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# On the Branch Loci of Moduli Spaces of Riemann Surfaces

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## Abstract

The spaces of conformally equivalent Riemann surfaces, $\mathcal{M}_g$ where $g \geq 1$, are not manifolds. However the spaces of the weaker Teichmüller equivalence, $T_g$ are known to be manifolds. The Teichmüller space $T_g$ is the universal covering of $\mathcal{M}_g$ and $\mathcal{M}_g$ is the quotient space by the action of the modular group. This gives $\mathcal{M}_g$ an orbifold structure with a branch locus $\mathcal{B}_g$. The branch loci $\mathcal{B}_g$ can be identified with Riemann surfaces admitting non-trivial automorphisms for surfaces of genus $g \geq 3$. In this thesis we consider the topological structure of $\mathcal{B}_g$. We study the connectedness of the branch loci in general by considering families of isolated strata and we establish that connectedness is a phenomenon for low genera. Further, we give the orbifold structure of the branch locus of surfaces of genus 4 and genus 5 in particular, by studying the equisymmetric stratification of the branch locus.

## Acknowledgments

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Gabriel Bartolini
2012

# Populärvetenskaplig sammanfattning

Ända sedan Bernhard Riemann introducerade konceptet Riemannyta i sin doktorsavhandling år 1851 har det varit ett centralt begrepp inom många matematiska discipliner. En av dess styrkor är att det finns många ekvivalenta sätt att definiera dem. Inom algebra talar man om Riemannytor som algebraiska kurvor, dvs lösningsmängder till polynomekvationer i komplexa variabler. Ett annat sätt att konstruera en Riemannyta kommer från differentialgeometrin. Här utgås det från en topologisk yta och den ges en extra struktur. Denna struktur är given av kartor som tilldelar komplexa koordinater till punkter på ytan och om en punkt ligger i flera kartor så ges övergången mellan kartorna av en deriverbar komplex funktion. En samling kompatibla kartor ger en atlas. Det finns många olika möjliga atlaser men två atlaser är ekvivalenta om deras kartor är kompatibla och en sådan klass av atlaser kallar vi för en komplex struktur. Dvs, en Riemannyta är en yta tillsammans med en komplex struktur.

Här är vi främst intresserade av kompakta ytor. Kompakta ytor karaktäriseras av dess matematiska genus, vilket räknar antalet hål eller handtag den har. Till exempel har en sfär genus noll, medan en torus (tänk ringmunk eller kaffekopp) har genus ett. För alla kompakta ytor utöver sfären, där redan Riemann visade att alla atlaser är ekvivalenta, finns det oändligt många komplexa strukturer. För att kunna studera mängden av komplexa strukturer försökte Riemann och andra matematiker att hitta en uppsättning parametrar som beskriver denna mängd. För Riemannytor av genus ett, där den underliggande ytan är en torus, fann de att mängden går att beskriva med en komplex parameter, även kallat modulus. För ytor av genus $g$ större än ett är det dock mycket svårare att beskriva dessa modulirum. Riemann förutspådde att detta rum har $3g - 3$ komplexa dimensioner, dock kunde han inte bevisa detta.

Lösningen vi använder oss av är att ta en omväg via ett annat rum. Genom att tillämpa ett mer strikt kompabilitetsvillkor för atlaser kan vi skapa det så kallade Teichmüllerrummet. Detta rum är ett metriskt rum av $6g - 6$ dimensioner och varje klass av Riemannyta i Teichmüllerrummet kan avbildas på dess motsvarande klass i modulirummet vilket ger en förgrenad övertäckning. Utanför förgreningsmängden kan modulirummet parametriseras på samma sätt som Teichmüllerrummet. Så för att förstå modulirummet är vi intresserade av att beskriva förgreningsmängden. För ytor av genus $g$ större än två kan de singulära punkterna, eller förgreningspunkterna identifieras med Riemannytor som har icke-triviala symmetrier. Denna mängd går att dela upp i mindre, icke-singulära delmängder där varje delmängd motsvarar ytor med symmetrigrupper som uppför sig på samma sätt. Att studera förgreningsmängden är därmed ekvivalent med att studera symmertrigrupper av Riemannytor.

År 1893 visade Adolf Hurwitz att för Riemannytor av genus $g$ större än ett är antalet olika symmetrier som mest $84(g - 1)$ stycken. Detta får bland annat konsekvensen att antalet möjliga symmetrigrupper är ändligt. För att studera symmetrier av Riemannytor använder vi oss av ett tredje synsätt på Riemannytor. Genom att skära upp ytan kan vi veckla ut den till en polygon. En torus kan vi till exempel veckla ut till ett parallellogram och detta parallellogram kan användas för att tessellera planet. Den funktion som tar en sida av en polygon till den sida den ska sitta ihop med ges av en symmetri av planet och symmetrierna av Riemannytan kan då beskrivas i termer av symmetrier av planet. Dock är det är välkänt att för att tessellera planet med likformiga konvexa polygoner får dessa ha högst sex sidor och en polygon från en yta av genus $g$ har minst $4g$ stycken sidor. Här utnyttjar vi istället det hyperboliska planet som kan tesselleras av alla polygoner med mer än sex sidor.

Om en Riemannyta har icke-triviala symmetrier innebär det att dess polygon kan delas upp i mindre, likformiga polygoner. Genom att studera de möjliga strukturerna för hur dessa uppdelningar kan se ut och de grupper av symmetrier som motsvarar dessa kan vi bestämma de möjliga typerna av symmetrigrupper för Riemannytor. Genom att studera undergrupper och extensioner av dessa grupper kan vi sedan avgöra hur de givna delmängderna av föregreningsmängden hänger samman.

I avhandlingen fokuserar vi på den topologiska karaktären av förgreningsmängden för att beskriva geometrin av modulirummet. Speciellt studerar vi om förgreningsmängden är sammanhängande eller om det finns isolerade komponenter. Om den är sammanhängande innebär det bland annat att varje Riemannyta med icke-triviala symmetrier kan kontinuerligt formas om till varje annan Riemannyta med icke-triviala symmetrier på ett sådant sätt att den hela tiden har symmetrier. Vi visar till exempel att de största delmängderna, vilka motsvarar Riemannytor med symmetrier av ordning två eller tre, sitter ihop. Samtidigt kan vi även se att med undantag för genus tre, fyra, tretton, sjutton, nitton och femtionio, är förgreningsmängden icke-sammanhängande.

# Contents

# Introduction

## Background

The first appearance of Riemann surfaces was in Riemann's dissertation *Foundations for a general theory of functions of a complex variable* in 1851. Riemann used the surfaces as a tool to study many-valued complex functions. The first abstract definition, which is still used today, was introduced by Weyl in 1913. Weyl considered a Riemann surface as a topological surface with a complex structure. If two homeomorphic Riemann surfaces admit a biholomorphic map they are not only topologically equivalent but also conformally equivalent. The famous Riemann mapping theorem states that any simply connected proper subset of the complex plane is conformally equivalent to the unit disc. Riemann also observed that the analogical statement holds for Riemann surfaces homeomorphic to the sphere. However, this does not hold for compact Riemann surface of higher genera. Thus, it is natural to consider the sets of classes of conformally equivalent compact Riemann surfaces of higher genera. For the tori, Riemann surfaces of genus 1, this set was studied in Riemann's time in terms of elliptic functions. It was shown that it is a one (complex-) dimensional space parametrized by the moduli of the elliptic functions, hence called the moduli space. In *Theory of abelian functions* [52], Riemann asserted the set of Riemann surfaces of genus $g(\geq 2)$, $\mathcal{M}_g$, could be parametrized by $3g-3$ (complex) parameters, which he called the moduli. Riemann substantiated this by considering Riemann surfaces as branched coverings of the Riemann sphere. However, precise statements of this structure require the construction of other spaces, the Teichmüller spaces, due to ideas of Teichmüller in 1939.

Another perspective of Riemann surfaces is the uniformization theory of Poincaré, Klein and Koebe. Poincaré's [50] idea presented in 1882 was as follows; lets say a polynomial equation $p(x, y) = 0$ defines $y$ as a multi-valued complex function of $x$, also known as a nonuniform function, then there exists Fuchsian functions, $f$ and $g$, defined on the unit disc such that $x = f(z)$ and $y = g(z)$, which are single-valued, or uniform functions. Later he extended this idea, proposing that for any nonuniform analytic function, one can find a variable uniformizing it. Through the work of Klein and Koebe, stating that the only simpy connected Riemann surfaces are the sphere, the plane and the disc, it was shown that the uniformizing variable takes the values in one of these. A result of this is that every Riemann surface admits a Riemann metric of constant curvature. There are three types of geometries, elliptic geometry with curvature $1$, euclidean geometry with curvature $0$ and hyperbolic geometry with curvature $-1$. A Riemann surface of genus $g(\geq 2)$ is uniformized by the unit disc, or the hyperbolic plane, and can be identified as the quotient space of the hyperbolic plane by the action of a fixed point free Fuchsian group, which is a group of automorphisms of the hyperbolic plane. For a surface of genus $g$ the corresponding group is generated by $2g$ elements, each having three unnormalized real parameters, satisfying one relation and with normalization this results in $6g-6$ parameters. This discovery reinforced Riemann's assertion.

A breakthrough in the study of the moduli spaces was due to Teichmüller who in 1939 related conformally equivalence to quasi-conformal maps between Riemann surfaces and quadratic differentials. It was already know to Riemann that $3g-3$ is the number of lineary independent analytic quadratic differentials on a Riemann surface of genus $g$. Each pair of conformally non-equivalent Riemann surfaces was associated with a unique pair of differentials. However, two conformally equivalent Riemann surfaces are identified only if a biholomorphic map between them can be continuously deformed to the identity map. Through this Teichmüller defined the set $T_g$ which covers the moduli space $\mathcal{M}_g$. With the metric $log\ D$, where $D > 1$ is the minimal dilation of quasi-conformal maps between the pair of conformally non-equivalent Riemann surfaces, Teichmüller showed that $T_g$ is homeomorphic to $\mathbb{R}^{6g-6}$ and that $\mathcal{M}_g$ is the quotient of $T_g$ by the action of a group isometries, the mapping class group.

The ideas of Teichmüller were continued by Ahlfors, Bers, Rauch and others [51], which led to the development of the whole complex analytic theory of the Teichmüller spaces. In particular, Ahlfors proved the existence of a natural structure of a complex analytic manifold on $T_g$. As a consequence, the mapping class group could be considered as a group of conformal homeomorphisms, endowing the moduli space $\mathcal{M}_g$ with a complex structure. Also, with the work of Fenchel and Nielsen, the Fuchsian group analogy of $T_g$ was developed and extended to larger classes of groups. The points fixed by the mapping class group correspond to Riemann surfaces with non-trivial automorphisms and it was shown that images of those are nonmanifold points (except some cases when $g = 2$ or $3$).

An automorphism of a Riemann surface is a biholomorphic self-map. At the end of the 19th century different properties of automorphisms where studied by Klein, Poincaré, Hurwitz, Clebsch and others. An important result due to Hurwitz [40] is that the total number of automorphisms for a surface of genus $g \geq 2$ is bounded by $84(g-1)$, i.e. any group of automorphims is finite. Wiman [58] improved the bound of the order of a single automorphism to $2(2g+1)$.

More recently automorphisms of Riemann surfaces as been studied due to their relation to moduli spaces of Riemann surfaces. By considering Riemann surfaces with topologically equivalent automorphism groups, called equisymmetric, Harvey [38] alluded to the equisymmetric stratification of the branch locus. Broughton [13] proved that this is indeed a stratification. Thus, to study the structure of the branch loci one can consider the equisymmetric Riemann surfaces. Broughton also studied the structure of branch locus for Riemann surfaces of genus $3$. Here we will present the structure of the branch loci for genera $4$ and $5$.

In addition to topological equivalence, other classifications have been considered. For genus $4$ and $5$ the automorphism groups has been studied by Kimura and Kuribayashi [42, 43] classifying them up to $GL(5,\mathbb{C})$-conjugacy. Breuer [11] generalized those ideas and classified the automorphism groups for $2 \leq g \leq 48$ by considering the character representations of their actions on the abelian differentials. This classification is known to be coarser, however not to what extent.

Riemann surfaces appear in several different mathematical fields. A reason for this is that there are several equivalent ways of studying them. Riemann showed that the following categories are isomorphic:

$$\left\{\begin{array}{c}\text{Smooth complex}\\ \text{algebraic curves}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{Compact}\\ \text{Riemann surfaces}\end{array}\right\}$$

The functor is sometimes called the Riemann functor. For Riemann surfaces of genus two or greater we also have the following isomorphism of categories:

$$\left\{\begin{array}{c}\text{Compact Riemann surfaces}\\ \text{of genus } g \geq 2\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{Conjugacy classes of}\\ \text{surface Fuchsian groups}\end{array}\right\}$$

## The moduli space of elliptic curves

The moduli space of elliptic curves, or tori, is well-known. Here we will construct this space and the Teichmüller space to give an intuitive understanding of the concept of moduli. Let $X$ be a flat torus. By cutting it along two geodesics we can identify the torus with a parallelogram in the complex plane. With rotation and magnification, which are symmetries of the complex plane $\mathbb{C}$, we may assume it is contained in the upper half-plane $\mathcal{H}$ and that one edge has end points $0$ and $1$. Thus we can identify the torus $X$ with the

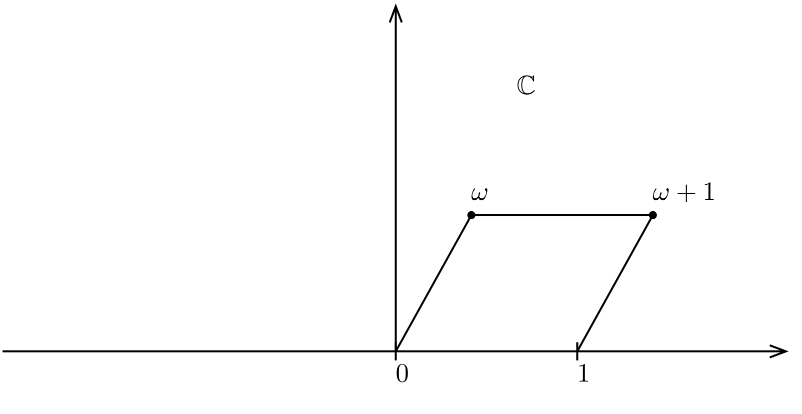


***Figure 1:*** *Parallelogram of a torus with modulus* $\omega$.

quotient space $\mathbb{C}/L$, where $L$ is a lattice $L = \{a + \omega b \,|a, b \in \mathbb{Z}\}$ which is determined by its modulus $\omega \in \mathcal{H}$. We clearly can't continuously deform one parallelogram into another through conformal maps. Thus we can identify the Teichmüller space of the tori as

$$T_1 = \mathcal{H} = \{z \in \mathbb{C} | Im(z) > 0\}.$$

However, we note that two different moduli $\omega_1$ and $\omega_2$ may define the same lattice. Thus resulting in two conformally equivalent tori. A map preserving lattices and mapping one modulus to the other is generated by the map $\omega \mapsto \omega + 1$. Further, we note that switching the sides of a parallelogram is just changing the orientation of the correponding torus, thus the map $\omega \mapsto -1/\omega$ implies conformal equivalence. The group generated by those maps has elements given by

$$\omega \mapsto \frac{a\omega + b}{c\omega + d}, \quad \text{where } a, b, c, d \in \mathbb{Z} \text{ such that } ad - bc = 1.$$

