if $AA^*A = A$ or, equivalently, if $A^\dagger = A^*$. Operator $A \in \mathcal{L}(H)$ is EP if $AA^\dagger = A^\dagger A$, or, in the other words, if $A^\dagger = A^D = A^\#$. Set of all EP operators on H will be denoted by $\mathcal{EP}(H)$. Self-adjoint and normal operators with closed range are important subset of set of all EP operators. However, converse is not true even in a finite dimensional case.

In this paper we consider pairs of generalized and hypergeneralized projections on a Hilbert space, whose concept for matrices $A \in C^{m \times n}$ was introduced by J. Gross and G. Trenkler in [6]. These operators extend the idea of orthogonal projections by deleting the idempotency requirement. Namely, we have the following definition:

Definition A.1. Operator $A \in \mathcal{L}(H)$ is

- (a) a generalized projection if $A^2 = A^*$;
- (b) a hypergeneralized projection if $A^2 = A^\dagger$.

The set of all generalized projection on H is denoted by $\mathcal{GP}(H)$ and the set of all hypergeneralized projection is denoted by $\mathcal{HGP}(H)$.

2 Characterization of generalized and hypergeneralized projections

We begin this section by giving characterizations of generalized and hypergeneralized projections. Similarly to Theorem 2 in [4] and Theorem 1 in [6], we have the following result:

Theorem A.1

Let $A \in \mathcal{L}(H)$. Then the following conditions are equivalent:

- (a) A is a generalized projection.
- (b) A is a normal operator and $A^4 = A$.
- (c) A is a partial isometry and $A^4 = A$.

Proof: (a \Rightarrow b) Since

$$AA^* = AA^2 = A^3 = A^2A = A^*A,$$

$$A^4 = (A^2)^2 = (A^*)^2 = (A^2)^* = (A^*)^* = A,$$

the implication is obvious.

(b \Rightarrow a) If $AA^* = A^*A$, recall that then exists a unique spectral measure E on the Borel subsets of $\sigma(A)$ such that $E(\Delta)$ is an orthogonal projection for every subset $\Delta \subset \sigma(A)$, $E(\emptyset) = 0$, $E(H) = I$ and if $\Delta_i \cap \Delta_j = \emptyset$ for $i \neq j$, then $E(\cup \Delta_i) = E(\sum \Delta_i)$. Moreover, A has the following spectral representation

$$A = \int_{\sigma(A)} \lambda dE_\lambda,$$

where $E_\lambda = E(\lambda)$ is the spectral projection associated with the point $\lambda \in \sigma(A)$.

From $A^4 = A$, we conclude $A^3 \mathcal{R}(A) = I_{\mathcal{R}(A)}$ and $\lambda^3 = 1$, or, equivalently $\sigma(A) \subseteq \{0, 1, e^{2\pi i/3}, e^{-2\pi i/3}\}$. Now,

$$A = 0E(0) \oplus 1E(1) \oplus e^{2\pi i/3}E(e^{2\pi i/3}) \oplus e^{-2\pi i/3}E(e^{-2\pi i/3}),$$

where $E(\lambda)$ is the spectral projection of A associated with the point $\lambda \in \sigma(A)$ such that $E(\lambda) \neq 0$ if $\lambda \in \sigma(A)$, $E(\lambda) = 0$ if $\lambda \in \{0, 1, e^{2\pi i/3}, e^{-2\pi i/3}\} \setminus \sigma(A)$ and $E(0) \oplus E(1) \oplus E(e^{2\pi i/3}) \oplus E(e^{-2\pi i/3}) = I$. From the fact that $\sigma(A^2) = \sigma(A^*)$ and from uniqueness of spectral representation, we get $A^2 = A^*$.

(a \Rightarrow c) If $A^* = A^2$, then $A^4 = AA^2A = AA^*A = A$. Multiplying from the left (from the right) by A^* , we get $A^*AA^*A = A^*A(AA^*AA^* = AA^*)$, which proves that $A^*A(AA^*)$ is the orthogonal projection onto $\mathcal{R}(A^*A) = \mathcal{R}(A^*) = \mathcal{N}(A)^\perp$ ($\mathcal{R}(AA^*) = \mathcal{R}(A) = \mathcal{N}(A^*)^\perp$), i.e., $A(A^*)$ is a partial isometry.

(c \Rightarrow a) If A is a partial isometry, we know that AA^* is orthogonal projection onto $\mathcal{R}(AA^*) = \mathcal{R}(A)$. Thus, $AA^*A = P_{\mathcal{R}(A)}A = A$ and $A^4 = AA^2A = A$ implies $A^2 = A^*$. \square

We give matrix representatons of generalized projections based on the previous characterizations.

Theorem A.2

Let $A \in \mathcal{L}(H)$ be a generalized projection. Then A is a closed range operator and A^3 is an orthogonal projection on $\mathcal{R}(A)$. Moreover, H has decomposition

$$H = \mathcal{R}(A) \oplus \mathcal{N}(A)$$

and A has the following matrix representaton

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix},$$

where the restriction $A_1 = A|_{\mathcal{R}(A)}$ is unitary on $\mathcal{R}(A)$.

Proof: If A is a generalized projection, then $AA^*A = A^4 = A$ and A is a partial isometry implying that

$$A^3 = AA^* = P_{\mathcal{R}(A)},$$

$$A^3 = A^*A = P_{\mathcal{N}(A)^\perp}.$$

Thus, A^3 is an orthogonal projection onto $\mathcal{R}(A) = \mathcal{N}(A)^\perp = \mathcal{R}(A^*)$. Consequently, $\mathcal{R}(A)$ is a closed subset in H as a range of an orthogonal projection on a Hilbert space. From Lemma (1.2) in [5] we get the following decomposition of the space

$$H = \mathcal{R}(A^*) \oplus \mathcal{N}(A) = \mathcal{R}(A) \oplus \mathcal{N}(A).$$

Now, A has the following matrix representation in accordance with this decomposition:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix},$$

where $A_1^2 = A_1^*$, $A_1^4 = A_1$ and $A_1A_1^* = A_1^*A_1 = A_1^3 = I_{\mathcal{R}(A)}$. □

Similar to Theorem 2 in [6], we have:

Theorem A.3

Let $A \in \mathcal{L}(H)$. Then the following conditions are equivalent:

- (a) A is a hypergeneralized projecton.
- (b) A^3 is an orthogonal projection onto $\mathcal{R}(A)$.
- (c) A is an EP operator and $A^4 = A$

Proof: (a \Rightarrow b) If $A^2 = A^\dagger$, then from $A^3 = AA^\dagger = P_{\mathcal{R}(A)}$ follows conclusion.

(b \Rightarrow a) If $A^3 = P_{\mathcal{R}(A)}$, a direct verificaton of the Moore-Penrose equations shows that $A^2 = A^\dagger$.

(a \Rightarrow c) Since

$$AA^\dagger = AA^2 = A^3 = A^2A = A^\dagger A,$$

we conclude that A is EP operator, $A^\dagger = A^\#$, $(A^\dagger)^n = (A^n)^\dagger$ and

$$A^4 = (A^2)^2 = (A^\dagger)^2 = (A^2)^\dagger = (A^\dagger)^\dagger = A.$$

(c \Rightarrow a) If A is an EP operator, then $A^\dagger = A^\#$ and $\text{ind}(A) = 1$ or, equivalently, $A^2 A^\dagger = A$. Since $A^4 = A^2 A^2 = A$, from uniqueness of A^\dagger follows $A^2 = A^\dagger$. \square

Theorem A.4

Let $A \in \mathcal{L}(H)$ be a hypergeneralized projection. Then A is a closed range operator and H has decomposition

$$H = \mathcal{R}(A) \oplus \mathcal{N}(A).$$

Also, A has the following matrix representaton with the respect to decomposition of the space

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix},$$

where restriction $A_1 = A|_{\mathcal{R}(A)}$ satisfies $A_1^3 = I_{\mathcal{R}(A)}$.

Proof: If A is a hypergeneralized projecton, then A is EP operator, and using Lemma (1.2) in [5], we get the following decomposition of the space $H = \mathcal{R}(A) \oplus \mathcal{N}(A)$ and A has the required representation. \square

Notice that since $\mathcal{R}(A)$ is closed for both generalized and hypergeneralized projections, these operators have the Moore-Penrose and Drazin inverses. Besides, they are EP operators, which implies that $A^\dagger = A^D = A^\# = A^2$. For generalized projections we can be more precise:

$$A^\dagger = A^D = A^\# = A^2 = A^*.$$

We can also write

$$\mathcal{GP}(H) \subseteq \mathcal{HGP}(H) \subseteq \mathcal{EP}(H).$$

Parts (b) and (c) of the following two theorems are known for matrices $A \in C^{m \times n}$, (see [2], [3]). Unlike their proof, which is based on representation of matrices, our proof relies only on properties of generalized and hypergeneralized projections and basic properties of the Moore-Penrose and the group inverse.

Theorem A.5

Let $A \in \mathcal{L}(H)$. Then the following holds:

- (a) $A \in \mathcal{GP}(H)$ if and only if $A^* \in \mathcal{GP}(H)$.
- (b) $A \in \mathcal{GP}(H)$ if and only if $A^\dagger \in \mathcal{GP}(H)$.
- (c) If $\text{ind}(A) \leq 1$, then $A \in \mathcal{GP}(H)$ if and only if $A^\# \in \mathcal{GP}(H)$.

Proof: (a) If $A \in \mathcal{GP}(H)$, then

$$(A^*)^* = A = A^4 = (A^2)^2 = (A^*)^2,$$

meaning that $A^* \in \mathcal{GP}(H)$. Conversely, if $A^* \in \mathcal{GP}(H)$, then $(A^*)^4 = A^*$ and $(A^*)^2 = (A^*)^* = A$, implying

$$A^2 = (A^*)^4 = A^*$$

and $A \in \mathcal{GP}(H)$.

