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An alternating iterative procedure for the Cauchy problem for the Helmholtz equation

Lydie Mpinganzima



Linköping University
INSTITUTE OF TECHNOLOGY

Department of Mathematics
Linköping University, SE-581 83 Linköping, Sweden

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Matematiska institutionen
Linköpings universitet
SE-581 83 Linköping, Sweden
lydie.mpinganzima@liu.se

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Abstract

Let Ω be a bounded domain in \mathbb{R}^n with a Lipschitz boundary Γ divided into two parts Γ_0 and Γ_1 which do not intersect one another and have a common Lipschitz boundary. We consider the following Cauchy problem for the Helmholtz equation:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega, \\ u = f & \text{on } \Gamma_0, \\ \partial_\nu u = g & \text{on } \Gamma_0, \end{cases}$$

where k , the wave number, is a positive real constant, ∂_ν denotes the outward normal derivative, and f and g are specified Cauchy data on Γ_0 . This problem is ill-posed in the sense that small errors in the Cauchy data f and g may blow up and cause a large error in the solution.

Alternating iterative algorithms for solving this problem are developed and studied. These algorithms are based on the alternating iterative schemes suggested by V.A. Kozlov and V. Maz'ya for solving ill-posed problems. Since these original alternating iterative algorithms diverge for large values of the constant k^2 in the Helmholtz equation, we develop a modification of the alternating iterative algorithms that converges for all k^2 . We also perform numerical experiments that confirm that the proposed modification works.

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Introduction

Inverse problems arise in many technical and scientific areas, such as medical and geophysical imaging [11], astrophysical problems [6], acoustic and electromagnetic scattering [5], and identification and location of vibratory sources [17]. Inverse problems often lead to mathematical models that are ill-posed. According to Hadamard's definition of well-posedness, a problem is well-posed if it satisfies the following three requirements [14]:

1. **Existence:** There exists a solution of the problem.
2. **Uniqueness:** There is at most one solution of the problem.
3. **Stability:** The solution depends continuously on the data.

If one or more of these requirements are not satisfied, then the problem is said to be ill-posed.

Example 0.1. Consider the Cauchy problem for the Laplace equation:

$$\begin{cases} \Delta u = 0, & 0 < x < \pi, \quad y > 0, \\ u(x, 0) = 0, & 0 \leq x \leq \pi, \\ \partial_y u(x, 0) = g_n(x), & 0 \leq x \leq \pi, \end{cases}$$

where $g_n(x) = n^{-1} \sin nx$, for $0 \leq x \leq \pi$ and $n > 0$. The solution to this problem is given by

$$u_n(x, y) = n^{-2} \sin nx \sinh ny.$$

We observe that g_n tends uniformly to zero as n tends to infinity, while for fixed $y > 0$ the value of $u_n(x, y)$ tends to infinity. Thus, the requirement that the solution depends continuously on the data does not hold.

Example 0.2. Consider the following Cauchy problem for the Helmholtz equation in the rectangle $\Omega = (0, a) \times (0, b)$:

$$\begin{cases} \Delta u(x, y) + k^2 u(x, y) = 0, & 0 < x < a, \quad 0 < y < b, \\ u(x, 0) = f(x), & 0 \leq x \leq a, \\ \partial_y u(x, 0) = g(x), & 0 \leq x \leq a, \\ u(0, y) = u(a, y) = 0, & 0 \leq y \leq b, \end{cases}$$

where k is the wave number, $f \in L^2(0, a)$, and $g \in L^2(0, a)$ are specified Cauchy data. The solution to this problem can be obtained using separation of variables

in the form

$$u(x, y) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{a} x \left(A_n \cosh \lambda_n y + \lambda_n^{-1} B_n \sinh \lambda_n y \right),$$

where $\lambda_n = \sqrt{a^{-2}n^2\pi^2 - k^2}$ and the coefficients A_n and B_n are given by

$$A_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi}{a} x dx \quad \text{and} \quad B_n = \frac{2}{a} \int_0^a g(x) \sin \frac{n\pi}{a} x dx.$$

Since the estimate $\|u\|_{L^2(\Omega)} \leq C (\|f\|_{L^2(0,a)} + \|g\|_{L^2(0,a)})$ cannot hold in general, the requirement that the solution depends continuously on the data does not hold and the problem is ill-posed. Note that this estimate cannot hold for any reasonable choice of norms. Another way of showing that the Helmholtz equation leads to ill-posed problem can be found in Lavrent'ev [18, 19].

More examples of ill-posed problems can be found in the literature such as Groetsch [8], Hadamard [9], Isakov [12], Kaipio [13], and Vogel [23].

The existence and the uniqueness parts in the Hadamard definition are important but they can be often circumvented by adding additional requirements to the solution or relaxing the notion of a solution. The requirement that the solution should depend continuously on the data is important in the sense that if one wants to approximate the solution to a problem, whose solution does not depend continuously on the data by a traditional numerical method, then one has to expect that the numerical solution becomes unstable. The computed solution thus has nothing to do with the true solution; see Engl et al. [7]. To obtain approximate solutions that are less sensitive to perturbations, one uses regularization methods.

Different regularization methods have been suggested in the literature [7, 10, 23]. In this thesis we investigate the so-called alternating iterative algorithms. Introduced by V.A. Kozlov and V. Maz'ya in [15], the alternating iterative algorithms are used for solving Cauchy problem for elliptic equations. The algorithm works by iteratively changing boundary conditions until a satisfactory result is obtained. Such algorithms preserve the differential equations, and every step reduces to the solution of well-posed problems for the original differential equation. The regularizing character of the algorithm is ensured solely by an appropriate choice of boundary conditions in each iteration. These methods have been applied by Kozlov et al. [16] to solve the Cauchy problem for the Laplace equation and the Lamé system. They also proved the convergence of the algorithms and established the regularizing properties. After that, different studies have been done using these algorithms for solving ill-posed problems originating from partial differential equations [1, 2, 3, 4, 20, 21].