The group of such Möbius transformations is called the modular group, and is usually identified with $PSL(2, \mathbb{Z})$. The moduli space of the tori can then be identified with the quotient space

$$\mathcal{M}_1 = T_1/PSL(2, \mathbb{Z}) = \mathcal{H}/PSL(2, \mathbb{Z}).$$

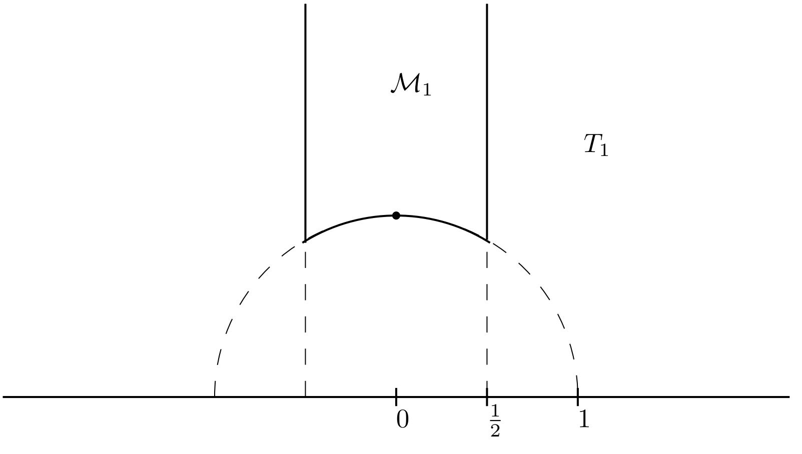


***Figure 2:*** *Teichmüller space and moduli space of tori.*

The point $i$ is fixed by $\omega \mapsto -1/\omega$, which has order $2$, and $e^{2i\pi/3}$ is fixed by $\omega \mapsto -1/(\omega + 1)$, which has order $3$, thus the branch locus is given by the set

$$\mathcal{B}_1 = \{i, e^{2i\pi/3}\}.$$

# Outline of the thesis

## Part I

### Chapter 1

In chapter 1 we introduce the basic concepts of Riemann surfaces, Fuchsian groups and automorphism groups of Riemann surfaces.

### Chapter 2

In chapter 2 we discuss a particular type of Riemann surfaces, the $p$-gonal and real $p$-gonal Riemann surfaces. We determine the automorphism groups of such surfaces.

### Chapter 3

In chapter 3 we consider Teichmüller spaces and moduli spaces of Fuchsian groups and the branch loci of the coverings. Further we introduce the concept of equisymmetric stratification of the branch loci and present theorems regarding the components of the branch loci.

## Part II

### Paper 1

In this paper we show that the strata corresponding to actions of order $2$ and $3$ belong to the same connected component for arbitrary genera. Further we show that the branch locus is connected with the exception of one isolated point for genera 5 and 6, it is connected for genus 7 and it is connected with the exception of two isolated points for genus 8.

### Paper 2

This paper contains a collection of results regarding components of the branch loci, some of them proved in detail in other papers. It is shown that for any integer $d$ if $p$ is a prime such that $p > (d+2)^2$, there there exist isolated strata of dimension $d$ in the moduli space of Riemann surfaces of genus $(d+1)(p-1)/2$. It is also shown that if we consider Riemann surfaces as Klein surfaces, the branch loci are connected for every genera due to reflections.

### Paper 3

Here we consider surfaces of genus $4$ and $5$. Here we study the automorphism groups of Riemann surfaces of genus $4$ and $5$ up to topological equivalence and determine the complete structure of the equisymmetric stratification of the branch locus. This can be divided into the following steps.

1. Determine all pairs of signatures $s$ and finite groups $G$ such that $G$ has an $s$-generating vector. This has been done [11, 43].
2. Determine the $Aut(G)$ classes of the generating vectors. Then determine the $\mathcal{B}$-orbits of the classes. This has been done with the aid of GAP [32].
3. Remove non-maximal actions. This has been done with the use of Singerman's list [55].
4. Compute actions determined by maximal subgroups of $G$. This has been done with the aid of GAP [32].

### Paper 4

In this paper we establish that the connectedness of the branch loci is a phenomenon for low genera. More precisely we prove, with the use of GAP [32], that the only genera $g$ where $\mathcal{B}_g$ is connected are $g = 3, 4, 13, 17, 19, 59$.

# Part I

# A Survey on the Branch Loci of Moduli Spaces

# 1

# Riemann surfaces and Fuchsian groups

The concept of the Riemann surface first appeared in Riemann's dissertation *Foundations for a general theory of functions of a complex variable* in 1851. While Riemann introduced the surfaces as a tool to study complex functions, they have grown to a subject of their own. Riemann surfaces can be studied in several different ways, for instance as manifolds, complex curves or quotient spaces. Here we will begin with a classical definition of Riemann surfaces due to Weyl, we will also introduce the concept of orbifolds due to Thurston and others. We will talk about the uniformization theory for Riemann surfaces by Poincaré, Klein and Koebe, which is the perspective we are mainly going to use throughout this thesis. Most Riemann surfaces are uniformized by so called Fuchsian groups and thus a large part of this chapter concerns those. Finally, we introduce the concept of automorphism groups of Riemann surface and their relation to Fuchsian groups. For details on Riemann surfaces and Fuchsian groups, see [8, 41], for basic groups theory see [31, 39], for covering maps see [46] and for a introduction to orbifolds see [56].

## 1.1 Riemann surfaces

In 1913 Weyl introduced the first abstract definition of a Riemann surface in *A concept of the Riemann surface*, stating that a *Riemann surface* is a Haussdorf space $X$ that is locally homeomorphic to the complex plane. This means that each point in $X$ has an open neighbourhood $U_i$ such that there exists a homeomorphism $\Phi_i : U_i \to V_i$, where $V_i$ is an open subset of $\mathbb{C}$. We call the pair $(\Phi_i, U_i)$ a *chart*. An *atlas* is a set of charts $A$ covering $X$ such that if $U_i \cap U_j \neq \emptyset$ then the *transition function*

$$\Phi_i \circ \Phi_j^{-1} : \Phi_j(U_i \cap U_j) \to \Phi_i(U_i \cap U_j)$$

is analytic. Further we call two analytic atlases $A, B$ *compatible* if all the transition functions of charts $(\Phi, U) \in A, (\Psi, V) \in B$ are analytic. Such atlases form an equivalence class called a *complex structure*. To sum this up; *a Riemann surface is a topological surface together with a complex structure*. Different complex structures on the same topological surface yield different Riemann surfaces. In particular we are interested in *compact* Riemann surfaces, which are modeled on a compact topological surface. Compact Riemann surfaces can be identified with smooth complex algebraic curves, i.e. sets of zeros of polynomial equations in two complex variables (or three if considered as projective curves). However, we will not use this fact except in some examples.

**Example 1.1**

(i) $\mathbb{C}$ with an atlas consisting of the identity map.

(ii) $\hat{\mathbb{C}}$ with an atlas consisting of the identity map of $\mathbb{C}$, $Id$, together with the map $\phi : \hat{\mathbb{C}} \setminus \{0\} \to \mathbb{C}$ defined as $\phi(z) = 1/z$ and $\phi(\infty) = 0$. This is indeed an atlas since $Id \circ \phi^{-1} = 1/z$ and $\phi \circ Id^{-1} = 1/z$ are analytic on $\mathbb{C} \cap \hat{\mathbb{C}} \setminus \{0\} = \mathbb{C} \setminus \{0\}$.

We are in particular interested in maps between Riemann surfaces and through the use of atlases we can define holomorphic maps between them. Let $X$ and $Y$ be Riemann surfaces with atlases $\{(\Phi_i, U_i)\}$ and $\{(\Psi_j, V_j)\}$ respectively. A map $f : X \to Y$ is called *holomorphic (or meromorphic)* if the maps

$$\Psi_j \circ f \circ \Phi_i^{-1} : \Phi_i(U_i \cap f^{-1}(V_j)) \to \mathbb{C}$$

are analytic (or meromorphic). An analytic map is sometimes called a *conformal* map. Conformal maps are the homomorphisms of the category of Riemann surfaces. Further, if $f$ is bijective and $f^{-1}$ is also holomorphic then we call $f$ *biholomorphic* (or sometimes a *conformal homeomorphism*). Two Riemann surfaces are *conformally equivalent* if there exists a biholomorphic map between them. If two Riemann surfaces are conformally equivalent, they are topologically equivalent and their complex structures are equivalent. Hence, we do not distinguish between conformally equivalent surfaces. A biholomorphism $f : X \to X$ is known as an *automorphism* of the Riemann surface $X$. We will take a closer look at automorphisms and groups of automorphisms of Riemann surfaces at the end of this chapter.

## Coverings and uniformization

Now we will consider another type of map between Riemann surfaces, *covering* maps (or simply *coverings*). A covering is a surjective continuous map $f : X \to Y$ such that for any point $y \in Y$ the preimage of some neighbourhood $V$ is a disjoint union of open subsets of $X$, each mapped homeomorphically to $V$ by $f$. The set of the preimages of $y$, $f^{-1}(y)$ is called a *fiber* and each fiber has the same cardinality. If $n = |f^{-1}(y)|$ is finite, we call $f$ an *n-sheeted* covering. A surjective continuous map $f : X \to Y$ such that

$$f : X \setminus f^{-1}(\{y_1, \ldots, y_k\}) \to Y \setminus \{y_1, \ldots, y_k\}$$

is a covering is called a *branched covering* and the points $y_i$ are called *branch points*. Two coverings $f : X \to Y$ and $f' : X' \to Y$ are considered *equivalent* if the exists a biholomorphism $g : X' \to X$ such that $f' = f \circ g$. The behaviour of a branched covering close to the branch points is well-known:

**Proposition 1.1.** *Let $f : X \to Y$ be a n-sheeted branched covering. Then for each point $x \in X$ there exist neighbourhoods $U$ of $x$ and $V$ of $0 \in \mathbb{C}$ and biholomorphisms $g : U \to V$ and $h : f(U) \to f(V)$ such that*

$$h \circ f \circ g^{-1} : V \to f(V) \text{ where } z \mapsto z^{m_x}$$

*and for each $y \in Y$*

$$\sum_{x \in f^{-1}(y)} m_x = n.$$

**Example 1.2**

(i) The map $\mathbb{C} \times \{0, 1\} \to \mathbb{C}$ defined by $(z, x) \mapsto z$ is a two-sheeted covering.

(ii) The map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by $z \mapsto z^n$ is a $n$-sheeted branched covering. The branch points are $0$ and $\infty$.

**Theorem 1.1.** *[46] Let $f : X \to Y$ be a covering, with Y path-wise connected. For each $p \in Y$, the fundamental group $\pi_1(Y, p)$ acts transitively on the right in the fiber $f^{-1}(p)$. The stabilizer of each point $x \in f^{-1}(p)$ is $H(x) = f_{\#}\pi_1(X, x)$.*

The permutations of a fiber $f^{-1}(p)$ by $g \in \pi_1(Y, p)$ forms a group called the *monodromy group* of $f$ at $p$. For Riemann surfaces all fibers are permuted in the same way so we might talk about the monodromy group of $f$, in this case it is sometimes called the *deck transformation group*. If $M$ is the monodromy group of an n-sheeted covering $f : X \to Y$, then $\pi_1(Y) : M = n$. Consider a closed path $A \subset Y$, then $A$ lifts to $n$ different paths $A'_i \subset X$ where each starting point $A'_i$ corresponds to a point in the fiber$f^{-1}(A(0))$. Now each endpoint $A'_i(1)$ is also a lift of the point $A(0)$ and thus is equal to $A_{\sigma(i)}(0)$, for some permutation $\sigma \in S_n$. Thus we get a permutation representation of the monodromy given by

$$\pi_1(Y) \to S_n.$$

Similarly for a branched covering $f : X \to Y$ the permutation representation of the monodromy is given by the induced (smooth) covering $f : X \setminus f^{-1}(\{y_1, \ldots, y_k\}) \to Y \setminus \{y_1, \ldots, y_k\}$. We note that around for each branch point $y$ there is a neighbourhood $U$, such that for each ramification point $x$, a loop in $U$ around $y$ lifts to $m_x$ paths forming a loop around $x$. If for each loop in $Y$ the lifts are either all closed or none of them is closed, then we call $f$ a *regular covering*.

Consider a Riemann surface $X$, then there exists a simply connected Riemann surface $\mathcal{U}$ such that we have covering $p : \mathcal{U} \to X$. Such a covering is known as a *universal covering*. We have the following theorem known as Poincaré's first theorem:

**Theorem 1.2.** *Let $p : \mathcal{U} \to X$ be the universal covering of a Riemann surface $X$. Then $\mathcal{U}$ is one of the following spaces:*

*(i) The complex plane $\mathbb{C}$*

*(ii) The Riemann sphere $\widehat{\mathbb{C}}$*

*(iii) The hyperbolic plane $\mathcal{H}$*

It follows that every simply connected Riemann surface is conformally equivalent to one of $\mathbb{C}$, $\widehat{\mathbb{C}}$ and $\mathcal{H}$. One property that distinguish the three simply connected Riemann surfaces is their curvature. The sphere has a constant positive curvature, the hyperbolic plane constant negative curvature and the Euclidean plane has everywhere vanishing curvature. Due to this we can restate the existence of a universal covering as follows: *every Riemann surface admits a Riemann metric of constant curvature.*

**Theorem 1.3.** *The only Riemann surface that does admit a universal covering given by $\hat{\mathbb{C}}$ is $\hat{\mathbb{C}}$ itself. The only Riemann surfaces that do admit a universal covering given by $\mathbb{C}$ are:*

*(i) The complex plane $\mathbb{C}$*

*(ii) The cylinder*

*(iii) The tori*

Every other Riemann surface has the hyperbolic plane $\mathcal{H}$ as universal covering space. In particular, every compact Riemann surface of genus greater than 1 is uniformized by $\mathcal{H}$.

## Orbifold structures

When working with maps between Riemann surfaces, in particular automorphisms of Riemann surfaces and coverings of Riemann surfaces, we sometimes get singularities, or cone points, as seen in Example 1.2. The previous definition of a Riemann surface is as a two dimensional manifold. However, it is useful to include the cone points when we consider coverings or group actions on Riemann surfaces. To do this we will use the notion of orbifolds introduced by Thurston and will define Riemann surfaces as 2-orbifolds.

A *two-dimensional orbifold* $X$ is a Haussdorf space (a topological surface) together with an atlas of *folding charts* $\{\Phi_i, G_i, U_i, V_i\}$ where $U_i \subset X$ is mapped homeomorphically to $V_i/G_i$, $V_i \subset \mathbb{C}$. Here $G_i$ is a finite cyclic or trivial group, since we only consider compact surfaces. Further, if $\Phi_i^{-1}(x) = \Phi_j^{-1}(y)$ then there exists $V_x \subset V_i$ and $V_y \subset V_j$ such that $x \in V_x$ and $y \in V_y$ and

$$\Phi_j \circ \Phi_i^{-1} : V_x \to V_y$$

is a diffeomorphism.