(b) If $A \in \mathcal{GP}(H)$, then $A^\dagger = A^* = A^2$ and

$$(A^\dagger)^2 = (A^2)^2 = A = (A^*)^* = (A^\dagger)^*,$$

implying $A^\dagger \in \mathcal{GP}$.

If $A^\dagger \in \mathcal{GP}(H)$, then $(A^\dagger)^2 = (A^\dagger)^* = (A^\dagger)^\dagger = A$ and $(A^\dagger)^4 = A^\dagger$. Thus,

$$A^2 = (A^\dagger)^4 = A^\dagger,$$

$$A^* = ((A^\dagger)^*)^* = A^\dagger$$

and $A \in \mathcal{GP}(H)$.

(c) If $A \in \mathcal{GP}(H)$, then $A^\dagger = A^\#$ and part (b) of this theorem implies that $A^\# \in \mathcal{GP}(\mathcal{H})$.

To prove converse, it is enough to see that $A^\# \in \mathcal{GP}(H)$ implies $(A^\#)^2 = (A^\#)^* = (A^\#)^\dagger = (A^\#)^\# = A$ and $(A^\#)^4 = A^\#$ and

$$A^2 = (A^\#)^4 = A^\# = ((A^\#)^*)^* = A^*.$$

□

Remark A.1. Let us mention an alternative proof for the previous theorem. If $A^\dagger \in \mathcal{GP}(H)$, then A is normal and $\mathcal{R}(A)$ is closed and A, A^\dagger have representations

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}, \quad A^\dagger = \begin{bmatrix} A_1^* B & 0 \\ A_2^* B & 0 \end{bmatrix},$$

where $B = (A_1 A_1^* + A_2 A_2^*)^{-1}$. From $(A^\dagger)^2 = (A^\dagger)^*$, we get

$$\begin{bmatrix} A_1^* B A_1^* B & 0 \\ A_2^* B A_1^* B & 0 \end{bmatrix} = \begin{bmatrix} B A_1 & B A_2 \\ 0 & 0 \end{bmatrix},$$

which implies $A_2 = 0$, $A_2^* = 0$ and $B = (A_1 A_1^*)^{-1}$ and

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $(A_1^{-1})^2 = (A_1^{-1})^*$, the same equality is also satisfied for A_1 and $A \in \mathcal{GP}(\mathcal{H})$.

Similarly, to prove that $A^\# \in \mathcal{GP}(H)$ implies $A \in \mathcal{GP}(H)$, assume that $H = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$. Then $A^\# = A^\dagger$

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}, \quad A^\# = \begin{bmatrix} A_1^\# & (A_1^\#)^2 A_2 \\ 0 & 0 \end{bmatrix}.$$

Since $(A^\#)^2 = (A^\#)^*$, we get $A_2 = 0$ and $(A_1^\#)^2 = (A_1^\#)^*$. From the fact that A_1 is surjective mapping on $\mathcal{R}(A)$ and $\mathcal{R}(A_1) \cap \mathcal{N}(A_1) = \{0\}$, we have $A_1^\# = A_1^{-1}$. Consequently, $(A_1^{-1})^2 = (A_1^{-1})^*$ and $A_1^2 = A_1^*$, which proves that $A \in \mathcal{GP}(H)$.

Theorem A.6

Let $A \in \mathcal{L}(H)$. Then the following holds:

- (a) $A \in \mathcal{HGP}(H)$ if and only if $A^* \in \mathcal{HGP}(H)$.
- (b) $A \in \mathcal{HGP}(H)$ if and only if $A^\dagger \in \mathcal{HGP}(H)$.
- (c) If $\text{ind}(A) \leq 1$, then $A \in \mathcal{HGP}(H)$ if and only if $A^\# \in \mathcal{HGP}(H)$.

Proof: Proofs of (a) and (b) are similar to proofs of Theorem A.5 (a) and (b).

(c) We should only prove that $A^\# \in \mathcal{HGP}(H)$ implies $A \in \mathcal{HGP}(H)$, since the " \Rightarrow " is analogous to the same part of Theorem A.5.

Let $H = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$ and $\text{ind}(A) \leq 1$. Then

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}, \quad A^\# = \begin{bmatrix} A_1^{-1} & (A_1^{-1})^2 A_2 \\ 0 & 0 \end{bmatrix}, \quad (A^\#)^\dagger = \begin{bmatrix} (A_1^{-1})^* B & 0 \\ (A_2^{-1})^* B & 0 \end{bmatrix},$$

where $B = (A_1^{-1}(A_1^{-1})^* + A_2^{-1}(A_2^{-1})^*)^{-1}$. From $(A^\#)^\dagger = (A^\#)^2$, we get $A_2 = 0$ and $A_1 = A_1^{-2}$. Multiplying with A_1^2 , the last equation becomes $A_1^3 = I_{\mathcal{R}(A)}$ and $A \in \mathcal{HGP}(H)$. \square

3 Properties of products, differences, and sums of generalized projections

In this section we study properties of products, differences, and sums of two generalized projections or of one orthogonal and one generalized projection. We begin with two very useful theorems which gives matrix representations of such pairs of operators. Also, we obtain basic properties of generalized projections which easily follow from their canonical representations.

Theorem A.7

Let $A, B \in \mathcal{GP}(H)$ and $H = \mathcal{R}(A) \oplus \mathcal{N}(A)$. Then A and B has the following representation with respect to decomposition of the space:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix},$$

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix},$$

where

$$\begin{aligned} B_1^* &= B_1^2 + B_2 B_3, \\ B_2^* &= B_3 B_1 + B_4 B_3, \\ B_3^* &= B_1 B_2 + B_2 B_4, \\ B_4^* &= B_3 B_2 + B_4^2. \end{aligned}$$

Furthermore, $B_2 = 0$ if and only if $B_3 = 0$.

Proof: Let $H = \mathcal{R}(A) \oplus \mathcal{N}(A)$. Then representation of A follows from Theorem A.2 and let B has representation

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}.$$

Then, from

$$B^2 = \begin{bmatrix} B_1^2 + B_2B_3 & B_1B_2 + B_2B_4 \\ B_3B_1 + B_4B_3 & B_3B_2 + B_4^2 \end{bmatrix} = \begin{bmatrix} B_1^* & B_3^* \\ B_2^* & B_4^* \end{bmatrix} = B^*,$$

the conclusion follows directly.

If $B_2 = 0$, then $B_3^* = B_1B_2 + B_2B_4 = 0$ and $B_3 = 0$. Analogously, $B_3 = 0$ implies $B_2 = 0$. \square

Corollary A.1

Under the assumptions of the previous theorem, operator $B \in \mathcal{GP}(H)$ has one of the following matrix representations:

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \quad \text{or} \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_4 \end{bmatrix}.$$

Theorem A.8

Let $P \in \mathcal{B}(H)$ be an orthogonal projection, $\mathcal{R}(P) = L$ and $H = L \oplus L^\perp$. If $A \in \mathcal{GP}(H)$, then P and A has the following matrix representation with the respect to decomposition of the space

$$P = \begin{bmatrix} I_L & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \rightarrow \begin{bmatrix} L \\ L^\perp \end{bmatrix},$$

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \rightarrow \begin{bmatrix} L \\ L^\perp \end{bmatrix},$$

where

$$\begin{aligned} A_1^* &= A_1^2 + A_2A_3, \\ A_2^* &= A_3A_1 + A_4A_3, \\ A_3^* &= A_1A_2 + A_2A_4, \\ A_4^* &= A_3A_2 + A_4^2. \end{aligned}$$

Moreover,

$$\begin{aligned} A_1 &= PAP|_L, \\ A_2 &= PA(I - P)|_{L^\perp}, \\ A_3 &= (I - P)AP|_L, \\ A_4 &= (I - P)A(I - P)|_{L^\perp}. \end{aligned}$$

Then $A_2 = 0$ if and only if $A_3 = 0$, or equivalently, $PA(I - P)|_{L^\perp} = 0$ if and only if $(I - P)AP|_L = 0$.

Opeators P and A commute if and only if either $A_2 = 0$ or $A_3 = 0$, or equivalently, if and only if either $PA(I - P)|_{L^\perp} = 0$ or $(I - P)AP|_L = 0$.

Proof: Matrix representation of $A \in \mathcal{GP}(H)$ and properties of A_i , $i = \overline{1,4}$ can be obtained like in the proof of Theorem A.7.

Using matrix multiplication, it is easy to see that

$$PAP = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$$

and $PAP|_L = A_1$. The rest of the equalities are obtained in an analogous way.

If $PA = AP$, again using matrix multiplication, we get $A_2 = A_3 = 0$ which is equivalent to $PA(I - P)|_{L^\perp} = 0$ or $(I - P)AP|_L = 0$. \square

Corollary A.2

Under the assumptions of the previous theorem, operator $A \in \mathcal{GP}(H)$ has the following matrix representations:

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

when P and A do not commute, or

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix},$$

when P and A commute.

As we know, if A is an orthogonal projection, $I - A$ is also an orthogonal projection. It is of interest to examine whether generalized projections keep the same property.

Example A.1

If $H = \mathbb{C}^2$ and $A = \begin{bmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & 0 \end{bmatrix}$, then $A^2 = A^*$, but $I - A = \begin{bmatrix} 1 - e^{\frac{2\pi i}{3}} & 0 \\ 0 & 1 \end{bmatrix}$ and, clearly, $I - A \neq (I - A)^4$ implying that $I - A$ is not a generalized projection.

Thus, we have the following theorem.

Theorem A.9

(Theorem 6 in [2]) Let $A \in \mathcal{L}(H)$ be a generalized projection. Then $I - A$ is a normal operator. Moreover, $I - A$ is a generalized projection if and only if A is an orthogonal projection.

If $I - A$ is a generalized projection, then A is a normal operator and A is a generalized projection if and only if $I - A$ is an orthogonal projection.