In our study, we generalize the problem in Example 0.2 as follows: let Ω be a bounded domain in \mathbb{R}^n with a Lipschitz boundary Γ divided into two parts Γ_0 and Γ_1 which do not intersect one another and have a common Lipschitz boundary. We denote by ν the outward unit normal to the boundary Γ . We

consider the following Cauchy problem for the Helmholtz equation:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega, \\ u = f & \text{on } \Gamma_0, \\ \partial_\nu u = g & \text{on } \Gamma_0, \end{cases} \quad (0.1)$$

where the wave number k^2 is a positive real constant, ∂_ν denotes the outward normal derivative, and f and g are specified Cauchy data on Γ_0 . We want to find real solutions to the problem (0.1). This problem is investigated in Paper 1. In the alternating iterative algorithm described in [16], for problem (0.1), one considers the following two auxiliary problems:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega, \\ u = f & \text{on } \Gamma_0, \\ \partial_\nu u = \eta & \text{on } \Gamma_1, \end{cases} \quad (0.2)$$

and

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega, \\ \partial_\nu u = g & \text{on } \Gamma_0, \\ u = \phi & \text{on } \Gamma_1, \end{cases} \quad (0.3)$$

where f and g are the original Cauchy data as seen in (0.1). The standard alternating iterative procedure for solving the problem (0.1) is as follows:

1. The first approximation u_0 to the solution u of (0.1) is obtained by solving (0.2), where η is an arbitrary initial approximation of the normal derivative on Γ_1 .
2. Having constructed u_{2n} , we find u_{2n+1} by solving (0.3) with $\phi = u_{2n}$ on Γ_1 .
3. We then find u_{2n+2} by solving (0.2) with $\eta = \partial_\nu u_{2n+1}$ on Γ_1 .

In Example 0.2, we show that for

$$k^2 \geq \pi^2(a^{-2} + (16b)^{-2})$$

this algorithm diverges and it thus cannot be applied for large values of the constant k^2 in the Helmholtz equation. The reason is that the bilinear form associated with the Helmholtz equation is not positive definite; see [22]. To guarantee the positivity of the bilinear form, we introduce an auxiliary interior boundary γ and a positive constant μ . We then assume that

$$\int_{\Omega} (|\nabla u|^2 - k^2 u^2) dx + \mu \int_{\gamma} u^2 dS > 0 \text{ for } u \in H^1(\Omega) \text{ such that } u \neq 0.$$

We denote by $[u]$ and by $[\partial_\nu u]$ the jump of the function u and the jump of the normal derivative $\partial_\nu u$ across γ , respectively. We thus propose a modified iterative algorithm that consists of solving the following boundary value problems alternatively:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega \setminus \gamma, \\ u = f & \text{on } \Gamma_0, \\ \partial_\nu u = \eta & \text{on } \Gamma_1, \\ [\partial_\nu u] + \mu u = \xi & \text{on } \gamma, \\ [u] = 0 & \text{on } \gamma, \end{cases} \quad (0.4)$$

and

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega \setminus \gamma, \\ \partial_\nu u = g & \text{on } \Gamma_0, \\ u = \phi & \text{on } \Gamma_1, \\ u = \varphi & \text{on } \gamma. \end{cases} \quad (0.5)$$

The modified alternating iterative algorithm for solving (0.1) is as follows:

1. The first approximation u_0 to the solution of (0.1) is obtained by solving (0.4), where η is an arbitrary initial approximation of the normal derivative on Γ_1 and ξ is an arbitrary approximation of $[\partial_\nu u] + \mu u$ on γ .
2. Having constructed u_{2n} , we find u_{2n+1} by solving (0.5) with $\phi = u_{2n}$ on Γ_1 and $\varphi = u_{2n}$ on γ .
3. We then obtain u_{2n+2} by solving the problem (0.4) with $\eta = \partial_\nu u_{2n+1}$ on Γ_1 and $\xi = [\partial_\nu u_{2n+1}] + \mu u_{2n+1}$ on γ .

In this thesis, the problems (0.4)–(0.5) are solved in the weak sense. This modification thus consists of solving well-posed mixed boundary value problems for the original equation. We denote the sequence of solutions to (0.1) obtained from the modified alternating algorithm above by $(u_n(f, g, \eta, \xi))_{n=0}^\infty$. The main result in this thesis concerning the convergence of the algorithm is as follows:

Theorem 0.3. *Let $f \in H^{1/2}(\Gamma_0)$ and $g \in H^{1/2}(\Gamma_0)^*$, and let $u \in H^1(\Omega)$ be the solution to problem (0.1). Then, for every $\eta \in H^{1/2}(\Gamma_1)^*$ and every $\xi \in H^{1/2}(\gamma)^*$, the sequence $(u_n)_{n=0}^\infty$, obtained from the modified alternating algorithm, converges to u in $H^1(\Omega)$.*

For the numerical implementation, we consider the problem presented in Example 0.2. We then solve well-posed boundary value problems in the modified algorithm using the finite difference method. We also make good choices of the interior boundary γ , the constant μ , and the initial approximations η of the normal derivative on Γ_1 and ξ of $[\partial_\nu u] + \mu u$ on γ . The numerical results confirm the convergence of the modified algorithm.

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