**Example 1.3**

(i) The surfaces in Example 1.1 can be seen as orbifolds with $G = Id$.

(ii) $\hat{\mathbb{C}}$ with the charts $\{z^{\frac{1}{n}}, C_n, \mathbb{C}, \mathbb{C}\}$ and $\{1/z^{\frac{1}{n}}, C_n, \hat{\mathbb{C}} \setminus \{0\}, \mathbb{C}\}$. This is the sphere with two cone points, 0 and $\infty$, of order $n$.

Let $X_g$ be a Riemann surface with an orbifold atlas inducing $k$ cone points with orders

$m_k$, the *orbifold structure* of $X_g$ is then given by $(g; m_1, \ldots, m_k)$. We consider two Riemann surfaces with orbifold structures to be equivalent if the they are conformally equivalent and the cone points are mapped to cone points of the same order. An orbifold $O$ such that there exists a branched covering $M \to O$, where $M$ is a manifold is known as a *good* orbifold.

**Example 1.4**

The only compact Riemann surfaces with orbifold structure that does not admit a universal covering are

(i) The teardrop: $\hat{\mathbb{C}}$ with the charts $\{Id, Id, \mathbb{C}, \mathbb{C}\}$ and $\{1/z, C_n, \hat{\mathbb{C}} \setminus \{0\}, \mathbb{C}\}$.

(ii) The rugby-ball: $\hat{\mathbb{C}}$ with the charts $\{Id, C_m, \mathbb{C}, \mathbb{C}\}$ and $\{1/z, C_n, \hat{\mathbb{C}} \setminus \{0\}, \mathbb{C}\}$.

Those are also known as *bad* orbifolds.

If we consider Riemann surfaces with orbifold structure then we have a result similar to Theorem 1.2 [48, 56].

**Theorem 1.4.** *Let the Euler number of a good orbifold be defined as*

$$\chi(X) = 2g - 2 + \sum \left(1 - \frac{1}{m_k}\right). \tag{1.1}$$

*Then the universal cover of $X$ is*

*(i) the hyperbolic plane if and only if $\chi(X) > 0$.*

*(ii) the complex plane if and only if $\chi(X) = 0$.*

*(iii) the sphere if and only if $\chi(X) < 0$.*

With the universal coverings we can define the *orbifold fundamental group* $\bar{\pi}_1(X)$ of a Riemann surface $X$ as the monodromy group of the universal (branched) covering. Consider a cone point $x \in X$ of order $m$, then a loop around $x$ lifts to $m$ paths which form a closed loop. Thus the monodromy group, and hence the orbifold fundamental group has elements of order $m$. We will give a more precise description of the structure later when we identify the orbifold fundamental group with a Fuchsian group.

## Hyperbolic geometry

As the hyperbolic plane is important in our work, it is useful to have a model for it. Two common models are the Poincaré disc model and the Poincaré upper half-plane model:

(i) The *Poincaré disc model* is given by the unit disc $\mathcal{D} = \{z||z| < 1\}$ with the metric given by $ds = \frac{2|dz|}{1-|z|^2}$. *Hyperbolic lines*, or geodesics, corresponds to lines and circles arcs perpendicular to the unit circle.

(ii) The *Poincaré half-plane model* is given by the upper half-plane $\mathcal{H} = \{z|Im(z) > 0\}$ with the metric given by $ds = \frac{|dz|}{|Im(z)|}$. Hyperbolic lines corresponds to lines and circles arcs perpendicular to the real line.

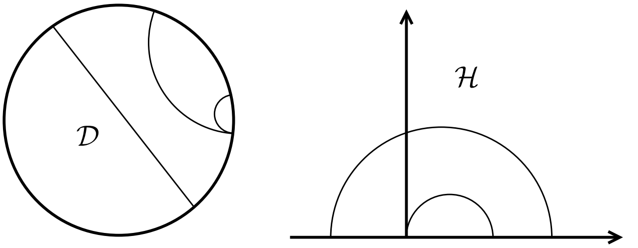


***Figure 1.1:*** *Hyperbolic lines in the Poincaré models of the hyperbolic plane.*

Note that while $\mathbb{C}$ has a "point of infinity", the hyperbolic plane has a "circle of infinity" given by the unit circle in the disc model and by the real line together with the point of infinity in the half-plane model. A biholomorphism between $\mathcal{H}$ and $\mathcal{D}$ is given by

$$z \mapsto \frac{z-i}{z+i}.$$

## 1.2 Fuchsian groups

Before we introduce Fuchsian groups we need some basic definitions of group actions on surfaces. Let $G$ be a group of homeomorphisms on a topological surface $X$. Then we say that $G$ acts *freely* or is *fixed point free* on $X$ if for each point $x \in X$ there is a neighbourhood $V$ such that $g(V) \cap V = \emptyset$ for a $g \in G$. However, we are going to consider groups with fixed points, so we also need a weaker condition. If each point $x \in X$ has a neighbourhood such that $g(V) \cap V \neq \emptyset$ for a finite number of $g \in G$ we say that $G$ acts *properly discontinuously* on $X$. The space of the orbits of the points of $X$ under the action $G$ is denoted by $X/G$.

**Theorem 1.5.** *Let $X$ be a topological space and $G$ a group of homeomorphisms acting on it. Then the following statements are equivalent:*

$(i)$ *The natural map $X \to X/G$ is a covering (branched covering).*

$(ii)$ *$G$ acts freely (properly discontinuously).*

First we are going to look at group actions on the hyperbolic plane $\mathcal{H}$ by conformal isometries. By considering the half-plane model of $\mathcal{H}$ we note that an isometry will map circles to circles in $\mathbb{C}$ and in particular preserve the real line. Thus we can identify the isometries, or automorphisms, as the set of Möbius transformations of the following kind;

$$z \to \frac{az+b}{cz+d}, \quad ad-bc=1, \; a,b,c,d \in \mathbb{R} \tag{1.2}$$

and this group is isomorphic to $PSL(2,\mathbb{R})$. The elements of $PSL(2,\mathbb{R})$ can be divided into three categories, characterized by their fixed point sets. We say that an element $g \in PSL(2,\mathbb{R})$ is

(i) hyperbolic if it has one fixed set $\{x,y\} \in \mathbb{R} \cup \{\infty\}$.

(ii) elliptic if it has one fixed point $z \in \mathcal{H}$.

(iii) parabolic if it has one fixed point $x \in \mathbb{R} \cup \{\infty\}$.

**Example 1.5**

(i) the map $z \mapsto \lambda z$, where $\lambda \in \mathbb{R}^+$, is hyperbolic with fixed points $0$ and $\infty$.

(ii) the map $z \mapsto -\frac{1}{z}$, is an elliptic element with fixed point $i$.

(iii) the map $z \mapsto z+1$, is a parabolic element with fixed point $\infty$.

Now, a group of conformal Möbius transformations is a *Fuchsian group* if it leaves a disc invariant on which it acts properly discontinuously. By identifying the hyperbolic plane $\mathcal{H}$ with the upper half-plane and $PSL(2,\mathbb{R})$ as a topological group with topology given by the matrix norm, we can consider a Fuchsian group as a discrete subgroup of $PSL(2,\mathbb{R})$.

**Example 1.6**

A famous Fuchsian group is the so called *modular group* $PSL(2,\mathbb{Z})$, consisting of the Möbius transformations with integer coefficients:

$$z \mapsto \frac{az+b}{cz+d}, \quad ad-bc=1, \ a,b,c,d \in \mathbb{Z}.$$

The surface $\mathcal{H}/PSL(2,\mathbb{Z})$ is a sphere with two cone points and one puncture.

We are only interested in Fuchsian groups $\Gamma$ such that the quotient space $\mathcal{H}/\Gamma$ is a compact Riemann surface. Then $\Gamma$ consists only of hyperbolic and elliptic elements and has the following presentation

$$\left\langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1 \ldots \gamma_k \mid \gamma_1^{m_1} = \cdots = \gamma_k^{m_k} = \prod \gamma_i \prod [\alpha_i, \beta_i] = 1 \right\rangle \tag{1.3}$$

where $x_i$ is a elliptic element and $a_i$ and $b_i$ are hyperbolic elements. We note that this group is isomorphic to the orbifold fundamental group of $\mathcal{H}/\Gamma$. If a group $\Gamma$ has presentation 1.3 then we say that $\Gamma$ has *signature*

$$s(\Gamma) = (g; m_1, \ldots, m_k) \tag{1.4}$$

where $g$ is called the *genus* of the topological surface $\mathcal{H}/\Gamma$ and $m_i, \ i = 1 \ldots k$ are the *orders* of the stabilizers of the cone points of the surface. The signature (1.4) gives us the algebraic structure of $\Gamma$ and the geometrical structure, or orbifold structure, of $\mathcal{H}/\Gamma$. We may assume that $m_i \leq m_{i+1}$ by noting that one can replace a pair of elliptic generators $\gamma_i$ and $\gamma_{i+1}$ with $\gamma_i' = \gamma_i \gamma_{i+1} \gamma_i^{-1}$ and $\gamma_{i+1}' = \gamma_i$. We also note that for each elliptic element $\gamma$ of $\Gamma$ there is a unique elliptic generator $\gamma_i$ such that $\gamma$ is conjugated to a unique element in $\langle \gamma_i \rangle$ [8].

**Theorem 1.6.** *[41] If $\Gamma$ is a Fuchsian group with signature (1.4) then the quotient space $\mathcal{H}/\Gamma$ is a Riemann surface.*

As we are interested in compact Riemann surfaces of genus $g \geq 2$ we would like to identify them with $\mathcal{H}/\Gamma$ for some Fuchsian group $\Gamma$. This is indeed possible:

**Theorem 1.7.** *[41] If $X$ is a Riemann surface not conformally equivalent to the sphere $\hat{\mathbb{C}}$, the plane $\mathbb{C}$, the punctured plane $\mathbb{C} \setminus \{0\}$ or the tori, then $X$ is conformally equivalent to $\mathcal{H}/\Gamma$, where $\Gamma$ is a Fuchsian group without elliptic elements.*

Based on this, a Fuchsian group without elliptic elements, i.e. the signature is $(g; -)$, is called a *surface group*. Surface groups are important for several reasons, one is concerning conformal equivalence:

**Theorem 1.8.** *[41] Let $\Gamma_1$ and $\Gamma_2$ be two surface Fuchsian groups, then the Riemann surfaces $\mathcal{H}/\Gamma_1$ and $\mathcal{H}/\Gamma_2$ are conformally equivalent if and only if $\Gamma_2 = \tau^{-1}\Gamma_1\tau$ for some $\tau \in PSL(2, \mathbb{R})$.*

Thus the category of Riemann surfaces of genus $g \geq 2$ is isomorphic to the category of conjugacy classes of surface Fuchsian groups. As a consequence of Theorem 1.6 and Theorem 1.7 we note that while the surface $\mathcal{H}/\Gamma$ may have cone points as an orbifold it is conformally equivalent to a smooth surface.

## Fundamental domains

While a Fuchsian group acts on the whole hyperbolic plane it is sometimes useful to consider a subset, usually polygon-shaped, of $\mathcal{H}$. Given a Fuchsian group $\Gamma$ a closed set $F \subset \mathcal{H}$ is called a *fundamental domain* to $\Gamma$ if it satisfies the following conditions:

(i) $\mathcal{H} = \bigcup_{\gamma \in \Gamma} \gamma(F)$.

(ii) If $p \in F$ and $\gamma(p) \in F$, where $\gamma \neq Id$, then $p \in \delta F$.

(iii) $\mu(\delta F) = 0$, where $\mu$ is the hyperbolic measure.

While any subset satisfying the three conditions suffices, for a Fuchsian group with signature (1.4) we can choose a polygonal-shaped fundamental domain, or a *fundamental polygon*. Now assume $F$ is a fundamental polygon for some Fuchsian group $\Gamma$. Then

(i) two sides, $A$ and $A'$ are called *congruent* if $A' = \gamma(A)$, for some $\gamma \in \Gamma$, also $\gamma(F) \cap F = A'$.

(ii) two vertices, $v$ and $v'$ are congruent if $v' = \gamma(v)$, for some $\gamma \in \Gamma$.

(iii) each elliptic element $\gamma_i$ is conjugated to an elliptic element with a fixed point $v$ corresponding to set of congruent vertices with a sum of angles equal to $2\pi/m_i$.

(iv) $F/\sim$ has the same hyperbolic structure as $\mathcal{H}/\Gamma$, where $\sim$ is the set of side-pairings defined above.

As state above, for a Fuchsian group we can construct a fundamental polygon. A faschinating and famous theorem by Poincaré states that we can go in the other direction as well; starting with any hyperbolic polygon satisfying a few constrains the group generated by the side pairings is a Fuchsian group:

**Theorem 1.9.** *Let $F$ be a hyperbolic polygon with side pairings generating a group $\Gamma$ satisfying*

$(i)$ *for each vertex $p$ of $F$ there are vertices $p$,$p_1$,…, $p_n$ and elements $\gamma$ $(= id)$, $\gamma_1$,…, $\gamma_n$ of $\Gamma$ such that $\gamma_i(U_i)$ are non-overlapping and $\bigcup \gamma_i(U_i) = B(p, \varepsilon)$, where $U_i = \{q \in F \mid d(p, q) < \varepsilon\}$.*

$(ii)$ *each $\gamma_{i+1} = \gamma_i\gamma_s$, where $\gamma_s$ is a side pairing and $\gamma_{n+1} = id$.*

*Further there exists $\varepsilon$ such that for each $p \in F$, $B(p, \varepsilon)$ is in a union of images of $F$. Then $\Gamma$ is a Fuchsian group and $F$ is a fundamental polygon of $\Gamma$.*

While there are many different fundamental domains for a Fuchsian group they all have the same area, i.e. if $F$ and $F'$ are fundamental domains of $\Gamma$ then $\mu(F) = \mu(F')$. Thus we define the *hyperbolic area* of $\Gamma$, denoted by $\mu(\Gamma)$, as $\mu(F)$ for some fundamental domain $F$ of $\Gamma$. The area, $\mu(F)$, of a fundamental polygon $F$ of a Fuchsian group with signature (1.4) is given by the Gauss-Bonnet formula:

$$\mu(F) = 2\pi(2g - 2 + \sum_i \left(1 - \frac{1}{m_i}\right)). \tag{1.5}$$

We note that the area is equal to $2\pi\chi(\mathcal{H}/\Gamma)$, where $\chi(\mathcal{H}/\Gamma)$ is the Euler number of the orbifold $\mathcal{H}/\Gamma$.

**Example 1.7**

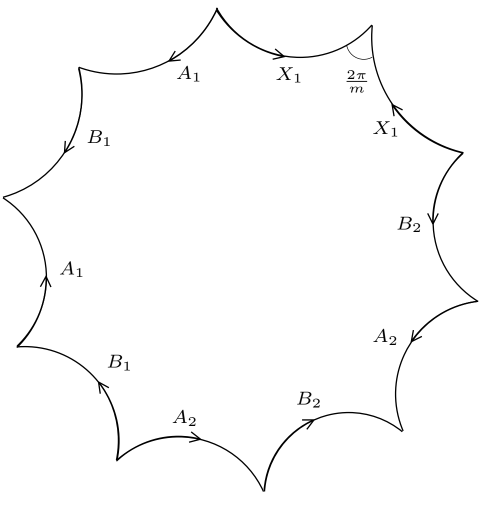


***Figure 1.2:*** *Fundamental polygon*

A fundamental polygon of a Fuchsian group with signature $(2; m)$ is given by the ten-sided fundamental polygon such that the sides are paired as in Figure 1.2. The sides $A_i$, $A_i'$ and $B_i$, $B_i'$ are paired by the hyperbolic elements and the sides $X_1$, $X_1'$ are paired by an elliptic element of order $m$. The area of the fundamental polygon is $2\pi(3 - \frac{1}{m})$.