Proof: If A is a generalized projection, then A is a normal operator and $A^4 = A$, which implies

$$A = 0E(0) \oplus 1E(1) \oplus e^{2\pi i/3}E(e^{2\pi i/3}) \oplus e^{-2\pi i/3}E(e^{-2\pi i/3}),$$

where $E(\lambda)$ is the orthogonal projection such that $E(\lambda) \neq 0$ if $\lambda \in \sigma(A)$, $E(\lambda) = 0$ if $\lambda \in \{0, 1, e^{2\pi i/3}, e^{-2\pi i/3}\} \setminus \sigma(A)$ and $E(0) \oplus E(1) \oplus E(e^{2\pi i/3}) \oplus E(e^{-2\pi i/3}) = I$. Thus,

$$I - A = (1 - 0)E(0) \oplus (1 - 1)E(1) \oplus (1 - e^{2\pi i/3})E(e^{2\pi i/3}) \oplus (1 - e^{-2\pi i/3})E(e^{-2\pi i/3}),$$

and

$$(I - A)^2 = E(0) \oplus (1 - e^{2\pi i/3})^2 E(e^{2\pi i/3}) \oplus (1 - e^{-2\pi i/3})^2 E(e^{-2\pi i/3}),$$

$$(I - A)^* = E(0) \oplus (1 - e^{2\pi i/3})^* E(e^{2\pi i/3}) \oplus (1 - e^{-2\pi i/3})^* E(e^{-2\pi i/3}).$$

Hence,

$$(I - A)^2 = (I - A)^*$$

if and only if

$$(1 - e^{2\pi i/3})^2 E(e^{2\pi i/3}) = (1 - e^{2\pi i/3})^* E(e^{2\pi i/3})$$

and

$$(1 - e^{-2\pi i/3})^2 E(e^{-2\pi i/3}) = (1 - e^{-2\pi i/3})^* E(e^{-2\pi i/3}).$$

This is true if and only if $E(e^{2\pi i/3}) = 0$ and $E(e^{-2\pi i/3}) = 0$, which is equivalent to $\sigma(A) = \{0, 1\}$ and A is an orthogonal projection. \square

Remark A.2. We can give another proof for this theorem. Let $H = \mathcal{R}(A) \oplus \mathcal{N}(A)$ and according to Theorem A.7 generalized projection A has representation

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix}.$$

Then

$$I - A = \begin{bmatrix} I_{\mathcal{R}(A)} - A_1 & 0 \\ 0 & I_{\mathcal{N}(A)} \end{bmatrix}$$

and it is obvious that normality of A implies normality of $I - A$. Also,

$$(I - A)^2 = \begin{bmatrix} (I_{\mathcal{R}(A)} - A_1)^2 & 0 \\ 0 & I_{\mathcal{N}(A)} \end{bmatrix} = \begin{bmatrix} (I_{\mathcal{R}(A)} - A_1)^* & 0 \\ 0 & I_{\mathcal{N}(A)} \end{bmatrix} = (I - A)^*$$

holds if and only if $(I_{\mathcal{R}(A)} - A_1)^2 = (I_{\mathcal{R}(A)} - A_1)^*$. Since $A^2 = A^*$, we get

$$I_{\mathcal{R}(A)} - 2A_1 + A_1^2 = I_{\mathcal{R}(A)} - 2A_1 + A_1^* = I_{\mathcal{R}(A)} - A_1^*,$$

which is satisfied if and only if $A_1 = A_1^*$. Hence, $A = A^* = A^2$.

Theorem A.10

If P is an orthogonal projection and A is a generalized projection, then AP is a generalized projection if and only if either $PA(I - P) = 0$ or $(I - P)AP = 0$.

Proof: Let $\mathcal{R}(P) = L$ and $H = L \oplus L^\perp$. Then

$$P = \begin{bmatrix} I_L & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}.$$

From

$$AP = \begin{bmatrix} A_1 & 0 \\ A_3 & 0 \end{bmatrix}, \quad (AP)^2 = \begin{bmatrix} A_1^2 & 0 \\ A_3 A_1 & 0 \end{bmatrix}, \quad (AP)^* = \begin{bmatrix} A_1^* & A_3^* \\ 0 & 0 \end{bmatrix}$$

we conclude that $(AP)^2 = (AP)^*$ if and only if $A_3 = 0$, which is equivalent to $(I - P)AP = 0$.

Theorem A.7 provide us with the condition that $A_3 = 0$ if and only if $A_2 = 0$, or, in the other words $(I - P)AP = 0$ if and only if $PA(I - P) = 0$. \square

Theorem A.11

Let P be an orthogonal projection and let A be a generalized projection. Then $P - A$ is a generalized projection if and only if A is an orthogonal projection commuting with P .

Furthermore, if P is an orthogonal projection and A is a generalized projection such that PAP is orthogonal projection and either $PA(I - P) = 0$ or $(I - P)AP = 0$, then $P - A$ is a generalized projection.

Proof: Obviously $(P - A)^2 = P - PA - AP + A^2 = P^* - A^* = (P - A)^*$ if and only if $PA = AP = A^*$. Since $A^2 = A^*$, we conclude that A is an orthogonal projection.

To prove the second part of the theorem, let $\mathcal{R}(P) = L$ and $H = L \oplus L^\perp$. If $PA(I - P) = 0$ or $(I - P)AP = 0$, then

$$P = \begin{bmatrix} I_L & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix}, \quad P - A = \begin{bmatrix} I_L - A_1 & 0 \\ 0 & -A_4 \end{bmatrix}.$$

From the orthogonality of PAP comes $A_1 = A_1^2 = A_1^*$ and

$$(P - A)^2 = \begin{bmatrix} (I_L - A_1)^2 & 0 \\ 0 & A_4^2 \end{bmatrix} = \begin{bmatrix} (I_L - A_1)^* & 0 \\ 0 & A_4^* \end{bmatrix} = (P - A)^*.$$

□

Theorem A.12

Let P be an orthogonal projection and let A be a generalized projection. Then $P + A$ is a generalized projection if $PAP = 0$. Moreover, if $P + A$ is a generalized projection, then $PAP = 0$, $PA(I - P) = 0$ and $(I - P)AP = 0$.

Proof: From $(P + A)^2 = P + PA + AP + A^2 = P + A^*$ we conclude that $AP = PA = 0$. This is equivalent to

$$PA = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ A_3 & 0 \end{bmatrix} = 0,$$

which holds if and only if $A_1 = A_2 = A_3 = 0$. Thus, $PAP = 0$, $PA(I - P) = 0$, and $(I - P)AP = 0$.

Conversely, if $PAP = 0$, then $A_1 = 0$ and by Theorem A.7 $A_2 = 0$, and $A_3 = 0$. Clearly,

$$P + A = \begin{bmatrix} I_L & 0 \\ 0 & A_4 \end{bmatrix}$$

is a generalized projection. □

Theorem A.13

If P is an orthogonal projection and A is a generalized projection, then $AP - PA$ is a generalized projection if and only if $PA(I - P) = 0$ or $(I - P)AP = 0$.

Proof: The matrix representations of the operators A , P , and AP imply that

$$PA - AP = \begin{bmatrix} 0 & A_2 \\ -A_3 & 0 \end{bmatrix},$$

and it is clear that

$$(PA - AP)^2 = \begin{bmatrix} -A_2A_3 & 0 \\ 0 & -A_3A_2 \end{bmatrix} = \begin{bmatrix} 0 & -A_3^* \\ A_2^* & 0 \end{bmatrix} = (PA - AP)^*$$

if and only if $A_2 = A_3 = 0$ which is equivalent to $PA(I-P) = 0$ or $(I-P)AP = 0$. \square

The next theorem is not new. Actually, it is proved for matrices $A \in C^{m \times n}$ by J. Gross and G. Trenkler in [6] and it appears again in [2].

Theorem A.14

Let $A, B \in \mathcal{GP}(H)$. Then $AB \in \mathcal{GP}(H)$ if $AB = BA$.

Proof: If

$$AB = \begin{bmatrix} A_1B_1 & A_1B_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B_1A_1 & 0 \\ B_3A_1 & 0 \end{bmatrix} = BA,$$

it is clear that $A_1B_1 = B_1A_1$, $B_2 = 0$ and $B_3 = 0$. Form Theorem A.7 we conclude that $B_1^* = B_1^2$, $B_4^* = B_4^2$, and

$$(AB)^2 = \begin{bmatrix} (A_1B_1)^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (A_1B_1)^* & 0 \\ 0 & 0 \end{bmatrix} = (AB)^*.$$

\square

Theorem A.15

Let $A, B \in \mathcal{GP}(H)$. Then $A + B \in \mathcal{GP}(H)$ if and only if $AB = BA = 0$.

Proof: If $A, B \in \mathcal{GP}(H)$ have the representations given in Theorem A.7, then

$$A + B = \begin{bmatrix} A_1 + B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

and if

$$\begin{aligned} (A + B)^2 &= \begin{bmatrix} (A_1 + B_1)^2 + B_2B_3 & (A_1 + B_1)B_2B_4 \\ B_3(A_1 + B_1) + B_4B_3 & B_3B_2 + B_4^2 \end{bmatrix} \\ &= \begin{bmatrix} (A_1 + B_1)^* & B_3^* \\ B_2^* & B_4^* \end{bmatrix} = (A + B)^*, \end{aligned}$$

it is clear that

$$(A_1 + B_1)^2 = A_1^2 + A_1B_1 + B_1A_1 + B_1^2 + B_2B_3 = (A_1 + B_1)^*.$$

Since $B_1^* = B_1^2 + B_2B_3$, we get $A_1B_1 + B_1A_1 = 0$. Thus, $B_1 = 0$ which implies $B_2 = B_3 = 0$, $B_4^2 = B_4^*$. In this case we obtain $AB = BA = 0$.

Conversely, if $AB = BA = 0$, then $A_1B_1 = B_1A_1 = 0$, implying $B_1 = B_2 = B_3 = 0$, $B_4^2 = B_4^*$, and obviously, $(A + B)^2 = (A + B)^*$. \square

The next theorem gives an answer when the difference of two generalized projections is a generalized projection itself. It can be proved using partial ordering on $\mathcal{GP}(H)$, like in [6] or [2]. We prefer using only the basic properties of the generalized projections and their matrix representation.