## Fuchsian groups and subgroups

We note that $\mu(\Gamma) > 0$ for any Fuchsian group. Actually, given an arbitrary signature, the existence of a Fuchsian group with the signature depends only on the induced area of the signature:

**Theorem 1.10.** *[41] If $g \geq 0$, $m_i \geq 2$ are integers and if*

$$2g - 2 + \sum_i \left(1 - \frac{1}{m_i}\right) > 0 \tag{1.6}$$

*then there exists a Fuchsian group with signature $(g; m_1, \ldots, m_k)$.*

**Example 1.8**

From the discussion above one can note that there exists a minimum of the possible values of $\mu(\Gamma)$. Indeed this is given by the signature $(0; 2, 3, 7)$ and the area is

$$2\pi(2g - 2 + \sum_i \left(1 - \frac{1}{m_i}\right)) = \frac{\pi}{21}.$$

**Example 1.9**

Let $\Delta$ be a hyperbolic triangle with angles $\pi/m_1, \pi/m_2, \pi/m_3$ and let $r_i$, $i = 1, 2, 3$ be the reflections in the sides of $\Delta$. If $\Gamma^*$ is the group generated by $r_1, r_2, r_3$ then $\Gamma = \Gamma^* \cap PSL(2, \mathbb{R})$ is a Fuchsian group called a *triangle group* and is generated by the elliptic elements $r_1 r_2$, $r_2 r_3$, $r_3 r_1$ where $(r_1 r_2)^{m_3} = (r_2 r_3)^{m_1} = (r_3 r_1)^{m_2} = Id$. $\Gamma$ has the following presentation

$$\left\langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_1^{m_1} = \gamma_2^{m_2} = \gamma_3^{m_3} = \gamma_1\gamma_2\gamma_3 = 1 \right\rangle.$$

Now consider two Riemann surfaces $\mathcal{H}/\Delta$ and $\mathcal{H}/\Delta'$, where $s(\Delta) = (g; m_1, \ldots, m_k)$. Assume that $\Delta' \subset \Delta$, then we have a (branched) covering

$$\mathcal{H}/\Delta' \to \mathcal{H}/\Delta.$$

The monodromy of the covering is tied to the relation between $\Delta$ and $\Delta'$. Here we will consider properties of subgroups of Fuchsian groups and later see how those are related to automorphism groups of Riemann surfaces. The possible algebraic structures of a subgroup are as follows:

**Theorem 1.11.** *([54]) Let $\Delta$ be a Fuchsian group with signature (1.4) and canonical presentation (1.3). Then $\Delta$ contains a subgroup $\Delta'$ of index $N$ with signature*

$$s(\Gamma') = (h; m'_{11}, m'_{12}, ..., m'_{1s_1}, ..., m'_{k1}, ..., m'_{ks_k}).$$

*if and only if there exists a transitive permutation representation $\theta : \Delta \to S_N$ satisfying the following conditions:*

*1. The permutation $\theta(\gamma_i)$ has precisely $s_i$ cycles of lengths less than $m_i$, the lengths of these cycles being $m_i/m'_{i1}, ..., m_i/m'_{is_i}$.*

*2. The Riemann-Hurwitz formula*

$$\mu(\Gamma')/\mu(\Gamma) = N.$$

*where $\mu(\Delta),\ \mu(\Delta')$ are the hyperbolic areas of the surfaces $\mathcal{H}/\Delta$, $\mathcal{H}/\Delta'$.*

Another way to look at subgroups of Fuchsian groups is to consider the fundamental domains of the groups. Let $\Delta'$ be a subgroup of a Fuchsian group $\Delta$. Then $\Delta = \bigcup g_i\Delta'$, where $\{g_i\}$ is a set of transversals. If $F$ is a fundamental domain for $\Delta$ then

$$F' = \bigcup g_i(F)$$

is a fundamental domain for $\Delta'$. Let us return to the (branched) covering $\mathcal{H}/\Delta \to \mathcal{H}/\Delta'$. The sheets of the covering can be identified with the sets $\gamma_i(F)$ given be the transversals. $\Delta'$ permutes those as

$$\gamma\gamma_i(F) = \gamma_j(F),\ \text{for}\ \gamma \in \Delta,\ \text{and some}\ \gamma_j \in \{\gamma_i\},$$

inducing a transitive permutation representation $\theta : \Delta \to S_N$. Thus we can identify this with the monodromy, or deck transformations, of the covering $\mathcal{H}/\Delta \to \mathcal{H}/\Delta'$. Before we consider automorphism groups of Riemann surfaces we will look at an example.

**Example 1.10**

Consider the polygon $F_1$ in Figure 1.3. Let $\Delta_1$ be the Fuchsian group induced by the side pairings given by the clockwise rotations at the vertices. Let $\gamma_i$ be the rotation in $p_i$; $\gamma_1$, $\gamma_2$, and $\gamma_3$ has order $2$, and $\gamma_4$ has order $4$. Further we note that

$$\gamma_1\gamma_2\gamma_3\gamma_4 = 1.$$

Thus $\Delta_1$ has signature $(0; 2, 2, 2, 4)$. Let $F_2$ be the polygon we get by gluing $F_1$ together with its image under the involution $\gamma_1$ in $p_1$. Now the generators of $\Delta_1$ permutes $F_1$ and $\gamma_1(F_1)$, denoted by $1$ and $2$ respectively, resulting in the following permutation representation:

$$\gamma_1 \to (1\ 2)$$

$$\gamma_2 \to (1\ 2)$$

$$\gamma_3 \to (1)(2)$$

$$\gamma_4 \to (1)(2)$$

As stated in Theorem 1.11 this induce a subgroup $\Delta_2$ with signature $s(\Delta_2) = (0; 2, 2, 4, 4)$ of index $\mu(\Gamma_2)/\mu(\Gamma_1) = 2$. We also note that $F_2$ is a fundamental polygon of $\Delta_2$. A set of generators of $\Delta_2$ is given by; $\gamma_3$, $\gamma_4\gamma_2\gamma_3\gamma_2\gamma_4^{-1}$, $\gamma_4$ and $\gamma_2\gamma_4\gamma_2$.

Similarly we can consider the polygon $F_3$ in Figure 1.3. Here we get $F_3$ by gluing $F_2$ together with its image under the involution in $p_3$. Now the generators of $\Delta_1$ permutes $F_1$, $\gamma_1(F_1)$, $\gamma_3(F_1)$ and $\gamma_3\gamma_1(F_1)$, denoted by $1$, $2$, $3$ and $4$ respectively, resulting in the following permutation representation:

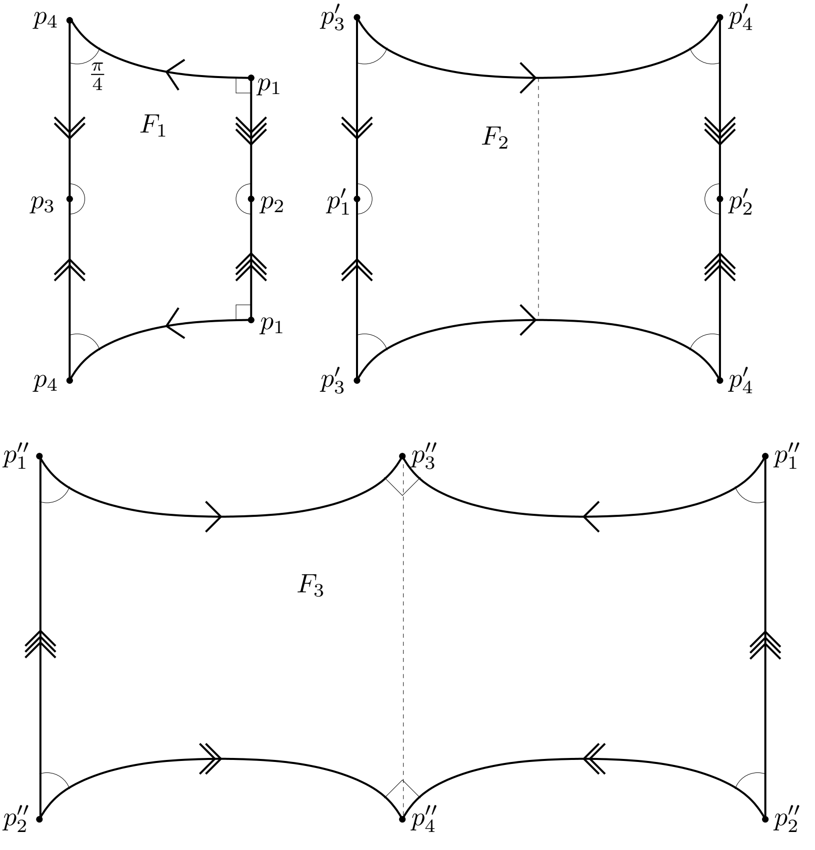


***Figure 1.3:*** *Fundamental polygons of a Fuchsian group and subgroups*

$$\gamma_1 \to (1\ 2)(3\ 4)$$
$$\gamma_2 \to (1\ 2)(3\ 4)$$
$$\gamma_3 \to (1\ 3)(2\ 4)$$
$$\gamma_4 \to (1)(2)(3)(4)$$

Thus this induce a subgroup of $\Delta_1$ of index $4$ and with signature $(0; 4, 4, 4, 4)$.

Subgroups structures of Fuchsian groups are also useful to study automorphism groups of Riemann surfaces. For the remainder of this chapter we will consider the automorphism groups and their relations to Fuchsian groups.

## 1.3 Automorphism groups of Riemann surfaces

An automorphism of a Riemann surface $X$ is a conformal homeomorphism $f : X \to X$. We denote the full group of automorphisms of a Riemann surface $Aut(X)$.

**Example 1.11**

The full groups of automorphisms of the simply connected Riemann surfaces:

(i) $Aut(\mathbb{C}) = \{az + b \mid a, b \in \mathbb{C}, a \neq 0\}$.

(ii) $Aut(\hat{\mathbb{C}}) = PSL(2, \mathbb{C})$.

(iii) $Aut(\mathcal{H}) = PSL(2, \mathbb{R})$.

We are mainly interested in Riemann surfaces of genus greater than one. In this case there is a well-known upper bound for the size of an automorphism group, due to Hurwitz:

**Theorem 1.12.** *[40] Let $X$ be a compact Riemann surface of genus $g$, $g \geq 2$. Then $|Aut(X)| \leq 84(g-1)$.*

A Riemann surface $X$ such that the bound is attained is known as a *Hurwitz surface* and the automorphism group $Aut(X)$ is known as a *Hurwitz group.*

**Example 1.12**

The smallest genera such that there exists a Hurwitz surface is $g = 3$. This well-known surface is called the *Klein quartic.* And is given as a curve by the polynomial equation $x^3y + y^3z + z^3x = 0$. The automorphism group of the Klein quartic, which is the smallest Hurwitz group, is $PSL(2, 7)$.

## Automorphism groups and Fuchsian groups

Automorphisms of Riemann surfaces of genus greater than 1 are closely related to Fuchsian groups. Consider a Riemann surface $X$, uniformized by a surface Fuchsian group $\Gamma$, and a group of automorphisms, $G$, of $X$. Then $G$ acts on $X$ and we can consider the quotient space $Y = X/G$, which is a Riemann surface (with cone points). We have the following diagram of coverings:

$$\begin{tikzcd}
\mathcal{H} \arrow[dr] \arrow[dd] & \\
& X = \mathcal{H}/\Gamma \arrow[dl] \\
\mathcal{H}/\Delta = Y = X/G &
\end{tikzcd}$$

As $Y$ is a Riemann surface (with cone points) there exists a Fuchsian group $\Delta$ uniformizing it, where $\Delta$ is the lift of $G$ to $\mathcal{H}$:

**Theorem 1.13.** *[41] If $\Gamma$ is a surface Fuchsian group then every group of automorphisms of $\mathcal{H}/\Gamma$ is isomorphic to $\Delta/\Gamma$ where $\Delta$ is a Fuchsian group such that $\Gamma \trianglelefteq \Delta$.*

Thus we have an exact sequence

$$1 \to \Gamma \xrightarrow{i} \Delta \xrightarrow{\theta} G \to 1,$$

where $\theta$ is called a *surface kernel epimorphism.*

**Example 1.13**

If $G$ is a Hurwitz group then the signature of $\Delta$ given in the sequence above has signature $(0; 2, 3, 7)$, since

$$|Aut(X)| = |\Delta : \Gamma| = \frac{\mu(\Gamma)}{\mu(\Delta)} = \frac{2\pi(2g-2)}{\pi/21} = 84(g-1).$$

This relation can be used to prove the Hurwitz bound.

Now since $\Gamma = ker(\theta)$ is a surface group, the epimorphisms $\theta$ has to preserve the orders of the elliptic generators, $\gamma_i$, of $\Delta$. With this in mind, assume that $s = (h; m_1, \ldots, m_k)$ is the signature of $\Delta$, then an *$s$-generating vector* of a finite group $G$ is a vector

$$(a_1, b_1, \ldots, a_h, b_h; c_1, \ldots, c_k), \text{ where } a_i, b_i, \text{ and } c_i \in G, \tag{1.7}$$

such that $ord(c_i) = m_i$ and $\prod[a_i, b_i] \prod x_i = 1$. In particular we note that

$$(\theta(\alpha_1), \theta(\beta_1), \ldots, \theta(\alpha_h), \theta(\beta_h); \theta(\gamma_1), \ldots, \theta(\gamma_k))$$

is an $s$-generating vector. We also note here that the choice of generating vector is not unique, as it depends on choice of generators of the groups. Thus given a group of automorphisms of a Riemann surface we can construct a generating vector. Now remember that Theorem 1.10 states that for any given signature inducing a positive area, there exists a Fuchsian group with the given signature. Thus if a finite group $G$ admits an $s$-generating vector for a signature $s$ satisfying the Riemann-Hurwitz formula, then there exists a a Fuchsian group $\Delta$ admitting this signature. From this we can construct a surface kernel epimorphism $\theta : \Delta \to G$, with $s(ker(\theta)) = (g; -)$. Thus to determine if a group acts on a Riemann surface is equivalent to construct $s$-generating vectors:

**Theorem 1.14.** *[14](Riemann's Existence Theorem) A group $G$ acts on a Riemann surface of genus $g$ with branching data $(h; m_1, \ldots, m_k)$ if and only if the Riemann-Hurwitz formula is satisfied and $G$ has an $(h; m_1, \ldots, m_k)$-generating vector.*

In Theorem 1.14 we only consider the algebraic structure of a Fuchsian group, given by a signature, and from Theorem 1.10 we know that there exists some Fuchsian group with signature. Thus we will further on consider abstract Fuchsian groups $\Delta$ (sometimes denoted $\Delta(g; m_1, \ldots, m_k)$), i.e. groups with presentation (1.3), and abstract finite groups $G$. We also note that since the size of $G$ is bounded and there are only a finite number of signatures satisfying the Riemann-Hurwitz formula, the number of possible automorphism groups of a surface of genus $g \geq 2$ is finite. Two actions of the same group with the same signature may be topologically equivalent, we will investigate this further in chapter 3.