Theorem A.16

Let $A, B \in \mathcal{GP}(H)$. Then $A - B \in \mathcal{GP}(H)$ if and only if $AB = BA = B^*$.

Proof: If A, B have the representations given in Theorem A.7, then

$$A - B = \begin{bmatrix} A_1 - B_1 & -B_2 \\ -B_3 & -B_4 \end{bmatrix}.$$

From

$$\begin{aligned} (A - B)^2 &= \begin{bmatrix} (A_1 - B_1)^2 + B_2B_3 & -(A_1 - B_1)B_2 + B_2B_4 \\ -B_3(A_1 - B_1) + B_4B_3 & B_3B_2 + B_4^2 \end{bmatrix} \\ &= \begin{bmatrix} (A_1 - B_1)^* & -B_3^* \\ -B_2^* & -B_4^* \end{bmatrix} = (A - B)^*, \end{aligned}$$

we get $B_2 = 0$, $B_3 = 0$, $B_4^2 = -B_4^*$ and from Theorem A.7 comes $B_4^2 = B_4^*$. Now, $B_4 = 0$ and

$$(A_1 - B_1)^2 = A_1^2 - A_1B_1 - B_1A_1 + B_1^2 = A_1^* - B_1^*$$

follows. This is true if and only if $A_1B_1 = B_1A_1 = B_1^*$, and in that case $AB = BA = B^*$. \square

Theorem A.17

Let A and B be two commuting generalized projections. Then $A(I - B) \in \mathcal{GP}(H)$ if and only if $ABA = (AB)^*$.

Proof: Since $AB = BA$, we know that AB is a generalized projection. Now,

$$\begin{aligned} (A(I - B))^2 &= (A - AB)^2 = A^2 - 2ABA + (AB)^2 \\ &= A^* - 2ABA + (AB)^* = (A(I - B))^* \end{aligned}$$

if and only if $ABA = (AB)^*$. \square

Theorem A.18

If $A \in \mathcal{GP}(H)$ and $\alpha \in \{1, e^{2\pi i/3}, e^{-2\pi i/3}\}$, then $\alpha A \in \mathcal{GP}(H)$.

Proof: Since $(\alpha A)^3 = \alpha^3 A^3 = A^3$, then $(\alpha A)^3|_{\mathcal{R}(A)} = I_{\mathcal{R}(A)}$ and αA is a normal operator, which completes the proof. \square

4 Properties of products, differences, and sums of hypergeneralized projections

For hypergeneralized projections we have results similar to those for generalized projections. In some theorems we are not able to establish equivalency like the one we establish for the generalized projections because we need additional conditions to ensure that $(A + B)^\dagger = A^\dagger + B^\dagger$ and $(A - B)^\dagger = A^\dagger - B^\dagger$.

We start with properties of pair of one orthogonal and one hypergeneralized projection.

Theorem A.19

Let $P \in \mathcal{B}(H)$ be an orthogonal projection and let $A \in \mathcal{HGP}(H)$. Then AP is a hypergeneralized projection if and only if $(I - P)AP = 0$. Similarly, PA is a hypergeneralized projection if and only if $PA(I - P) = 0$.

Proof: Let $H = L \oplus L^\perp$, where $\mathcal{R}(P) = L$. Then

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad AP = \begin{bmatrix} A_1 & 0 \\ A_3 & 0 \end{bmatrix}.$$

If

$$(AP)^2 = \begin{bmatrix} (A_1)^2 & 0 \\ A_3 A_1 & 0 \end{bmatrix} = \begin{bmatrix} D^\dagger A_1^* & D^\dagger A_3^* \\ 0 & 0 \end{bmatrix} = (PA)^\dagger,$$

then $A_3 = 0$, which is equivalent to $(I - P)AP = 0$.

Conversely, if $(I - P)AP = 0$ i.e. $A_3 = 0$, then A has matrix form

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}, \quad A^2 = \begin{bmatrix} A_1^2 & A_1 A_2 + A_2 A_4 \\ 0 & A_4^2 \end{bmatrix},$$

and it is easy to see that

$$A^\dagger = \begin{bmatrix} A_1^\dagger & (2I - A_1)^\dagger (I - A_1)^\dagger A_2 A_4^\dagger \\ 0 & A_4^\dagger \end{bmatrix}.$$

Since A is a hypergeneralized projection, it is clear that $A_1^2 = A_1^\dagger$ and consequently

$$(AP)^2 = \begin{bmatrix} A_1^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^\dagger & 0 \\ 0 & 0 \end{bmatrix} = (AP)^\dagger.$$

□

The following example shows that Theorem A.9 does not hold for hypergeneralized projections.

Example A.2

Let $H = \mathcal{C}^2$ and $A = \begin{bmatrix} 1 & 1 \\ 0 & e^{\frac{2\pi i}{3}} \end{bmatrix}$. Then $A^2 = \begin{bmatrix} 1 & 1 + e^{\frac{2\pi i}{3}} \\ 0 & e^{-\frac{2\pi i}{3}} \end{bmatrix}$, $A^3 = I_{\mathcal{R}(A)}$,

$A^4 = A$ and A is a hypergeneralized projection. However, $I - A = \begin{bmatrix} 0 & -1 \\ 0 & 1 - e^{\frac{2\pi i}{3}} \end{bmatrix}$

and it is not normal.

Theorem A.20

Let $A, B \in \mathcal{HGP}(H)$. If $AB = BA$, then $AB \in \mathcal{HGP}(H)$.

Proof: Let $H = \mathcal{R}(A) \oplus \mathcal{N}(A)$ and $A, B \in \mathcal{HGP}(H)$ have representations

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} A_1B_1 & A_1B_2 \\ 0 & 0 \end{bmatrix}, \quad (AB)^2 = \begin{bmatrix} A_1B_1A_1B_1 & A_1B_1A_1B_2 \\ 0 & 0 \end{bmatrix}.$$

A straightforward calculation using formula $A^\dagger = A^*(AA^*)^\dagger$ shows that

$$(AB)^\dagger = \begin{bmatrix} (A_1B_1)^*D^{-1} & 0 \\ (A_1B_2)^*D^{-1} & 0 \end{bmatrix},$$

where $D = A_1B_1(A_1B_1)^* + A_1B_2(A_1B_2)^* > 0$ is invertible. Assume that hypergeneralized projections A, B commute, i.e., that

$$AB = \begin{bmatrix} A_1B_1 & A_1B_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B_1A_1 & 0 \\ B_3A_1 & 0 \end{bmatrix} = BA.$$

This implies $B_2 = 0$, $B_3 = 0$, $A_1B_1 = B_1A_1$ and it is easy to see that $(AB)^2 = (AB)^\dagger$. \square

Theorem A.21

Let $A, B \in \mathcal{HGP}(H)$. If $AB = BA = 0$, then $A + B \in \mathcal{HGP}(H)$.

Proof: Let $H = \mathcal{R}(A) \oplus \mathcal{N}(A)$. Then

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}.$$

From these matrix representations it is easy to see that $AB = BA = 0$ implies $B_1 = B_2 = B_3 = 0$ and $B_4^2 = B_4^\dagger$. Now,

$$(A + B)^2 = A^2 + B^2 = A^\dagger + B^\dagger = (A + B)^\dagger.$$

\square

Theorem A.22

Let $A, B \in \mathcal{HGP}(H)$. If $AB = BA = B^2$, then $A - B \in \mathcal{HGP}(H)$.

Proof: Let $H = \mathcal{R}(A) \oplus \mathcal{N}(A)$. Then

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}.$$

From the condition $AB = BA = B^2$, we get $A_1B_1 = B_1A_1 = B_1^2$, $B_2 = B_3 = 0$, $B_4^2 = B_4^\dagger$, which implies $(A - B)^\dagger = A^\dagger - B^\dagger$. Hence

$$(A - B)^2 = A^2 - AB - BA + B^2 = A^2 - B^2 = A^\dagger - B^\dagger = (A - B)^\dagger.$$

\square

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Paper B

On the Moore-Penrose and the Drazin inverse of two projections on a Hilbert space

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On the Moore-Penrose and the Drazin inverse of two projections on a Hilbert space

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Abstract

For two given orthogonal, generalized or hypergeneralized projections P and Q on a Hilbert space H , we give their matrix representation. We also give canonical forms of the Moore-Penrose and the Drazin inverses of their products, differences, and sums. In addition, it is showed when these operators are EP operators and some simple relations between the mentioned operators are established.

Keywords: Orthogonal projection, generalized projection, hypergeneralized projection, Moore-Penrose inverse, Drazin inverse

1 Introduction

Motivation for writing this paper comes from publications of C. Deng and Y. Wei ([5], [6]) and O. M. Baksalary and G. Trenkler, ([1], [2]). Namely, Deng and Wei studied Drazin invertibility for products, differences, and sums of idempotents and Baksalary and Trenkler used matrix representation of the Moore-Penrose inverse of products, differences, and sums of orthogonal projections. Our main goal is to give canonical form of the Moore-Penrose and the Drazin inverse for products, differences, and sums of two orthogonal, generalized or hypergeneralized projections on an arbitrary Hilbert space. Using the canonical forms, we examine when the Moore-Penrose and the Drazin inverse exist. Also, we describe the relation between inverses (if any), estimate the Drazin index and establish necessary and sufficient conditions under which these operators are EP operators. Although some of the results are the same or similar to the results in the mentioned papers, there are differences since we use generalized and hypergeneralized projections, and not only orthogonal projections. We also examine different properties.

Throughout the paper, H stands for the Hilbert space and $\mathcal{L}(H)$ stands for set of all bounded linear operators on space H . The symbols $\mathcal{R}(A)$, $\mathcal{N}(A)$ and A^* denote range, null space and adjoint operator of the operator $A \in \mathcal{L}(H)$.

Operator $P \in \mathcal{L}(H)$ is an idempotent if $P = P^2$ and it is an orthogonal projection if $P = P^2 = P^*$.

Generalized and hypergeneralized projections were introduced in [7] by J. Gross and G. Trenkler.

Definition B.1. Operator $G \in \mathcal{L}(H)$ is

- (a) a generalized projection if $G^2 = G^*$,
- (b) a hypergeneralized projection if $G^2 = G^\dagger$.

The set of all generalized projection on H is denoted by $\mathcal{GP}(H)$ and the set of all hypergeneralized projection is denoted by $\mathcal{HGP}(H)$.

Here A^\dagger is the Moore-Penrose inverse of $A \in \mathcal{L}(H)$, i.e., the unique solution to the equations

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$

Notice that A^\dagger exists if and only if $\mathcal{R}(A)$ is closed. Then AA^\dagger is the orthogonal projection onto $\mathcal{R}(A)$ parallel to $\mathcal{N}(A^*)$, and $A^\dagger A$ is the orthogonal projection onto $\mathcal{R}(A^*)$ parallel to $\mathcal{N}(A)$. Consequently, $I - AA^\dagger$ is the orthogonal projection onto $\mathcal{N}(A^*)$ and $I - A^\dagger A$ is the orthogonal projection onto $\mathcal{N}(A)$. An essential property of any $P \in \mathcal{L}(H)$ is that P is an orthogonal projection if and only if it is expressible as AA^\dagger , for some $A \in \mathcal{L}(H)$.

For $A \in \mathcal{L}(H)$, an element $B \in \mathcal{L}(H)$ is the Drazin inverse of A , if the following hold:

$$BAB = B, \quad BA = AB, \quad A^{n+1}B = A^n,$$

for some non-negative integer n . The smallest such n is called the Drazin index of A . By A^D we denote the Drazin inverse of A and by $\text{ind}(A)$ we denote Drazin index of A .

If such n does not exist, $\text{ind}(A) = \infty$ and operator A is generalized Drazin invertible. Its inverse is denoted by A^d .

Operator A is invertible if and only if $\text{ind}(A) = 0$.

If $\text{ind}(A) \leq 1$, A is group invertible and A^D is group inverse, usually denoted by $A^\#$.

Notice that if the Drazin inverse exists, it is unique. Operator $A \in \mathcal{L}(H)$ is Drazin invertible if and only if $\text{asc}(A) < \infty$ and $\text{dsc}(A) < \infty$, where $\text{asc}(A)$ is the minimal integer such that $\mathcal{N}(A^{n+1}) = \mathcal{N}(A^n)$ and $\text{dsc}(A)$ is the minimal integer such that $\mathcal{R}(A^{n+1}) = \mathcal{R}(A^n)$. In this case, $\text{ind}(A) = \text{asc}(A) = \text{dsc}(A) = n$.

Recall that if $\mathcal{R}(A^n)$ is closed for some integer n , then $\text{asc}(A) = \text{dsc}(A) < \infty$.

Operator $A \in \mathcal{L}(H)$ is EP operator if $AA^\dagger = A^\dagger A$, or, in the other words, if $A^\dagger = A^D = A^\#$. There are many characterization of EP operators. In this paper, we use results from D. Djordjević and J. Koliha, (see [4]).

In what follows, \overline{A} stands for $I - A$ and P_A stands for AA^\dagger .

2 Auxiliary results

Let $P, Q \in \mathcal{L}(H)$ be orthogonal projectons and $\mathcal{R}(P) = L$. Since $H = \mathcal{R}(P) \oplus \mathcal{R}(P)^\perp = L \oplus L^\perp$, we have the following representaton of the projections $P, \overline{P}, Q, \overline{Q} \in \mathcal{L}(H)$ with respect to the decomposition of space:

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_L & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \rightarrow \begin{bmatrix} L \\ L^\perp \end{bmatrix}, \quad (1)$$

$$\overline{P} = \begin{bmatrix} 0 & 0 \\ 0 & I_{L^\perp} \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \rightarrow \begin{bmatrix} L \\ L^\perp \end{bmatrix}, \quad (2)$$

$$Q = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \rightarrow \begin{bmatrix} L \\ L^\perp \end{bmatrix}, \quad (3)$$

$$\overline{Q} = \begin{bmatrix} I_L - A & -B \\ -B^* & I_{L^\perp} - D \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \rightarrow \begin{bmatrix} L \\ L^\perp \end{bmatrix}, \quad (4)$$

with $A \in \mathcal{L}(L)$ and $D \in \mathcal{L}(L^\perp)$ being self adjoint and non-negative.

The next two theorems are known for matrices on C^m , (see [2]).

Theorem B.1

Let $Q \in \mathcal{L}(H)$ be represented as in (3). Then the following holds:

- (a) $A = A^2 + BB^*$, or, equivalently, $A\overline{A} = BB^*$,
- (b) $B = AB + BD$, or, equivalently, $B^* = B^*A + DB^*$,
- (c) $D = D^2 + B^*B$, or, equivalently, $D\overline{D} = B^*B$.

Proof: Since $Q = Q^2$, we obtain

$$\begin{bmatrix} A & B \\ B^* & D \end{bmatrix} \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} = \begin{bmatrix} A^2 + BB^* & AB + BD \\ B^*A + DB^* & B^*B + D^2 \end{bmatrix} = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$$

implying that $A = A^2 + BB^*$, $B = AB + BD$ and $D = D^2 + B^*B$. \square

Theorem B.2

Let $Q \in \mathcal{L}(H)$ be represented as in (3). Then:

- (a) $\mathcal{R}(B) \subseteq \mathcal{R}(A)$,
- (b) $\mathcal{R}(B) \subseteq \mathcal{R}(\bar{A})$,
- (c) $\mathcal{R}(B^*) \subseteq \mathcal{R}(D)$,
- (d) $\mathcal{R}(B^*) \subseteq \mathcal{R}(\bar{D})$,
- (e) $A^\dagger B = B\bar{D}^\dagger$,
- (f) $\bar{A}^\dagger B = BD^\dagger$,
- (g) A is a contraction,
- (h) D is a contraction,
- (i) $A - BD^\dagger B^* = I_L - \bar{A}\bar{A}^\dagger$.

Proof: (a) Since $A = A^2 + BB^*$, we have

$$\mathcal{R}(A) = \mathcal{R}(A^2 + BB^*) = \mathcal{R}(AA^* + BB^*).$$

To prove that $\mathcal{R}(AA^* + BB^*) = \mathcal{R}(A) + \mathcal{R}(B)$, observe the operator matrix

$$M = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}.$$

For any $x \in \mathcal{R}(MM^*)$, there exists $y \in H$ such that $x = MM^*y = M(M^*y)$ and $x \in \mathcal{R}(M)$. On the other hand, for $x \in \mathcal{R}(M)$, there is $y \in H$ and $x = My$. Besides, $MM^\dagger x = MM^\dagger My = My = x$ and $MM^\dagger = MM^*(MM^*)^\dagger = P_{\mathcal{R}(MM^*)}$ implying $x \in \mathcal{R}(MM^*)$. Hence, $\mathcal{R}(M) = \mathcal{R}(MM^*)$ and

$$\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(M) = \mathcal{R}(MM^*) = \mathcal{R}(AA^* + BB^*),$$

and we have

$$\mathcal{R}(A) = \mathcal{R}(A) + \mathcal{R}(B),$$

implying $\mathcal{R}(B) \subseteq \mathcal{R}(A)$.

(b) Since $A = I - \bar{A}$, from Theorem B.1 (a), we get $\bar{A} = \bar{A}^2 + BB^*$. The rest of the proof is analogously to the point (a) of this theorem.

(c), (d) Similarly.

(e) Since $B = AB + BD$, we have $A^\dagger B = A^\dagger(AB + BD) = A^\dagger AB + A^\dagger BD$, and using the facts that $A^\dagger A = P_{\mathcal{R}(A^*)}$ and $\mathcal{R}(B) \subseteq \mathcal{R}(A^*)$, we get $A^\dagger AB = B$ and $A^\dagger B = B + A^\dagger BD$, or, equivalently $B = A^\dagger B\bar{D}$. Postmultiplying this equation by \bar{D}^\dagger and using item (d) of this Theorem, in its equivalent form $B\bar{D}\bar{D}^\dagger = B$, we obtain (e).

(f) Analogously to the previous proof.

(g) Since $A = A^*$, from Theorem A.1 (a), we have that

$$I_L - AA^* = I_L - (A - BB^*) = \bar{A} + BB^*,$$

and the right hand side is nonnegative as a sum of two nonnegative operators implying that A is a contraction.

(h) This part of the proof is dual to the part (g).

(i) From Theorem B.1 (a), item (f) of this theorem and the fact that self adjoint operator A commutes with its MP-inverse, it follows that

$$BD^\dagger B^* = \bar{A}^\dagger BB^* = \bar{A}^\dagger A\bar{A} = \bar{A}^\dagger (I - \bar{A})\bar{A} = \bar{A}^\dagger \bar{A} - \bar{A}^\dagger \bar{A}\bar{A} = \bar{A}^\dagger \bar{A} - \bar{A},$$

by taking into account that $\bar{A}\bar{A}^\dagger = \bar{A}^\dagger\bar{A}$. Now we get

$$A - BD^\dagger B^* = I - \bar{A}^\dagger \bar{A},$$

establishing the condition. \square

Following the results of J. Gross and G. Trenkler for matrices, we formulate a few theorems for the generalized and hypergeneralized projections on arbitrary Hilbert space. We start with the result which is very similar to Theorem (1) in [7].

Theorem B.3

Let $G \in \mathcal{L}(H)$ be a generalized projection. Then G is a closed range operator and G^3 is an orthogonal projection on $\mathcal{R}(G)$. Moreover, H has the decomposition

$$H = \mathcal{R}(G) \oplus \mathcal{N}(G)$$

and G has the following matrix representaton

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix},$$

where the restriction $G_1 = G|_{\mathcal{R}(G)}$ is unitary on $\mathcal{R}(G)$.