**Example 1.14**

Let $G$ be isomorphic to the cyclic group $C_6$ and consider the signature $s = (0, 2, 2, 3, 6, 6)$. By Theorem 1.14 $C_6$ acts on a surface of genus $5$ since with the Riemann-Hurwitz formula (see Theorem 1.11) we find that

$$2g - 2 = 6\left(-2 + \frac{1}{2} + \frac{1}{2} + \frac{2}{3} + \frac{5}{6} + \frac{5}{6}\right) = 8$$

and $(a^3, a^3, a^4, a, a)$, where $\langle a \rangle = C_6$, is an s-generating vector of $C_6$. Thus there exist a Fuchsian group $\Delta$ with signature $s(\Delta) = (0, 2, 2, 3, 6, 6)$ and a presentation

$$\left\langle \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \mid \gamma_1^2 = \gamma_2^2 = \gamma_3^3 = \gamma_4^6 = \gamma_5^6 = \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5 = 1 \right\rangle$$

such that there exists an epimorphism $\theta : \Delta \to C_6$ by $\theta(\gamma_1) = a^3$, $\theta(\gamma_2) = a^3$, $\theta(\gamma_3) = a^4$, $\theta(\gamma_4) = a$ and $\theta(\gamma_5) = a$. Note that any elliptic element $\gamma$ of $\Delta$ is conjugate to some $\gamma_i$ thus $\theta(\gamma) = \theta(x^{-1}\gamma_i x) = \theta(\gamma_i) \neq 1$. So $ker(\theta)$ is a surface group of genus $g$. Also, $\theta$ is the monodromy of the covering

$$X = \mathcal{U}/\Gamma \to \mathcal{U}/\Delta = X/G.$$

# 2

# Automorphism groups of $p$-gonal Riemann surfaces

An important type of Riemann surfaces for our work are the $p$-gonal surfaces. In particular in Paper 4, we are considering $p$-gonal and elliptic-$p$-gonal Riemann surfaces with unique $p$-gonal morphisms. The $p$-gonal Riemann surfaces has been widely studied, and for the hyperelliptic and trigonal surfaces, where $p$ is equal to $2$ respectively $3$, the automorphism groups have been found by Bujalance et al. [20, 19]. We will in this chapter discuss some definitions and properties of such Riemann surfaces, in particular for $p \geq 3$ and such that the $p$-gonal morphism is unique. Similar studies have been done for algebraically closed fields of characteristic $q$ [44], and Riemann surfaces admitting multiple $p$-gonal morphisms are classified in [59]. The $(h, p)$-gonal surfaces have been in studied in [37].

## 2.1 $p$-gonal Riemann surfaces

A *$p$-gonal* morphism is a $p$-sheeted covering $f : X \to \hat{\mathbb{C}}$, where $p$ is some prime. If $f$ is regular, we call $f$ *cyclic $p$-gonal*. For the primes 2 and 3, *hyperelliptic* and *trigonal* are respectively used. If $X$ is a cyclic $p$-gonal Riemann surface, then there exists an automorphism $\phi$ of $X$ of order $p$ such that

$$\begin{tikzcd}
\mathcal{H} \arrow[dr] \arrow[dd] & \\
& X = \mathcal{H}/\Gamma \arrow[dl, "f"] \\
\mathcal{H}/\Delta = Y = X/\langle\phi\rangle &
\end{tikzcd}$$

where $s(\Delta) = (0; p, \overset{\frac{2(g-1+p)}{p-1}}{\ldots}, p)$. Here $\Gamma$ is a normal subgroup of $\Delta$. If $f$ is non-cyclic then we call $X$ a *generic* $p$-gonal Riemann surface. In the generic case the group $\Delta$ has signature

$$s(\Delta) = (0; 2, \overset{u}{\ldots}, 2, p, \overset{v}{\ldots}, p) \text{ where } u + 2v = \frac{2(g+p-1)}{(p-1)/2}, u \equiv 0 \bmod 2, u \neq 0.$$

Further $X$ is conformally equivalent to $\mathcal{H}/\Delta'$, where $\Delta'$ is a non-normal index $p$ subgroup of $\Delta$ with signature $(g; 2, \overset{u}{\ldots}, 2)$.

We can also extend this definition to coverings of other surfaces than the sphere. A $(h, p)$-*gonal* morphism is a $p$-sheeted covering $f : X \to X_h$, where $X_h$ has genus $h$. If $h = 1$ then we call $X$ an *elliptic*-$p$-gonal Riemann surface. An important property of $(h, p)$-gonal surfaces to our work is the Castelnuovo-Severi inequality:

**Theorem 2.1.** *[2] Let $X_g$, $X_h$ and $X_k$ be compact Riemann surfaces of genus $g$, $h$ and $k$, respectively, such that $X_g$ is a covering of $X_h$ of prime degree $p$, and a covering of $X_k$ of prime degree $q$, then*

$$g \leq ph + qk + (p-1)(q-1). \tag{2.1}$$

As a consequence, if $g > 2ph + (p-1)^2$, there is at most one covering $X_g \to X_h$ of degree $p$.

**Lemma 2.1.** *Let $X$ be a cyclic $p$-gonal Riemann surface of genus $g \geq (p-1)^2 + 1$. Then $G \leq Aut(X)$ is an extension of $C_p$ by a group of automorphisms of the Riemann sphere.*

**Proof:** By Lemma 2.1 in [1] the $p$-gonal morphism is induced by an automorphism $\varphi$ of $X$ of order $p$ such that $X/\langle\varphi\rangle = \hat{\mathbb{C}}$ and for any other automorphism $\alpha$ of $X$, there is an automorphism $\overline{\alpha}$ of the Riemann sphere such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"] & X \arrow[d, "f"] \\ \hat{\mathbb{C}} \arrow[r, "\bar{\alpha}"] & \hat{\mathbb{C}} \end{tikzcd}$$

Then $C_p = \langle\varphi\rangle$ is normal in $G = Aut(X)$ and the quotient group $\overline{G} = Aut(X)/\langle\varphi\rangle$ is a finite group of automorphisms of the Riemann sphere. $\square$

A finite group $\overline{G}$ of conformal automorphisms of the Riemann sphere is a subgroup of the following groups: $C_q$, $D_q$, $A_4$, $S_4$, $A_5$. A natural question is; what are the possible automorphism groups of $p$-gonal Riemann surfaces when the $p$-gonal morphism is unique as above? Since any finite group of automorphisms of the sphere extends to the groups listed, we only need to consider those. The answer is given in the following theorem:

**Theorem 2.2.** *Let $X_g$ be a cyclic $p$-gonal Riemann surface of genus $g \geq (p-1)^2 + 1$ with $p$ an odd prime integer. Then the possible full groups of conformal automorphisms of $X_g$ are:*

1. $C_{pq}$
2. $D_{pq}$
3. $C_p \rtimes C_q$

4. $C_p \rtimes D_q$
5. $C_p \times A_4$, $(C_p \times A_4) \rtimes C_2$, $C_p \times S_4$, $C_p \times A_5$
6. *Exceptional Case 1.* $((C_2 \times C_2) \rtimes C_9) \rtimes C_2$ *for* $p = 3$ *and* $\overline{G} = S_4$
7. *Exceptional Case 2.* $((C_2 \times C_2) \rtimes C_9)$ *for* $p = 3$ *and* $\overline{G} = A_4$
8. *Exceptional Case 3.* $(C_p \times C_2 \times C_2) \rtimes C_3$ *for* $p \equiv 1$ *mod* $6$*,* $\overline{G} = A_4$

To prove this we need to consider some properties of group extensions and cohomology groups of finite groups. Lemma 2.1 says that any group $G$ of automorphisms of a real cyclic $p$-gonal Riemann surface is a (central) extension

$$1 \longrightarrow C_p \xrightarrow{\mu} G \xrightarrow{\epsilon} \overline{G} \longrightarrow 1$$

of $C_p$ by a group $\overline{G}$ of automorphisms of the Riemann sphere listed above. Consider an extension

$$1 \longrightarrow N \xrightarrow{\mu} G \xrightarrow{\epsilon} Q \longrightarrow 1. \tag{2.2}$$

It defines a *transversal function* (in general no homomorphism) $\tau : Q \to G$ satisfying $\tau\epsilon = 1$. This yields a function (in general no homomorphism) $\lambda : Q \to Aut(N)$, two such functions $\lambda, \lambda' : Q \to Aut(N)$ differ by an inner automorphism of $N$. So an extension of a normal subgroup $N$ of a group $G$ by a quotient group $Q$ induces a homomorphism $\eta : Q \to Out(N)$, called the *coupling of* $Q$ *to* $N$. Two equivalent extensions (in the natural sense) induce the same coupling. A coupling $\eta : Q \to Out(N)$ induces a structure as $Q$-module on $Z(N)$, where $Z(N)$ is the center of $N$, and we have:

**Theorem 2.3.** *[3, 39] Let $N$ and $Q$ be groups and let $\eta : Q \to Out(N)$ be a coupling of $Q$ to $N$. Assume that $\eta$ is realized by at least one extension of $N$ by $Q$. Then there is a bijection between the equivalence classes of extensions of $N$ by $Q$ with coupling $\eta$ and the elements of $H^2_\eta(Q, Z(N))$, with $Z(N)$ the center of $N$ with structure of $Q$-module given by $\eta$.*

We say that an extension (2.2) *splits* if the transversal function $\tau : Q \to G$ is an (injective) homomorphism, in this case the function $\lambda : Q \to Aut(N)$ is a homomorphism and $Q$ acts as a group of automorphisms of $N$. An extension splits if and only if $Q$ is a complement to $N$ in $G$, i.e. $G$ is a semi-direct product $N \rtimes Q$. In case of $N$ being Abelian the classes of extensions of $N$ by $Q$ is in bijection with $H^2_\eta(Q, N)$ and the classes of complements of $N$ in $G = N \rtimes Q$ is in bijection with $H^1_\eta(Q, N)$. See [3, 39, 57].

**Proof of Theorem 2.2:** Let $(X_g, f)$ be a cyclic $p$-gonal Riemann surface with $p \geq 3$ prime and $g \geq (p-1)^2 + 1$. Then $(X_g, f)$ is a cyclic $p$-gonal Riemann surface with $p$-gonal morphism $f$ induced by the automorphism $\varphi$ of $X_g$ of order $p$ such that the cyclic group $C_p = \langle \varphi \rangle$ is normal in $G = Aut(X_g)$ with quotient group

$$\overline{G} = C_q,\ D_q,\ A_4,\ S_4 \text{ or } A_5.$$

By Lemma 2.1 we have to find all the equivalence classes of extensions

$$1 \longrightarrow C_p \xrightarrow{\mu} G \xrightarrow{\epsilon} \overline{G} \longrightarrow 1.$$

First of all (Zassenhaus Lemma), if $(|\overline{G}|, p) = 1$, then the extension splits and all the complements of $C_p$ in $G$ are conjugated, since $C_p$ is solvable. Further, by Shur-Zassenhaus Lemma, an extension splits if and only if all the extensions of $C_p$ by any $t$-Sylow subgroup of $\overline{G}$ splits, with $t \mid |\overline{G}|$ [39].

Since $C_p$ is an Abelian group, by Theorem 2.3, the coupling $\eta : Q \to Aut(N)$ will be realized by an extension given by an element of $H^2(\overline{G}, C_p)$ with the $\overline{G}$-module structure of $C_p$ given by $\eta$. The split extension $G = C_p \rtimes \overline{G}$ corresponds to $1 \in H^2(\overline{G}, C_p)$ (see [3, 39, 57]). Now we determine the possible extensions of $C_p$ by the spherical groups.

$A_5$: $H^2(A_5, C_p) = \{1\}$ for $p \geq 3$ and since the only homomorphism $\lambda : A_5 \to C_{p-1}$ is trivial, $G = C_p \times A_5$ (see [39]).

$S_4$: $H_i^2(S_4, C_p) = \{1\}$ for $p \geq 5$, $i = 1, 2$, where the possible epimorphisms $\lambda_i : S_4 \to C_{p-1}$ are $\lambda_1 \equiv 1$ and $\lambda_2$ with $Ker(\lambda_2) = A_4$. Thus we have two cases, $G = C_p \times S_4$ and $G = (A_4 \times C_p) \rtimes C_2$.

If $p = 3$, then $H_2^2(S_4, C_3) = C_3 = \langle b \rangle$, and there are two extensions, $G = (A_4 \times C_p) \rtimes C_2$, corresponding to $1 \in C_3$, and $G = ((C_2 \times C_2) \rtimes C_9) \rtimes C_2$ (corresponding to $b$ and $b^2$ in $C_3$). This last case is the Exceptional Case 1.

$A_4$: $H_i^2(A_4, C_p) = \{1\}$ for $p \geq 5$, $i = 1, 2$, where the possible epimorphisms $\lambda_i : S_4 \to C_{p-1}$ are $\lambda_1 \equiv 1$ and $\lambda_2$ with $Ker(\lambda_2) = C_2 \times C_2$ if $p \equiv 1 \bmod 6$. Then we have two cases $G = C_p \times A_4$ and $G = (C_2 \times C_2 \times C_p) \rtimes C_3$. This last case is Exceptional Case 3.

If $p = 3$, then $H_1^2(A_4, C_3) = C_3 = \langle b \rangle$ and similarly to above there are two extensions, $G = C_p \times A_4$, corresponding to $1 \in C_3$, and $G = (C_2 \times C_2) \rtimes C_9$ (again, corresponding to $b$ and $b^2$ in $C_3$). This last case is the Exceptional Case 2.

$C_q$: Consider extensions of $C_p = \langle \varphi \rangle$ by $C_q = \langle b \rangle$. By Zassenhaus Lemma if $(p, q) = 1$ then $G = C_p \times C_q = C_{pq}$ or in general $G = C_p \rtimes C_q$ when $(q, p-1) = d > 1$. In this case the action of $C_q$ on $C_p$ has order a divisor of $d$.

Consider now extensions of $C_p$ by $C_q$ with $q = p^k m$, $(p, m) = 1$. Here we have the following possibilities [3, 39]:

- $H^2(C_q, C_p) = \{1\}$ if $\varphi^b \neq \varphi$
- $H^2(C_q, C_p) = C_p$ if $\varphi^b = \varphi$.

In the first case we have $G = C_p \rtimes C_q$ with the action of $C_q$ on $C_p$ has order a divisor of $d = (q, p-1)$. In the second case we have the extensions $G = C_p \times C_q$ and $G = C_{pq}$ since the extensions given by non-trivial elements of $H^2(C_q, C_p) = C_p$ are isomorphic.

$D_q$: Finally consider extensions of $C_p = \langle \varphi \rangle$ by $D_q = \langle s, b \rangle = \langle s, b \mid s^2 = b^q = (sb)^2 = 1 \rangle$. Again by Zassenhaus Lemma if $(p, q) = 1$ then $G = C_p \rtimes D_q$.