Proof: If G is a generalized projection, then $G^4 = (G^2)^2 = (G^*)^2 = (G^2)^* = (G^*)^* = G$. From $GG^*G = G^4 = G$, it follows that G is a partial isometry implying that

$$G^3 = GG^* = P_{\mathcal{R}(G)},$$

$$G^3 = G^*G = P_{\mathcal{N}(G)^\perp}.$$

Thus, G^3 is the orthogonal projection onto $\mathcal{R}(G) = \mathcal{N}(G)^\perp = \mathcal{R}(G^*)$. Consequently, $\mathcal{R}(G)$ is a closed subset in H as a range of an orthogonal projection on a Hilbert space. From Lemma (1.2) in [4] we get the following decomposition of the space

$$H = \mathcal{R}(G^*) \oplus \mathcal{N}(G) = \mathcal{R}(G) \oplus \mathcal{N}(G).$$

Now, G has the following matrix representation in accordance with this decomposition:

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix},$$

where $G_1^2 = G_1^*$, $G_1^4 = G_1$ and $G_1G_1^* = G_1^*G_1 = G_1^3 = I_{\mathcal{R}(G)}$. \square

Theorem B.4

Let $G, H \in \mathcal{GP}(H)$ and $H = \mathcal{R}(G) \oplus \mathcal{N}(G)$. Then G and H has the following representations with respect to the decomposition of the space:

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix},$$

$$H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix},$$

where

$$\begin{aligned} H_1^* &= H_1^2 + H_2H_3, \\ H_2^* &= H_3H_1 + H_4H_3, \\ H_3^* &= H_1H_2 + H_2H_4, \\ H_4^* &= H_3H_2 + H_4^2. \end{aligned}$$

Furthermore, $H_2 = 0$ if and only if $H_3 = 0$.

Proof: Let $H = \mathcal{R}(G) \oplus \mathcal{N}(G)$. Then representation of G follows from Theorem (1) in [7], and let H has the representaton

$$H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}.$$

Then, from

$$H^2 = \begin{bmatrix} H_1^2 + H_2H_3 & H_1H_2 + H_2H_4 \\ H_3H_1 + H_4H_3 & H_3H_2 + H_4^2 \end{bmatrix} = \begin{bmatrix} H_1^* & H_3^* \\ H_2^* & H_4^* \end{bmatrix} = H^*,$$

conclusion follows directly.

If $H_2 = 0$, then $H_3^* = H_1H_2 + H_2H_4 = 0$ and $H_3 = 0$. Analogously, $H_3 = 0$ implies $H_2 = 0$. \square

Theorem B.5

Let $G \in \mathcal{L}(H)$ be a hypergeneralized projection. Then G is a closed range operator and H has the decomposition

$$H = \mathcal{R}(G) \oplus \mathcal{N}(G).$$

Also, G has the following matrix representaton with respect to the decomposition of the space

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix},$$

where the restriction $G_1 = G|_{\mathcal{R}(G)}$ satisfies $G_1^3 = I_{\mathcal{R}(G)}$.

Proof: If G is a hypergeneralized projection, then G and G^\dagger commute and G is EP. Using Lemma (1.2) in [4], we get decomposition of the space $H = \mathcal{R}(G) \oplus \mathcal{N}(G)$, and G has the required representation. \square

3 The Moore-Penrose and the Drazin inverse of two orthogonal projections

We start this section with theorem which gives the matrix representation of the Moore-Penrose inverse of products, differences, and sums of orthogonal projections.

Theorem B.6

Let orthogonal projections $P, Q \in \mathcal{L}(H)$ be represented as in (1) and (2). Then the Moore-Penrose inverse of PQ , $P - Q$ and $P + Q$ exists and the following holds:

$$(a) \quad (PQ)^\dagger = \begin{bmatrix} AA^\dagger & 0 \\ B^*A^\dagger & 0 \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \rightarrow \begin{bmatrix} L \\ L^\perp \end{bmatrix} \text{ and} \\ \mathcal{R}(PQ) = \mathcal{R}(A)$$

$$(b) \quad (P - Q)^\dagger = \begin{bmatrix} \overline{AA}^\dagger & -BD^\dagger \\ -B^*\overline{A}^\dagger & -DD^\dagger \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \rightarrow \begin{bmatrix} L \\ L^\perp \end{bmatrix} \text{ and} \\ \mathcal{R}(P - Q) = \mathcal{R}(\overline{A}) \oplus \mathcal{R}(D)$$

$$(c) \quad (P + Q)^\dagger = \begin{bmatrix} \frac{1}{2}(I + \overline{AA}^\dagger) & -BD^\dagger \\ -D^\dagger B^* & 2D^\dagger - DD^\dagger \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \rightarrow \begin{bmatrix} L \\ L^\perp \end{bmatrix} \text{ and} \\ \mathcal{R}(P + Q) = L \oplus \mathcal{R}(D).$$

Proof: (a) Using representatons (1) and (3) for orthogonal projections $P, Q \in \mathcal{L}(H)$, the well known Harte-Mbekhta formula $(PQ)^\dagger = (PQ)^*(PQ(PQ)^*)^\dagger$ and Theorem B.1(a), we obtain

$$(PQ)^\dagger = \begin{bmatrix} A & 0 \\ B^* & 0 \end{bmatrix} \begin{bmatrix} A^2 + BB^* & 0 \\ 0 & 0 \end{bmatrix}^\dagger = \begin{bmatrix} AA^\dagger & 0 \\ B^*A^\dagger & 0 \end{bmatrix}.$$

From $PQ(PQ)^\dagger = P_{\mathcal{R}(PQ)}$, we obtain

$$(PQ)(PQ)^\dagger = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} AA^\dagger & 0 \\ B^*A^\dagger & 0 \end{bmatrix} = \begin{bmatrix} AA^\dagger & 0 \\ 0 & 0 \end{bmatrix},$$

or, in the other words, $\mathcal{R}(PQ) = \mathcal{R}(A)$.

(b) Similarly to part (a), we can calculate the Moore-Penrose inverse of $P - Q$ as follows

$$\begin{aligned} (P - Q)^\dagger &= (P - Q)^*((P - Q)(P - Q)^*)^\dagger \\ &= \begin{bmatrix} \overline{A} & -B \\ -B^* & -D \end{bmatrix} \begin{bmatrix} \overline{A}^2 + BB^* & -\overline{A}B + BD \\ -B^*\overline{A} + DB^* & B^*B + D^2 \end{bmatrix}^\dagger \\ &= \begin{bmatrix} \overline{A} & -B \\ -B^* & -D \end{bmatrix} \begin{bmatrix} \overline{A}^\dagger & 0 \\ 0 & D^\dagger \end{bmatrix} \\ &= \begin{bmatrix} \overline{AA}^\dagger & -BD^\dagger \\ -B^*\overline{A}^\dagger & -DD^\dagger \end{bmatrix}. \end{aligned}$$

For the range of $P - Q$ we have

$$\begin{aligned} P_{\mathcal{R}(P-Q)} &= (P - Q)(P - Q)^\dagger \\ &= \begin{bmatrix} \overline{A} \overline{A} \overline{A}^\dagger + BD^\dagger B^* & -\overline{A} B D^\dagger + B D D^\dagger \\ -B^* \overline{A} \overline{A}^\dagger + D D^\dagger B^* & B^* B D^\dagger + D D D^\dagger \end{bmatrix} \\ &= \begin{bmatrix} \overline{A} \overline{A}^\dagger & 0 \\ 0 & D D^\dagger \end{bmatrix}, \end{aligned}$$

implying

$$\mathcal{R}(P - Q) = \mathcal{R}(\overline{A}) \oplus \mathcal{R}(D).$$

(c) The Moore-Penrose inverse of $P + Q$ has the following representation with the respect to decomposition of the space:

$$(P + Q)^\dagger = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \rightarrow \begin{bmatrix} L \\ L^\perp \end{bmatrix}.$$

In order to calculate $(P + Q)^\dagger$, we use the Moore-Penrose equations. From the first Moore-Penrose equation, $(P + Q)(P + Q)^\dagger(P + Q) = P + Q$, we have

$$\begin{aligned} ((I + A)X_1 + B X_3)(I + A) + ((I + A)X_2 + B X_4)B^* &= I + A, \\ ((I + A)X_1 + B X_3)B + ((I + A)X_2 + B X_4)D &= B, \\ (B^* X_1 + D X_3)(I + A) + (B^* X_2 + D X_4)B^* &= B^*, \\ (B^* X_1 + D X_3)B + (B^* X_2 + D X_4)D &= D. \end{aligned}$$

The second Moore-Penrose equation, $(P + Q)^\dagger(P + Q)(P + Q)^\dagger = (P + Q)^\dagger$, implies

$$\begin{aligned} (X_1(I + A) + X_2 B^*)X_1 + (X_1 B + X_2 D)X_3 &= X_1, \\ (X_1(I + A) + X_2 B^*)X_2 + (X_1 B + X_2 D)X_4 &= X_2, \\ (X_3(I + A) + X_4 B^*)X_1 + (X_3 B + X_4 D)X_3 &= X_3, \\ (X_3(I + A) + X_4 B^*)X_2 + (X_3 B + X_4 D)X_4 &= X_4, \end{aligned}$$

while the third and fourth Moore-Penrose equations, $((P + Q)(P + Q)^\dagger)^* = (P + Q)(P + Q)^\dagger$ and $((P + Q)^\dagger(P + Q))^* = (P + Q)^\dagger(P + Q)$, give $X_3 = X_2^*$. Further calculations show that

$$\begin{aligned} (I + A)X_1 + B X_2^* &= I_L, \\ (I + A)X_2 + B X_4 &= 0, \\ B^* X_1 + D X_2^* &= 0, \\ B^* X_2 + D X_4 &= D D^\dagger. \end{aligned}$$

According to Theorem B.2 (b), (c), from $B^* X_1 + D X_2^* = 0$ we get $D^\dagger B^* X_1 + X_2 = 0$, or equivalently, $X_2^* = -D^\dagger B^* X_1$.

From $(I + A)X_1 + BX_2^* = I_L$ and Theorem B.2 (i), we get $(2I - \overline{A}\overline{A}^\dagger)X_1 = I_L$ i.e. $X_1 = (2I - \overline{A}\overline{A}^\dagger)^{-1} = \frac{1}{2}(I + \overline{A}\overline{A}^\dagger)$. Theorem B.1 (c) and $B^*X_2 + DX_4 = DD^\dagger$ imply $-B^*BD^\dagger + DX_4 = DD^\dagger$. Finally, we have $X_2 = -BD^\dagger$, $X_3 = -D^\dagger B^*$, $X_4 = 2D^\dagger - DD^\dagger$ and

$$(P + Q)^\dagger = \begin{bmatrix} \frac{1}{2}(I + \overline{A}\overline{A}^\dagger) & -BD^\dagger \\ -D^\dagger B^* & 2D^\dagger - DD^\dagger \end{bmatrix}.$$

Like in the proof of part (b) of this theorem,

$$\begin{aligned} P_{\mathcal{R}(P+Q)} &= (P + Q)(P + Q)^\dagger \\ &= \begin{bmatrix} \frac{1}{2}(I + A)(I + \overline{A}\overline{A}^\dagger) - BD^\dagger B^* & -(I + A)BD^\dagger + 2BD^\dagger - BDD^\dagger \\ \frac{1}{2}B^*(I + \overline{A}\overline{A}^\dagger) - DD^\dagger B^* & -B^*BD^\dagger + 2DD^\dagger - DDD^\dagger \end{bmatrix} \\ &= \begin{bmatrix} I_L & 0 \\ 0 & DD^\dagger \end{bmatrix}, \end{aligned}$$

which asserts

$$\mathcal{R}(P + Q) = L \oplus \mathcal{R}(D).$$

To prove the existence of the Moore-Penrose inverse of PQ , $P - Q$ and $P + Q$, it is sufficient to prove that these operators have closed range. Since Q is the orthogonal projection, $\mathcal{R}(Q)$ is closed subset of H . Also,

$$\mathcal{R}(Q) = Q(H) = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} \begin{bmatrix} L \\ L^\perp \end{bmatrix} = \begin{bmatrix} \mathcal{R}(A) + \mathcal{R}(B) \\ \mathcal{R}(B^*) + \mathcal{R}(D) \end{bmatrix} = \mathcal{R}(A) + \mathcal{R}(D),$$

because items (a), (c) of Theorem A.1 state that $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(B^*) \subseteq \mathcal{R}(D)$. This implies that $\mathcal{R}(A)$ and $\mathcal{R}(D)$ are closed subsets of L and L^\perp , respectively. If $\mathcal{R}(A)$ is closed, then for every sequence $(x_n) \subseteq L$, $x_n \rightarrow x$ and $Ax_n \rightarrow y$ imply $x \in L$ and $Ax = y$. Now, $(I - A)x_n \rightarrow x - y$ and $x - y \in L$, $(I - A)x = x - y$ which proves that $\mathcal{R}(I - A)$ is closed. Consequently, $\mathcal{R}(PQ)$, $\mathcal{R}(I - A)$ and $\mathcal{R}(I + A)$ are closed which completes the proof. \square

Similarly to Theorem 3.1 in [6], we have the following result.

Theorem B.7

Let orthogonal projections $P, Q \in \mathcal{L}(H)$ be represented as in (1) and (3). Then the Drazin inverses of PQ , $P - Q$ and $P + Q$ exist, $P - Q$ and $P + Q$ are EP operators and the following holds:

- (a) $(PQ)^D = \begin{bmatrix} A^D & (A^D)^2 B \\ 0 & 0 \end{bmatrix}$ and $\text{ind}(PQ) \leq \text{ind}(A) + 1$,
- (b) $(P - Q)^D = (P - Q)^\dagger$ and $\text{ind}(P - Q) \leq 1$,
- (c) $(P + Q)^D = (P + Q)^\dagger$ and $\text{ind}(P + Q) \leq 1$.

Proof: (a) Theorem B.6 proves that $\mathcal{R}(PQ)$ is the closed subset of H . Thus, the Drazin inverse for this operators exists. According to representations (1) and (3) of projections P and Q , their product PQ and the Drazin inverse $(PQ)^D$ can be written in the following way:

$$PQ = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \quad (PQ)^D = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \rightarrow \begin{bmatrix} L \\ L^\perp \end{bmatrix}.$$

Equations that describe Drazin inverse are

$$(PQ)^D PQ (PQ)^D = \begin{bmatrix} X_1 A X_1 & X_1 A X_2 \\ X_3 A X_1 & X_3 A X_2 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} = (PQ)^D,$$

$$(PQ)^D PQ = \begin{bmatrix} X_1 A & X_1 B \\ X_3 A & X_3 B \end{bmatrix} = \begin{bmatrix} A X_1 & A X_2 \\ 0 & 0 \end{bmatrix} = PQ (PQ)^D,$$

$$(PQ)^{n+1} (PQ)^D = \begin{bmatrix} A^{n+1} X_1 & A^{n+1} X_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A^n & A^{n-1} B \\ 0 & 0 \end{bmatrix} = (PQ)^n.$$

Thus, from the first equation we have

$$X_1 A X_1 = X_1, \quad X_1 A X_2 = X_2, \quad X_3 A X_1 = X_3, \quad X_3 A X_2 = X_4,$$

from the second equation

$$X_1 A = A X_1, \quad A X_2 = X_1 B, \quad X_3 A = 0, \quad X_3 B = 0,$$

and the third equation implies

$$A^{n+1} X_1 = A^n, \quad A^{n+1} X_2 = A^{n-1} B.$$

It is easy to conclude that $X_1 = A^D$, $X_3 = 0$, $X_4 = 0$. Equations $X_1 A X_2 = X_2$ and $A X_2 = X_1 B$ give $X_1^2 B = X_2$. Finally,

$$(PQ)^D = \begin{bmatrix} A^D & (A^D)^2 B \\ 0 & 0 \end{bmatrix}.$$

To estimate the Drazin index of PQ , suppose that $\text{ind}(A) = n$. Then

$$\begin{aligned} (PQ)^{n+2} (PQ)^D &= \begin{bmatrix} A^{n+2} & A^{n+1} B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A^D & (A^D)^2 B \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} A^{n+1} & A^{n+1} A^D B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A^{n+1} & A^n B \\ 0 & 0 \end{bmatrix} = (PQ)^{n+1} \end{aligned}$$

implying that $\text{ind}(PQ) \leq \text{ind}(A) + 1$.

(b) Since $(P - Q)(P - Q)^* = (P - Q)^*(P - Q)$ and $\mathcal{R}(P - Q) = \mathcal{R}(\bar{A}) \oplus \mathcal{R}(D)$ is closed, $P - Q$ is the EP operator as a normal operator with the closed range and $(P - Q)^\dagger = (P - Q)^D$. Besides,

$$(P - Q)^2 (P - Q)^D = (P - Q)(P - Q)^\dagger (P - Q) = P - Q$$

and $\text{ind}(P - Q) \leq 1$.

(c) Similarly to (b), $P + Q$ is EP operator and $(P + Q)^\dagger = (P + Q)^D$, $\text{ind}(P + Q) \leq 1$. □

Theorem B.8

Let orthogonal projections $P, Q \in \mathcal{L}(H)$ be represented as in (1) and (3). Then the following holds:

(a) If $PQ = QP$ or $PQP = PQ$, then

$$(P + Q)^D = \begin{bmatrix} I_L - \frac{1}{2}A & 0 \\ 0 & D \end{bmatrix}, \quad (P - Q)^D = \begin{bmatrix} \bar{A} & 0 \\ 0 & -D \end{bmatrix}.$$

(b) If $PQP = P$, then

$$(P + Q)^D = \begin{bmatrix} \frac{1}{2}I_L & 0 \\ 0 & D \end{bmatrix}, \quad (P - Q)^D = \begin{bmatrix} 0 & 0 \\ 0 & -D \end{bmatrix}.$$

(c) If $PQP = Q$, then

$$(P + Q)^D = \begin{bmatrix} I_L - \frac{1}{2}A & 0 \\ 0 & 0 \end{bmatrix}, \quad (P - Q)^D = \begin{bmatrix} \bar{A} & 0 \\ 0 & 0 \end{bmatrix} = P - Q.$$

(d) If $PQP = 0$, then

$$(P + Q)^D = \begin{bmatrix} I_L & 0 \\ 0 & D \end{bmatrix} = P + Q, \quad (P - Q)^D = \begin{bmatrix} I_L & 0 \\ 0 & -D \end{bmatrix} = P - Q.$$

Proof: Let

$$(P + Q)^D = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \rightarrow \begin{bmatrix} L \\ L^\perp \end{bmatrix}.$$

(a) If

$$PQ = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ B^* & 0 \end{bmatrix} = QP$$

or

$$PQP = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} = PQ,$$

then $B = B^* = 0$, $I_L + A$ is invertible and $(I_L + A)^{-1} = I_L - \frac{1}{2}A$ and according to Theorem B.1 (c), $D = D^2$. Thus, we can write

$$Q = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \quad P + Q = \begin{bmatrix} I_L + A & 0 \\ 0 & D \end{bmatrix}, \quad (P + Q)^n = \begin{bmatrix} (I_L + A)^n & 0 \\ 0 & D \end{bmatrix}.$$

Verifying the equation

$$\begin{aligned} (P + Q)^2(P + Q)^D &= \begin{bmatrix} (I_L + A)^2X_1 & (I_L + A)^2X_2 \\ DX_3 & DX_4 \end{bmatrix} \\ &= \begin{bmatrix} I_L + A & 0 \\ 0 & D \end{bmatrix} = P + Q \end{aligned}$$

we get

$$X_2 = X_3 = 0, \quad DX_4 = D.$$

The other two equations, $(P + Q)^D(P + Q)(P + Q)^D = (P + Q)^D$ and $(P + Q)^D(P + Q) = (P + Q)(P + Q)^D$, give

$$X_4DX_4 = X_4, \quad X_4D = DX_4$$

i.e. $X_4 = D$. Thus,

$$(P + Q)^D = \begin{bmatrix} I_L - \frac{1}{2}A & 0 \\ 0 & D \end{bmatrix}.$$

Formula

$$(P - Q)^D = \begin{bmatrix} \bar{A} & 0 \\ 0 & -D \end{bmatrix}$$

follows from Theorem B.7 (b) and the fact that $A = A^2$ implies $\bar{A}^D = \bar{A}^2 = \bar{A}$.