Consider now extensions of $C_p = \langle \varphi \rangle$ by $D_q$ with $q = p^k m$, $(p, m) = 1$. First we note that the diagram

$$\begin{tikzcd} H^2(D_q, C_p) \arrow[r, "Res_1"] \arrow[rr, bend left, "Res"] & H^2(C_q, C_p) \arrow[r, "Res_2"] & H^2(C_{p^k}, C_p) \end{tikzcd}$$

commutes and the restrictions to the $p$-Sylow subgroup ($\simeq C_{p^k}$ in this case) are injective. Thus if $\varphi^b \neq \varphi$ then $H^2(D_q, C_p) = \{1\}$. Further, the following diagram commutes (see [57]):

$$\begin{tikzcd} H^2(D_q,C_p) \arrow[r, "Res_1"] \arrow[d, "id"] & H^2(C_q,C_p) \arrow[d, "s^*"] \\ H^2(D_q,C_p) \arrow[r, "Res_1"] & H^2(C_q,C_p) \end{tikzcd}$$

Here, the conjugation $s^*$ is given by $s^* : f \mapsto \lambda_s \circ f \circ (\rho_s \times \rho_s)$, where $\lambda_s(a) = a^s$, $a \in C_p$ and $\rho_s(b) = b^{-1}$. Now, if $|H^2(D_q, C_p)| > 1$ then $s^*$ has to be trivial thus

– $H^2(D_n, C_p) = \langle 1 \rangle$ if $\varphi^b \neq \varphi$ or $\varphi^s = \varphi$.
– $H^2(D_n, C_p) = C_p$ if $\varphi^b = \varphi$ and $\varphi^s = \varphi^{-1}$.

In the first case we have the extension $G = C_p \rtimes C_q$. In the second case we have the extensions $G = C_p \rtimes D_q$ and $G = D_{pq} = \langle s, a \mid s^2 = a^{pq} = (sa)^2 = 1\rangle$ since the extensions given by non-trivial elements of $H^2(D_q, C_p) = C_p$ are isomorphic.

This concludes our proof. $\square$

## 2.2 Real *p*-gonal Riemann surfaces

Often one is also interested in anti-conformal automorphisms, i.e. orientation reversing automorphisms, of Riemann surfaces. In general, a cocompact, discrete subgroup $\Delta$ of $Aut(\mathcal{H}) \simeq PSL(2,\mathbb{R})$ is called a *non-euclidean crystallographic group (NEC).* The subgroup of $\Delta$ consisting of the orientation-preserving elements is called the *canonical Fuchsian subgroup of* $\Delta$, and it is denoted $\Delta^+$.
If an NEC group $\Delta$ is isomorphic to an abstract group with presentation with generators:
$x_1, \ldots, x_r, e_i, c_{ij}, 1 \le i \le k, 0 \le j \le s_i$ and $a_1, b_1, \ldots, a_g, b_g$ if $\mathcal{H}/\Delta$ is orientable, or $d_1, \ldots, d_g$ otherwise, and relators:
$x_i^{m_i}, i = 1, \ldots, r,\ c_{ij}^2, (c_{ij-1}c_{ij})^{n_{ij}}, c_{i0}e_i^{-1}c_{is_i}e_i,\ i = 1, \ldots, k, j = 0, \ldots, s_i$
and $x_1 \ldots x_r e_1 \ldots e_k a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}$ or $x_1 \ldots x_r e_1 \ldots e_k d_1^2 \ldots d_g^2$,
according to whether $\mathcal{H}/\Delta$ is orientable or not, we say that $\Delta$ has signature

$$(g; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}). \tag{2.3}$$

Similarly to Fuchsian groups, an NEC group $\Gamma$ without elliptic elements is called a *surface group*; it has signature given by $s(\Gamma) = (g; \pm; -; \{(-) \ldots (-)\})$. In such a case $\mathcal{H}/\Gamma$ is a *Klein surface*, that is, a surface of topological genus $g$ with a dianalytical structure, orientable or not according to the sign $+$ or $-$ and possibly with boundary. Any Klein surface of genus greater than one can be identified with the qoutient space $\mathcal{H}/\Gamma$, for some surface group $\Gamma$.

Now, a cyclic $p$-gonal Riemann surface $X$ is called *real cyclic p-gonal* if there is an anti-conformal involution (symmetry) $\sigma$ of $X$ commuting with the $p$-gonal morphism, i.e.

$f \cdot \sigma = c \cdot f$, with $c$ the complex conjugation, $\sigma$ is the lift of the complex conjugation by the covering $f$. We denote it by the triple $(X, f, \sigma)$. Costa and Izquierdo [27] gave the following characterization of real $p$-gonal Riemann surfaces: Let $X$ be a Riemann surface of genus $g$. The surface $X$ admits a symmetry $\sigma$ and a meromorphic function $f$ such that $(X, f, \sigma)$ is a real cyclic $p$-gonal surface if and only if there is an NEC group $\Lambda$ with signature

$$(0; +; [p^r]; \{(p^s)\}),\ r, s \geq 0$$

and an epimorphism

$$\theta : \Lambda \to D_p,\ \text{or}\ \theta : \Lambda \to C_{2p},$$

such that $X$ is conformally equivalent to $\mathcal{H}/Ker(\theta)$ with $Ker(\theta)$ a surface Fuchsian group. In the case of $\theta : \Lambda \to C_{2p}$, then $s(\Lambda) = (0; +; [p^r]; \{(-)\})$.

Further, a finite group $\overline{G}$ of conformal and anti-conformal automorphisms of the Riemann sphere is a subgroup of: $D_q,\ C_q \times C_2,\ D_q \rtimes C_2,\ A_4 \times C_2,\ S_4,\ S_4 \times C_2,\ A_5 \times C_2$. As a consequence of Theorem 2.2 and Shur-Zassenhaus Lemma the full groups of conformal and anti-conformal automorphisms of a real cyclic $p$-gonal Riemann surfaces with unique $p$-gonal morphism where $p \geq 3$ are as follows:

**Corollary 2.1.** *Let $(X_g, f, \sigma)$ be a real cyclic $p$-gonal Riemann surface with $p$ an odd prime integer, $g \geq (p-1)^2 + 1$. Then the possible automorphisms groups of $X_g$ are*

1. $C_{pq} \times C_2$ *if* $\langle \varphi, \sigma \rangle = C_{2p}$
   $D_{pq}$ *if* $\langle \varphi, \sigma \rangle = D_p$
2. $D_{pq} \rtimes C_2$
3. $(C_p \rtimes C_q)) \rtimes C_2$
4. $(C_p \rtimes D_q) \rtimes C_2$
5. $C_p \rtimes S_4,\ D_p \times A_4\ D_p \times S_4,\ D_p \times A_5$ *if* $\langle \varphi, \sigma \rangle = D_p$
   $C_p \times S_4,\ C_{2p} \times A_4,\ C_{2p} \times S_4,\ C_{2p} \times A_5$ *if* $\langle \varphi, \sigma \rangle = C_{2p}$
6. *Exceptional Case 1.* $((C_2 \times C_2) \rtimes C_9) \times C_2$ *for* $p = 3$ *and* $\overline{G} = S_4$
7. *Exceptional Case 2.* $(C_p \times C_2 \times C_2) \rtimes C_6$ *for* $p \equiv 1$ *mod* $6$, $\overline{G} = A_4 \times C_2$ *and* $\langle \varphi, \sigma \rangle = D_p$
   *or* $((C_p \times C_2 \times C_2) \rtimes C_3) \times C_2$ *for* $p \equiv 1$ *mod* $6$, $\overline{G} = A_4 \times C_2$ *and* $\langle \varphi, \sigma \rangle = C_{2p}$

# 3

# Equisymmetric stratification of branch loci

In chapter 1 we constructed Riemann surfaces on a topological surface $X$ and classified them up to conformal equivalence. The space of equivalence classes is known as the moduli space, which we are interested in studying. The moduli space is an orbifold in its natural topology, and the orbifold structure is in particular what we are examining. As a tool to study the moduli spaces we consider the Teichmüller spaces of Riemann surfaces. Here we restrict the equivalence to surfaces such that there exists a biholomorphism between them that is homotopic to the identity map. Classically the Teichmüller space is constructed as a space of Riemann surfaces marked by so-called quasi-conformal maps. However, when a Riemann surface is uniformized by a surface Fuchsian group we can construct the Teichmüller space by classes of surface Fuchsian groups. One benefit of the second approach is that we do not have to explicitly use quasi-conformal maps. For details on quasi-conformal maps and Teichmüller theory, see [49]. For details on the equisymmetric stratification see [13, 38].

## 3.1 The Teichmüller space and the moduli space

Consider a topological surface $X_g$ of genus $g$. We define the *moduli space* of $X_g$ as the space of classes of conformally equivalent Riemann surfaces modeled on $X_g$ and denote it by

$$\mathcal{M}(X_g) \text{ or } \mathcal{M}_g. \tag{3.1}$$

Now if there exists a biholomorphism homotopic to the identity map between two Riemann surfaces way say that those are *Teichmüller equivalent*. The *Teichmüller space* $T_g$ is defined as the space of classes of Teichmüller equivalent Riemann surfaces modeled on $X_g$ and is denoted by

$$T(X_g) \text{ or } T_g. \tag{3.2}$$

Now we will construct the Teichmüller spaces and the moduli spaces of Fuchsian groups which generalize the definitions above. Let $\Delta$ be an abstract Fuchsian group (not necessary a surface group), then consider the the set of monomorphisms

$$R(\Delta) = \{r \mid r : \Delta \to PSL(2,\mathbb{R}), r(\Delta) \text{ is Fuchsian}\}. \tag{3.3}$$

Since $PSL(2,\mathbb{R})$ is a topological group, $R(\Delta)$ has a natural topology given by

$$r \mapsto (r(\alpha_1), \ldots, r(\beta_g), r(\gamma_1), \ldots, r(\gamma_k)).$$

The *Teichmüller space of* $\Delta$, denoted by $T(\Delta)$, is defined as

$$T(\Delta) = R(\Delta)/\sim_c \tag{3.4}$$

where $\sim_c$ is equivalence up to conjugation in $PSL(2,\mathbb{R})$. The Teichmüller space $T(\Delta)$ is in fact a manifold and its dimension is well-known.

**Theorem 3.1.** *[36] The Teichmüller space $T(\Delta)$ is a complex analytic manifold of dimension $3g-3+k$, diffeomorphic to $\mathbb{R}^{6g-6+2k}$.*

Based on this result we define the dimension of a Fuchsian group $\Delta$ to be $d(\Delta) = d(T(\Delta)) = 3g-3+k$. Also, if $\Delta \subset \Delta'$, then the inclusion $i : \Delta \to \Delta'$ induces a map $i^* : T(\Delta') \to T(\Delta)$ as $[r'] \mapsto [r' \circ i]$. Thus there is a natural embedding of one Teichmüller space into another as follows:

**Theorem 3.2.** *[36] Let $\Delta$ and $\Delta'$ be two Fuchsian groups such that there exists a monomorphism $i : \Delta \to \Delta'$ then the induced map*

$$i^* : T(\Delta') \to T(\Delta),\ [r] \mapsto [r \circ i], \tag{3.5}$$

*is an isometric embedding.*

The group of outer automorphisms of $\Delta$ acts on $T(\Delta)$ where the action of an outer automorphism $\omega$ on $T(\Delta)$ is given by $[r] \mapsto [r \circ \omega]$. The group of such actions, $M(\Delta)$, is known as the *Teichmüller modular group* or the *mapping class group.* The *moduli space of* $\Delta$ is defined as the quotient space

$$\mathcal{M}(\Delta) = T(\Delta)/M(\Delta).$$

Recall Theorem 1.8 in Chapter 1, stating that surface groups $\Gamma$ and $\Gamma'$, with signature $s(\Gamma) = s(\Gamma') = (g; -)$, corresponds to conformally equivalent Riemann surfaces if and only if $\Gamma$ and $\Gamma'$ and conjugated in $PSL(2,\mathbb{R})$. With this in mind we can identify the Teichmüller space $T(\Gamma)$ by $T(\mathcal{H}/r(\Gamma)) = T_g$ and the moduli space $\mathcal{M}(\Gamma)$ by $\mathcal{M}(\mathcal{H}/r(\Gamma)) = \mathcal{M}_g$. Here we usually denote $M(\Gamma)$ by $M_g$. Let us return to the the canonical covering $T_g \to \mathcal{M}_g$. As one might except this is a branched covering; consider the set of fixed points for an finite subgroup $G \subset M_g$,

$$T_g^G = \{[r] \in T_g \mid [r] = [r \circ \omega], \omega \in G\}. \tag{3.6}$$

Due to the Nielsen Realization Theorem [13, 38] we have

**Proposition 3.1.** *If $G \subset M_g$ is finite, then $T_g^G$ is non empty.*

Let $G \subset M_g$ be a finite subgroup and $[r] \in T_g^G$, then there is a Fuchsian group $\Delta$ such that (recall Theorem 1.13)

$$\Delta/\Gamma \simeq Aut(\mathcal{H}/r(\Gamma)) \to M_g,$$

where the image in $M_g$ is conjugate to $G$. Thus the image in Theorem 3.2 can be identified as follows [36, 38]:

**Proposition 3.2.** *Let $G$ and $\Delta$ be defined as above, then*

$$i^*(\Delta) = T_g^G.$$

As we have seen, the moduli spaces are quotient spaces of a manifold by a group action properly discontinuously, or with other words, they are good orbifolds. Outside of the branch loci they are well described by the Teichmüller spaces. Thus, we are in particular interested in the *branch locus* $\mathcal{B}_g \subset \mathcal{M}_g$ of the covering $T_g \to \mathcal{M}_g$. From the discussion above we note that (for details see [13, 38])

$$\mathcal{B}_g = \{X \in \mathcal{M}_g \mid |Aut(X)| > 1\},\ g \geq 3. \tag{3.7}$$

## 3.2 Equisymmetric Riemann surfaces

As we have seen the branch loci $\mathcal{B}_g$ consist of Riemann surfaces with non-trivial automorphism groups. Remember that the action of an automorphism group $Aut(X)$ is given a monodromy representation $\theta : \Delta \to G$, where $G \simeq Aut(X)$. Two actions $\theta, \theta'$ of $G$ on a surface $X$ are *topologically equivalent* if there is an $w \in \mathrm{Aut}(G)$ and an $h \in Hom^+(X)$ such that $\theta'(g) = h\theta w(g)h^{-1}$. We can formulate this for Riemann surfaces as follows.