(b) If $PQP = P$, then $A = I_L$ and Theorem B.1 implies $B = B^* = 0$. Then,

$$Q = \begin{bmatrix} I_L & 0 \\ 0 & D \end{bmatrix}$$

and from part (a) of this Theorem we conclude

$$(P + Q)^D = \begin{bmatrix} \frac{1}{2}I_L & 0 \\ 0 & D \end{bmatrix}, \quad (P - Q)^D = \begin{bmatrix} 0 & 0 \\ 0 & -D \end{bmatrix}.$$

(c) From $PQP = Q$ we get $B = B^* = D = 0$ and $A = A^2$. Now, $I_L + A$ is invertible and

$$(P + Q)^D = (P + Q)^{-1} = \begin{bmatrix} (I_L + A)^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_L - \frac{1}{2}A & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$(P - Q)^D = \begin{bmatrix} \bar{A} & 0 \\ 0 & 0 \end{bmatrix}.$$

(d) If $PQP = 0$, then $A = 0$ and since $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, we conclude $B = B^* = 0$. In this case,

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}, \quad P + Q = \begin{bmatrix} I_L & 0 \\ 0 & D \end{bmatrix}$$

implying

$$(P + Q)^D = P + Q = \begin{bmatrix} I_L & 0 \\ 0 & D \end{bmatrix}, \quad (P - Q)^D = P - Q = \begin{bmatrix} I_L & 0 \\ 0 & -D \end{bmatrix}.$$

□

Theorem B.9

Let orthogonal projections $P, Q \in \mathcal{L}(H)$ be represented as in (1) and (3). Then

$$(PQ)^D = (QP)^\dagger(PQ)^\dagger(QP)^\dagger.$$

Moreover, if $PQ = QP$, then PQ is the EP operator and

$$(PQ)^D = (PQ)^\dagger, \quad \text{ind}(PQ) \leq 1.$$

Proof: Corollary 5.2 in [8] states that $(PQ)^\dagger$ is idempotent for every orthogonal projections P and Q . Thus we can write

$$(PQ)^\dagger = \begin{bmatrix} I & 0 \\ K & 0 \end{bmatrix}$$

and

$$PQ = (PQ)^{\dagger\dagger} = \begin{bmatrix} (I + K^*K)^{-1} & (I + K^*K)^{-1}K^* \\ 0 & 0 \end{bmatrix}.$$

Denote by $A = (I + K^*K)^{-1}$ and $B = (I + K^*K)^{-1}K^* = AK^*$. Then

$$PQ = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$$

and according to Theorem B.8 (a),

$$\begin{aligned} (PQ)^D &= \begin{bmatrix} A^D & (A^D)^2B \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} I + K^*K & (I + K^*K)^2(I + K^*K)^{-1}K^* \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} I + K^*K & (I + K^*K)K^* \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I + K^*K & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & K^* \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} I & K^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ K & 0 \end{bmatrix} \begin{bmatrix} I & K^* \\ 0 & 0 \end{bmatrix} \\ &= (QP)^\dagger (PQ)^\dagger (QP)^\dagger, \end{aligned}$$

where we used $((PQ)^\dagger)^* = ((PQ)^*)^\dagger = (QP)^\dagger$.

If P and Q commute, then PQ is the normal operator with the closed range, which means that it is an EP operator and $(PQ)^\dagger = (PQ)^D$. \square

4 The Moore-Penrose and the Drazin inverse of the generalized and hypergeneralized projections

Some of the results obtained in the previous section we extend to generalized and hypergeneralized projections.

Theorem B.10

Let $G, H \in \mathcal{L}(H)$ be two generalized or hypergeneralized projections. Then the Moore-Penrose inverse of GH exists and has the following matrix representation

$$(GH)^\dagger = \begin{bmatrix} (G_1H_1)^*D^{-1} & 0 \\ (G_1H_2)^*D^{-1} & 0 \end{bmatrix},$$

where $D = G_1H_1(G_1H_1)^* + G_1H_2(G_1H_2)^* > 0$ is invertible.

Proof: From Theorems B.3, B.4 and B.5, we see that $\mathcal{R}(G) = \mathcal{R}(G_1)$ is closed and pair of generalized or hypergeneralized projections has the matrix form

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}.$$

Then,

$$GH = \begin{bmatrix} G_1H_1 & G_1H_2 \\ 0 & 0 \end{bmatrix}$$

and analogously to the proof of Theorem B.6 (a), we obtain the mentioned matrix form. Since $\mathcal{R}(GH) = \mathcal{R}(G_1)$ is closed, the Moore-Penrose inverse $(GH)^\dagger$ exists. \square

Theorem B.11

Let $G, H \in \mathcal{L}(H)$ be two generalized or hypergeneralized projections. Then the Drazin inverse of GH exists and has the following matrix representation

$$(GH)^D = \begin{bmatrix} (G_1H_1)^D & ((G_1H_1)^D)^2G_1H_2 \\ 0 & 0 \end{bmatrix}.$$

Proof: Similarly to the proof of Theorem B.7 (a) and using Theorem A.18. \square

Theorem B.12

Let $G, H \in \mathcal{L}(H)$ be two generalized projections.

(a) If $GH = HG$, then GH is EP operator and

$$(GH)^\dagger = (GH)^D = (GH)^* = (GH)^2 = (GH)^{-1},$$

$$(GH)^\dagger = \begin{bmatrix} (G_1H_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

(b) If $GH = HG = 0$, then $G + H$ is EP operator and

$$(G + H)^\dagger = (G + H)^D = (GH)^* = (G + H)^2 = (G + H)^{-1},$$

$$(G + H)^\dagger = \begin{bmatrix} G_1^{-1} & 0 \\ 0 & H_4^{-1} \end{bmatrix}.$$

(c) If $GH = HG = H^*$, then $G - H$ is EP operator and

$$(G - H)^\dagger = (G - H)^D = (GH)^* = (G - H)^2 = (G - H)^{-1},$$

$$(G - H)^\dagger = \begin{bmatrix} (G_1 - H_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof: (a) If $G, H \in \mathcal{L}(H)$ are two commuting generalized projections, then from

$$(GH)^* = H^*G^* = H^2G^2 = (HG)^2 = (GH)^2$$

we conclude that GH is also generalized projection, and therefore EP operator. Checking the Moore-Penrose equations for $(GH)^2$, we see that they hold. From the uniqueness of the Moore-Penrose inverse follows $(GH)^2 = (GH)^\dagger$ and

$$(GH)^\dagger = (GH)^D = (GH)^2$$

From $GH(GH)^\dagger = P_{\mathcal{R}(GH)}$, using matrix form of GH , we get $G_1H_1(G_1H_1)^\dagger = I$, or equivalently, $(G_1H_1)^\dagger = (G_1H_1)^{-1}$. Finally,

$$(GH)^\dagger = (GH)^D = (GH)^* = (GH)^2 = (GH)^{-1}.$$

(b) If $GH = HG = 0$, then $(G + H)^2 = G^2 + H^2 = G^* + H^* = (G + H)^*$ and $G + H$ is a generalized projection. The rest of the proof is similar to part (a).

(c) If $GH = HG = H^*$, then $(G - H)^2 = G^2 - H^2 = G^* - H^* = (G - H)^*$ and the rest of the proof is similar to part (a).

Matrix representations are easily obtained by using canonical forms of G and H given in Theorem B.4. \square

Theorem B.13

Let $G, H \in \mathcal{L}(H)$ be two hypergeneralized projections.

(a) If $GH = HG$, then GH is EP operator and

$$(GH)^\dagger = (GH)^D = (GH)^2 = (GH)^{-1},$$

$$(GH)^\dagger = \begin{bmatrix} (G_1H_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

(b) If $GH = HG = 0$, then $G + H$ is EP operator and

$$(G + H)^\dagger = (G + H)^D = (G + H)^2 = (G + H)^{-1},$$

$$(G + H)^\dagger = \begin{bmatrix} G_1^{-1} & 0 \\ 0 & H_4^{-1} \end{bmatrix}.$$

(c) If $GH = HG = H^*$, then $G - H$ is EP operator and

$$(G - H)^\dagger = (G - H)^D = (G - H)^2 = (G - H)^{-1},$$

$$(G - H)^\dagger = \begin{bmatrix} (G_1 - H_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof: (a) If $GH = HG$, then GH is an EP operator and $(GH)^4 = GH$, so it is a hypergeneralized projection. Since $(GH)^2 = (GH)^\dagger$, operator GH commutes with its Moore-Penrose inverse and $(GH)^\dagger = (GH)^D$. From

$$GH(GH)^\dagger = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

follows $(GH)^\dagger = (GH)^{-1}$. Thus,

$$(GH)^\dagger = (GH)^D = (GH)^2 = (GH)^{-1}.$$

(b) If $GH = HG = 0$, then $(G + H)^2 = (G + H)^\dagger$ and $H_1 = H_2 = H_3 = 0$ implies

$$G + H = \begin{bmatrix} G_1 & 0 \\ 0 & H_4 \end{bmatrix}, \quad (G + H)^\dagger = \begin{bmatrix} G_1^\dagger & 0 \\ 0 & H_4^\dagger \end{bmatrix}.$$

From $(G + H)(G + H)^\dagger = P_{\mathcal{R}(G+H)} = P_{\mathcal{R}(G)} + P_{\mathcal{R}(H)}$ and

$$(G + H)(G + H)^\dagger = \begin{bmatrix} G_1 G_1^\dagger & 0 \\ 0 & H_4 H_4^\dagger \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

we conclude that $G_1^\dagger = G_1^{-1}$, $H_4^\dagger = H_4^{-1}$ and $(G + H)^\dagger = (G + H)^{-1}$.

(c) Similarly to (b). □

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