**Lemma 3.1.** *[14] Two epimorphisms $\theta_1, \theta_2 : \Delta \to G$ define two topologically equivalent actions of $G$ on $X$ if there exists two automorphisms $\phi : \Delta \to \Delta$ and $w : G \to G$ such that the following diagram commutes:*

$$\begin{tikzcd} \Delta \arrow[r, "\theta_1"] & G \arrow[d, "w"] \\ \Delta \arrow[u, "\phi"] \arrow[r, "\theta_2"] & G \end{tikzcd}$$

$\phi$ is the induced automorphism by the lifting $h^*$ of $h$ to the universal orbifold covering. Remember from chapter 1, that from an epimorphism $\theta : \Delta \to G$ we can construct generating vectors. In terms of generating vectors, two actions are topologically equivalent if the $s(\Delta)$-generating vector

$$(\theta_1(\alpha_1^\phi)^w, \theta_1(\beta_1^\phi)^w, \ldots, \theta_1(\alpha_h^\phi)^w, \theta_1(\beta_h^\phi)^w; \theta_1(\gamma_1^\phi)^w, \ldots, \theta_1(\gamma_k^\phi)^w)$$

is equal to the generating vector

$$(\theta_2(\alpha_1), \theta_2(\beta_1), \ldots, \theta_2(\alpha_h), \theta_2(\beta_h); \theta_2(\gamma_1), \ldots, \theta_2(\gamma_k))$$

after applying automorphisms $\phi \in Aut(\Delta)$ and $w \in Aut(G)$. Let $\mathcal{B}$ be the subgroup of $Aut(\Delta)$ composed of such automorphisms. We will later consider the elements of $\mathcal{B}$ in more detail.

## Equisymmetric stratification

Consider the automorphism group $Aut(X)$ of a Riemann surface $X = \mathcal{H}/\Gamma$. $G = Aut(X)$ determines a conjugacy class of subgroups, $\bar{G}$, of $M_g$ given by topologically equivalent actions of $G$ [13]. We call $\bar{G}$, or simply $G$, the *symmetry type* of $X$. Two surfaces are *equisymmetric* is they have the same symmetry type, that is their automorphism groups are conjugate in $\mathcal{M}_g$. Conformally equivalent surfaces clearly have the same symmetry type so we can talk about the symmetry type of points in the moduli space.

We note here that any conjugacy class of finite subgroups of $M_g$ corresponds to an automorphism group of a Riemann surface. On the other hand, due to the Hurwitz bound (Theorem 1.12) and the fact there is only a finite number of $s$-generating vectors of a finite group with a given signature, there are only finitely many possible symmetry types of Riemann surfaces and hence finitely many conjugacy classes of finite subgroups of $M_g$.

With the definitions made above we can divide the branch loci $\mathcal{B}_g$ into subsets of classes of equisymmetric Riemann surfaces. We denote the set of equisymmetric Riemann surfaces with a given symmetry type by

$$\mathcal{M}_g^{G,\theta} = \{X \in \mathcal{B}_g \mid Aut(X) \text{ top. equiv. to } G \text{ (given by } \theta)\}.$$

We are also interested general automorphism groups of Riemann surfaces thus we denote the set of Riemann surfaces with a group of automorphisms of a given type by

$$\overline{\mathcal{M}}_g^{G,\theta} = \{X \in \mathcal{B}_g \mid G' \subset Aut(X), G' \text{ top. equiv. to } G \text{ (given by } \theta)\}.$$

We also note that $\overline{\mathcal{M}}_g^{G,\theta}$ is the image of $i^*(\Delta) = T_g^G$ under the covering $T_g \to \mathcal{M}_g$. This decomposition of $\mathcal{B}_g$ is a stratification as shown by Broughton:

**Theorem 3.3.** *([13]) Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus $g$, $G$ a finite subgroup of the corresponding modular group $M_g$. Then:*

*(1) $\overline{\mathcal{M}}_g^{G,\theta}$ is a closed, irreducible algebraic subvariety of $\mathcal{M}_g$.*

*(2) $\mathcal{M}_g^{G,\theta}$, if it is non-empty, is a smooth, connected, locally closed algebraic subvariety of $\mathcal{M}_g$, Zariski dense in $\overline{\mathcal{M}}_g^{G,\theta}$.*

While it is possible to find a finite generating set for $\mathcal{B}$ [9], we rarely need to do it. Here we will list some actions of which automorphisms of $\Delta$ used in our computations can be composed of, more can be found in [12, 38].

We note that if a composition of the actions in Table 3.1-3.3 is an automorphism of $\Delta$, then the orders of the elliptic elements has to be preserved. While we use the actions above to show that two automorphism groups of Riemann surfaces are topologically equivalent, we can often distinguish non-equivalent automorphism groups without explicitly calculate all actions of $\mathcal{B}$ by considering the conjugacy classes of the elliptic elements.

**Example 3.1**

Let us consider the quaternion group $Q = \langle i, j \mid i^2 = j^2 = -1, ij = -ji \rangle$ and the signature $s = (0; 4, 4, 4, 4)$. With the Riemann-Hurwitz formula

$$2g - 2 = 8\left(-2 + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4}\right) = 8$$

| | $\alpha_i$ | $\beta_i$ | $\alpha_{i+1}$ | $\beta_{i+1}$ |
|---|---|---|---|---|
| $A_{1,i}$ | $\alpha_i$ | $\beta_i\alpha_i^n$ | $\alpha_{i+1}$ | $\beta_{i+1}$ |
| $A_{2,i}$ | $\alpha_i\beta_i^n$ | $\beta_i$ | $\alpha_{i+1}$ | $\beta_{i+1}$ |
| $A_{3,i}$ | $\alpha_i\beta_i\alpha_i^{-1}$ | $\alpha_i^{-1}$ | $\alpha_{i+1}$ | $\beta_{i+1}$ |
| $A_{4,i}$ | $\alpha_i\alpha_{i+1}$ | $\alpha_{i+1}^{-1}\beta_i\alpha_{i+1}$ | $[\alpha_{i+1}^{-1}\beta_i]\alpha_{i+1}$ | $\beta_{i+1}\beta_i^{-1}[\alpha_{i+1}^{-1}\beta_i]^{-1}$ |

***Table 3.1:*** *Actions on the hyperbolic elements.*

| | $\gamma_i$ | $\gamma_{i+1}$ |
|---|---|---|
| $B_{1,i}$ | $\gamma_i\gamma_{i+1}\gamma_i^{-1}$ | $\gamma_i$ |
| $B_{2,i}$ | $\gamma_{i+1}$ | $\gamma_{i+1}^{-1}\gamma_i\gamma_{i+1}$ |

***Table 3.2:*** *Actions on the elliptic elements.*

| | $\alpha_1$ | $\beta_1$ | $\gamma_i$ |
|---|---|---|---|
| $C_{1,i}$ | $x_1\alpha_1$ | $\beta_1$ | $y_1\gamma_i y_1^{-1}$ |
| $C_{2,i}$ | $\alpha_1$ | $x_2\beta_1$ | $y_2\gamma_i y_2^{-1}$ |

***Table 3.3:*** *Here $x_1 = \beta_1^{-1}wz$, $y_1 = z\beta_1^{-1}w$, $x_2 = wz\alpha_1$, $y_2 = z\alpha_1 w$ where $w = \prod_{j<i}\gamma_j$, $z = \prod_{j>i}\gamma_j$*

we find that $Q$ can act with the given signature on Riemann surfaces with genus $5$. There are a total of $144$ group actions induced by $Q$ with signature $s$. However, letting $Aut(Q) \simeq S_4$ act on the set of generating vectors it is easy to see that we get $144/|S_4| = 6$ classes, represented by the following vectors:

$$(i,i,j,j),(i,j,-i,j),(i,j,j,i)$$
$$(i,j,i-j),(i,j,-j,-i),(i,-i,-j,j)$$

Now we use the actions $B_{2,i}$ in Table 3.2 to show that all the six classes are topologically equivalent:

$$\begin{aligned}
B_{2,2}(i,i,j,j) &= (i,j,-jij,j) = (i,j,-i,j)\\
B_{2,3}(i,j,-i,j) &= (i,j,j,jij) = (i,j,j,i)\\
B_{2,3}(i,j,j,i) &= (i,j,i,-iji) = (i,j,i,-j)\\
B_{2,3}(i,j,i,-j) &= (i,j,-j,-jij) = (i,j,-j,-i)\\
B_{2,2}(i,j,-i,j) &= (i,-i,-iji,j) = (i,-i,-j,j)
\end{aligned}$$

We can conclude that there is only one symmetry type of Riemann surface given by the quaternion group $Q$ with signature $(0,4,4,4,4)$.

In the case of cyclic actions it turns out to be much simpler [38]. We have to following theorem regarding the actions of the cyclic group $C_n$:

**Theorem 3.4.** *[38] Let $\Delta$ be a Fuchsian group with signature $(g; m_1, \ldots, m_k)$ and $lcm(m_i) = m$. Then each symmetry type is given by an epimorphism $\theta$ such that if $k \geq 2$ then $\theta(\alpha_i) = Id$, $i = 1, \ldots, k$, $\theta(\beta_1) = a$, where $a$ generates $C_n/C_m$ and $\theta(\beta_i) = Id$, $i = 2, \ldots, k$. If $k = 0$, there exists exactly one symmetry type given by $C_n$ and the signature $(g; -)$.*

Abelian groups, and in particular elementary abelian groups have been studied in [12].

**Example 3.2**

From [12] it follows that there exists two classes of actions of $C_2 \times C_2$ on Riemann surfaces of genus 5 with signature $(2; -)$. Those are given by the generating vectors

$$(a, b, 1, 1) \text{ and } (a, 1, b, 1).$$

## Full automorphism groups

So far we have mainly considered arbitrary groups of automorphisms of Riemann surfaces. However, we are interested in the full groups of automorphisms. A natural question that arises is; does it exists a group of automorphisms of a Riemann surface, given by an epimorphism $\theta : \Delta \to G$, for some abstract Fuchsian group $\Delta$ and a finite group $G$, such that it is the full group of automorphisms? Or formulated in terms of equisymmetric strata; when is the stratum $\mathcal{M}^{\theta,G}$ non-empty? To determine this we consider maximal actions and maximal Fuchsian groups. A Fuchsian group $\Delta$ is called a *finitely maximal* Fuchsian group if there is no other Fuchsian group $\Delta'$ containing $\Delta$ with finite index and such that $d(\Delta) = d(\Delta')$. We also call a signature *non-maximal* if it is the signature of some non-maximal Fuchsian group. The full list of pairs of signatures $s(\Delta)$, $s(\Delta')$ as above was obtained by Singerman in [55].

| $s(\Delta)$ | $s(\Delta')$ | $\|\Delta' : \Delta\|$ |
|---|---|---|
| $(2; -)$ | $(0; 2, 2, 2, 2, 2, 2)$ | 2 |
| $(1; t, t)$ | $(0; 2, 2, 2, 2, t)$ | 2 |
| $(1; t)$ | $(0; 2, 2, 2, 2t)$ | 2 |
| $(0; t, t, t, t)$ | $(0; 2, 2, 2, t)$ | 4 |
| $(0; t, t, u, u)$ | $(0; 2, 2, t, u)$ | 2 |
| $(0; t, t, t)$ | $(0; 3, 3, t)$ | 3 |
| $(0; t, t, t)$ | $(0; 2, 3, 2t)$ | 6 |
| $(0; t, t, u)$ | $(0; 2, t, 2u)$ | 2 |

***Table 3.4:*** *Normal pairs of non-maximal signatures.*

| $s(\Delta)$ | $s(\Delta')$ | $\lvert\Delta' : \Delta\rvert$ |
|---|---|---|
| $(0;7,7,7)$ | $(0;2,3,7)$ | 24 |
| $(0;2,7,7)$ | $(0;2,3,7)$ | 9 |
| $(0;3,3,7)$ | $(0;2,3,7)$ | 8 |
| $(0;4,8,8)$ | $(0;2,3,8)$ | 12 |
| $(0;3,8,8)$ | $(0;2,3,8)$ | 10 |
| $(0;9,9,9)$ | $(0;2,3,9)$ | 12 |
| $(0;4,4,5)$ | $(0;2,4,5)$ | 6 |
| $(0;n,4n,4n)$ | $(0;2,3,4n)$ | 6 |
| $(0;n,2n,2n)$ | $(0;2,4,2n)$ | 4 |
| $(0;3,n,3n)$ | $(0;2,3,3n)$ | 4 |
| $(0;2,n,2n)$ | $(0;2,3,2n)$ | 3 |

***Table 3.5:*** *Non-normal pairs of non-maximal signatures.*

A group action induced by a maximal signature corresponds to the full automorphism group of some Riemann surface. However, if the signature is non-maximal we have to make further investigations to determine whether or not it corresponds to a full group of automorphisms. This is summed up in the following proposition:

**Proposition 3.3.** *[4] Let $\Gamma$ be a surface Fuchsian group and $X = \mathcal{U}/\Gamma$ a Riemann surface. Assume that there exists a surface kernel epimorphism $\theta : \Delta \to G$ such that $ker(\theta) = \Gamma$ where $s(\Delta)$ is a non-maximal signature. Further assume that there exists another surface kernel epimorphism $\theta' : \Delta' \to G'$ such that $ker(\theta') = \Gamma$ where $\Delta' \geq \Delta$, $G' \geq G$ and $\theta'^{-1}(G) = \Delta$. Then $G = \Delta/\Gamma$ is the full group of automorphisms $Aut(X)$ if and only if $\theta'|_\Delta$ is not equivalent under automorphisms of $\Delta$ and $G$ to $\theta$ for any such extensions $\theta' : \Delta' \to G'$.*

Another way of stating it is that an action $\theta : \Delta \to G$ is non-maximal if and only if there exists an action $\theta' : \Delta' \to G'$ such that the following diagram commutes:

$$\begin{tikzcd}
\Delta' \arrow[r, "\theta'"] & G' \\
\Delta \arrow[u] \arrow[r, "\theta"] & G \arrow[u] \\
\Gamma \arrow[u] \arrow[r] & 1 \arrow[u]
\end{tikzcd}$$

**Example 3.3**

Let $\Delta$ be a group with signature $s(\Delta) = (0;4,4,4,4)$ such that there exists an epimorphism $\theta : \Delta \to C_4 \times C_2$ defined by $\theta(\gamma_1) = a$, $\theta(\gamma_2) = a$, $\theta(\gamma_3) = ab$ and $\theta(\gamma_4) = ab$. With Riemann-Hurwitz formula we find that $\theta$ induces a group of automorphisms of a surface of genus $5$. This group is indeed not the full group of automorphisms. Consider a group $\Delta'$ with signature $s(\Delta') = (0;2,2,2,4)$ such that there exists an epimorphism

$\theta' : \Delta' \to (C_4 \times C_2 \times C_2) \rtimes C_2$ defined by $\theta'(\gamma_1') = ba$, $\theta'(\gamma_2') = b$, $\theta'(\gamma_3') = c$ and $\theta'(\gamma_4') = ac$ acting on a surface of genus 5. Consider the following commutative diagram

$$\begin{tikzcd}
(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \arrow[r, "\theta"] \arrow[d] & (a, a, ab, ab) \arrow[d] \\
(\gamma_3'\gamma_2'\gamma_4'^{-1}\gamma_2'\gamma_4'\gamma_2'\gamma_4'\gamma_2'\gamma_3', \gamma_4', \gamma_2'\gamma_4'\gamma_2', \gamma_3'\gamma_2'\gamma_4'\gamma_2'\gamma_3') \arrow[r, "\theta'"] & (ac, ac, (ac)^3(bc)^2, (ac)^3(bc)^2)
\end{tikzcd}$$

where $C_4 \times C_2 \simeq \langle ac, (ac)^2(bc)^2 \rangle \subset (C_4 \times C_2 \times C_2) \rtimes C_2$. As seen in the diagram the action $\theta$ can be extended to $\theta'$.

In order to find the inclusion $\Delta \subset \Delta'$ as in Example 3.3, we usually consider the monodromy of the covering $\mathcal{H}/\theta'^{-1}(G) \to \mathcal{H}/\Delta'$ to find out if an action $\theta : \Delta \to G$ extends to an action $\theta' : \Delta' \to G'$. We show the calculations by an example.

**Example 3.4**

We return to the actions of Example 3.3. Let $\Delta = \theta'^{-1} \langle ac, (ac)^2(bc)^2 \rangle$. To determine the monodromy group of the induced covering $\mathcal{H}/\Delta \to \mathcal{H}/\Delta'$ we consider the cosets of $\langle ac, (ac)^2(bc)^2 \rangle$ and by enumerating the cosets we have the following permutation representation of the monodromy:

| $\theta'(\gamma_i')$ | $\langle ac, (ac)^2(bc)^2 \rangle$ |
|---|---|
| $ba \to$ | $(1\ 4)(2\ 3)$ |
| $b \to$ | $(1\ 2)(3\ 4)$ |
| $c \to$ | $(1\ 3)(2\ 4)$ |
| $ac \to$ | $(1)(2)(3)(4)$ |

By Theorem 1.11 we see that the subgroup $\Delta$ of $\Delta'$ has signature $(0; 4, 4, 4, 4)$. Now, consider the set of transversals corresponding to the cosets: $\{1, b, c, bc\}$. We know that any generator $\gamma_i$ of $\Delta$ is conjugated to $\gamma_4'$ in $\Delta'$, $\gamma_i = \gamma^{-1}\gamma_4'\gamma$. As $C_4 \times C_2$ is abelian, $\theta'(\gamma^{-1}\gamma_4'\gamma)$ depends only on the transversal, thus we can conclude that a generating vector of the induced action is given by

$$(ac, bacb, cacc, bcacbc) = (ac, (ac)^3(bc)^2, ac, (ac)^3(bc)^2)$$

without explicitly determine a generating set for $\Delta$.

Sufficient conditions for non-maximal actions have been studied in [17].

## 3.3 Connected components of the branch loci

However, we are not only interested in knowing if actions correspond to the full automorphism group of some Riemann surface. We are also interested in the structure of the branch loci $\mathcal{B}_g$. From Theorem 3.3 we know there exists a stratification of the branch loci by equisymmetric Riemann surfaces. Now given a symmetry type $G$ if a given prime $p$ divides $|G|$ then by Lagrange's Theorem there is an element of $G$ with order $p$. Thus

$$\overline{\mathcal{M}}_g^{G,\theta} \subset \overline{\mathcal{M}}_g^{C_p,i}$$

and by considering all primes that are possible we have the following theorem:

**Theorem 3.5.** *[23] Let $\overline{\mathcal{M}}_g^{C_p,i}$ be the stratum corresponding to an action $i$ of the cyclic group $C_p$ of prime order $p$. Then*

$$\mathcal{B}_g = \bigcup_{p\in\mathbb{P}} \overline{\mathcal{M}}_g^{C_p,i}. \tag{3.8}$$

Cornalba [23] also give an algorithmic way of finding those strata. Further two (closures of) strata $\overline{\mathcal{M}}_g^{G_1,\theta_1}$ and $\overline{\mathcal{M}}_g^{G_2,\theta_2}$ are connected if and only if there exists some $\overline{\mathcal{M}}_g^{G,\theta}$ which is a subset of both closures:

$$\begin{tikzcd} & \overline{\mathcal{M}}_g^{G,\theta} \arrow[dl] \arrow[dr] & \\ \overline{\mathcal{M}}_g^{G_1,\theta_1} & & \overline{\mathcal{M}}_g^{G_2,\theta_2} \end{tikzcd}$$

Here clearly, $G_1, G_2 \subset G$. Here we consider coset permutations as in Example 3.4 and only do explicit constructions when necessary. If for a given stratum $\mathcal{M}_g^{G_1,\theta_1}$ there exists no strata $\mathcal{M}_g^{G_2,\theta_2}$ and $\mathcal{M}_g^{G,\theta}$ satisfying the above diagram we note that

$$\overline{\mathcal{M}}_g^{G_1,\theta_1} = \mathcal{M}_g^{G_1,\theta_1},$$

and this stratum is an isolated component of the branch locus $\mathcal{B}_g$. The necessary and sufficient conditions for the existence of isolated points, i.e. zero-dimensional strata was proved by Kulkarni[45], further the necessary and sufficient conditions for the existence of one dimensional was proven in [24]. We sum up these results on isolated strata in the following proposition:

**Proposition 3.4.**

$(i)$ *The number of isolated points in $\mathcal{B}_g$ is 1 if $g = 2$, $\lfloor (g-2)/3 \rfloor$ if $q = 2g+1$ is a prime $> 7$ and 0 otherwise [45].*

$(ii)$ *The branch locus $\mathcal{B}_g$ contains isolated connected components of complex dimension $1$ if and only if $g = p-1$, with $p$ prime $\geq 11$ [24].*

In fact, in Paper 2 it is proven that for any given dimension $d$, there exists infinitely many genera $g$ such that the $\mathcal{B}_g$ contains an isolated stratum of dimension $d$:

**Theorem 3.6.** *[6] Let $p$ be a prime and $d$ be an integer such that $p > (d+2)^2$, then there are isolated equisymmetric strata of dimension $d$ in $\mathcal{M}_{(d+1)(p-1)/2}$*

While there exists isolated strata, in Paper 1 it is shown that the strata of Riemann surfaces admitting automorphisms of order $2$ and Riemann surfaces admitting automorphisms of order $3$ belong to the same connected component [7]:

**Theorem 3.7.** *[7] Let $g \geq 3$. Then the strata $\overline{\mathcal{M}}_g^{C_2,i}$, $i = 0 \ldots \lfloor \frac{g+1}{2} \rfloor$, belong to the same connected component. In particular*

$$\overline{\mathcal{M}}_g^{C_2,i} \cap \overline{\mathcal{M}}_g^{C_2,\lfloor \frac{g+1}{2} \rfloor} \neq \emptyset.$$

**Theorem 3.8.** *[7] Let $g \geq 4$. Then for each stratum $\overline{\mathcal{M}}_g^{C_3,i}$ there exists a stratum $\overline{\mathcal{M}}_g^{C_2,j}$ such that*

$$\overline{\mathcal{M}}_g^{C_3,i} \cap \overline{\mathcal{M}}_g^{C_2,j} \neq \emptyset.$$

Thus the strata of highest dimensions belong to the same connected component of the branch loci. However, in general a branch locus $\mathcal{B}_g$ contains isolated components, as shown in Paper 4:

**Theorem 3.9.** *The branch locus $\mathcal{B}_g$ is connected if and only if $g$ is one of $3, 4, 7, 13, 17, 19$, and $59$.*

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# Appendix A

# GAP code

Here we will list some sample GAP code used to generate and classify actions. First a code to calculate admissible signatures.

```
findorders:= function(genus, index)
local orders;
orders:=[];
for d in DivisorsInt(index) do
 if d>1 then
  if d<4*genus+3 then
   Add(orders,d);
  fi;
 fi;
od;
return orders;
end;;


signcalc:= function(orders,position,sign,signatures,n,g,g0)
local tempsign, sigsum;
 tempsign:=ShallowCopy(sign);

 #Display([position,sign,tempsign]);

 if position < Length(orders) + 1 then
  sigsum:=[];

  Add(tempsign, orders[position]);

  for order in tempsign do
   Add(sigsum,1-1/order);
  od;

  if n*(Sum(sigsum)-2+2*g0)=2*g-2 then
   Add(signatures,[g0,tempsign]);
```

```
  elif n*(Sum(sigsum)-2+g0)<2*g-2 then
   #Display("recursion list position:");
   #Display([position,sign,tempsign]);
   signcalc(orders,position+1,sign,signatures,n,g,g0);
   #Display([position,sign,tempsign]);
   #Display("recursion number of elements:");
   signcalc(orders,position,tempsign,signatures,n,g,g0);

  fi;

 fi;

 return signatures;
end;;


findsignatures:= function(genus,index)
local signs;
 signs:=[];
 for a in [0.. genus] do
  Append(signs,signcalc(findorders(genus,index),
          1,[],[],index,genus,a));
  if index*(-2+2*a)=2*genus-2 then
   Add(signs,[a,[]]);
  fi;
 od;
 return signs;
end;;
```

Next one finds and classifies group actions with signature $(0; m_1, \ldots, m_k)$. A slightly modified code is used when $g = 1$.

```
elementproducts:=function(elements,position,products)
local templist, templists1, templists2;
templist:=[];
templists1:=[];
templists2:=[];
if position>Size(elements) then
 return products;
else
 templists1:=elementproducts(elements,position+1,products);
 for number in elements[position] do
  for list in templists1 do
   templist:=ShallowCopy(list);
   #Display(templist);
   Add(templist,number,1);
   Add(templists2,templist);
   #Display(templists1);
  od;
 od;
 return templists2;
fi;
end;;

#########################################
findepi:=function(group,signature)
local orders, elementsoforder, epimorphisms, autorbits,
      braidorbits, totalorbits, epiclasses, templist;
orders:=[];
```

```
elementsoforder:=[];
epimorphisms:=[];
epiclasses:=[];
templist:=[];
autorbits:=[];
braidorbits:=[];
totalorbits:=[];
for order in signature[2] do
 if (order in orders)<>true then
  Add(orders,order);
 fi;
od;
for order in orders do
 templist:=[];
 for g in group do if Order(g)=order then Add(templist,g); fi; od;
 Add(elementsoforder,[order,templist]);
od;
templist:=[];
for order in signature[2] do
 for set in elementsoforder do
  if set[1]=order then
   Add(templist, set[2]);
  fi;
 od;
od;
for product in elementproducts(templist,1,[[]]) do
 if (Product(product)=Identity(group)) and
    (Size(Subgroup(group,product))=Size(group)) then
  Add(epimorphisms,product);
 fi;
od;
autorbits:=OrbitsDomain(AutomorphismGroup(group),
                        epimorphisms, OnTuples);
#Display(["auto",autorbits]);
if Size(autorbits)=1 then
 epiclasses:=[Representative(autorbits[1])];
else
 for epiclass in autorbits do
  Add(braidorbits,braidorbit(Representative(epiclass)));
 od;
 totalorbits:=[1..Size(autorbits)];
 for braidclass in [1..Size(autorbits)-1] do
  for element in braidorbits[braidclass] do
   for class in [braidclass..Size(autorbits)-1] do
    if (element in autorbits[class+1]) then
     #Display([braidclass,element,class]);
     totalorbits[class+1]:=0;
    fi;
   od;
  od;
 od;
 for n in totalorbits do
  if n > 0 then
   Add(epiclasses,Representative(autorbits[n]));
  fi;
 od;
fi;
#Display(totalorbits);
```

```
return epiclasses;
end;;

##########################################

braid:=function(genvector,position)
local tempvector;
tempvector:=ShallowCopy(genvector);
Remove(tempvector,position);
Add(tempvector,genvector[position+1],position);
Remove(tempvector,position+1);
Add(tempvector,
    genvector[position+1]^-1*genvector[position]*genvector[position+1],
    position+1);
return tempvector;
end;;

#####################

braidorbit:=function(genvector)
local orbit;
orbit:=[];
Add(orbit, genvector);
for point in orbit do
 for position in [1..Size(genvector)-1] do
  if (braid(point,position) in orbit) <> true then
   Add(orbit, braid(point,position));
  fi;
 od;
od;
return orbit;
end;;
```

To calculate the signatures of subgroups of Fuchsian groups:

```
subgroupsignatures:=function(signature, epimorphism, orggroup)
local group, subgroup, indsig, subgroupinfo, cycles, order, i, j,
      k, sigsum1, sigsum2, genus, sggens;
indsig:=[];
cycles:=[];
order:=[];
subgroupinfo:=[];
sggens:=[];
group:=GroupWithGenerators(epimorphism);
for subgroup in MaximalSubgroupClassReps(group) do
  indsig:=[];
  PermG:=Action(group,RightCosets(group,subgroup),OnRight);
  Permgens:=GeneratorsOfGroup(PermG);
  #Display(Permgens);
  for k in [1..Size(Permgens)] do
   cycles:=CycleStructurePerm(Permgens[k]);
   for j in [1..Size(cycles)] do
    if IsBound(cycles[j]) then
     for i in [1..cycles[j]] do
      #Display([sign[2][k],(j+1)]);
      order:=sign[2][k]/(j+1);
      if order>1 then
       Add(indsig,sign[2][k]/(j+1));
      fi;
```

```
     od;
    fi;
   od;
   if Index(group,subgroup)>NrMovedPoints(Permgens[k]) then
    for i in [1..Index(group,subgroup)-NrMovedPoints(Permgens[k])] do
     Add(indsig,sign[2][k]);
    od;
   fi;
  od;
  sigsum1:=[];
  sigsum2:=[];
  for order in indsig do
   Add(sigsum1,1-1/order);
  od;
  for order in signature[2] do
   Add(sigsum2,1-1/order);
  od;
  genus:=1+(signature[1]-1+Sum(sigsum2)/2)*Index(group,subgroup)
          -Sum(sigsum1)/2;
  Sort(indsig);
  sggens:=[];
  for g in GeneratorsOfGroup(subgroup) do
   Add(sggens,Factorization(orggroup, g));
  od;
  Add(subgroupinfo,[IdSmallGroup(subgroup),StructureDescription(subgroup),
                    [genus,indsig],sggens,Permgens,
                    List(RightTransversal(group,subgroup))]);
od;
return subgroupinfo;
end;;
```

Here is a collection of examples of GAP code, and the output when running the code, used to find isolated strata in the case of $\mathcal{B}_{25}$ in Paper 4. The other cases presented in this article use similar code. First we find the possible signatures corresponding to an extension of an action of order $11$. Then we check the possible extensions of $C_{11}$ of the given orders, such that $C_{11}$ is a maximal subgroup.

```
for k in [1..1000] do
 if Size(findsignatures(25,11*k))>0 then
  Display([k,findsignatures(25,11*k)]);
 fi;
od;

>[ 1, [ [ 0, [ 11, 11, 11, 11, 11, 11, 11 ] ] ] ]
>[ 2, [ [ 0, [ 2, 11, 11, 11, 22 ] ] ] ]
>[ 3, [ [ 0, [ 3, 11, 11, 33 ] ] ] ]
>[ 5, [ [ 0, [ 11, 55, 55 ] ] ] ]
>[ 6, [ [ 0, [ 11, 11, 11 ] ], [ 0, [ 6, 11, 66 ] ] ] ]
>[ 7, [ [ 0, [ 7, 7, 11 ] ] ] ]
>[ 9, [ [ 0, [ 3, 11, 11 ] ] ] ]
>[ 10, [ [ 0, [ 2, 22, 55 ] ] ] ]
>[ 12, [ [ 0, [ 2, 11, 22 ] ] ] ]
>[ 14, [ [ 0, [ 2, 7, 22 ] ] ] ]
>[ 18, [ [ 0, [ 3, 3, 11 ] ], [ 0, [ 2, 6, 11 ] ] ] ]
>[ 36, [ [ 0, [ 2, 3, 22 ] ] ] ]

for k in [2,3,5,6,7,9,10,12,14,18,36] do
 for G in AllSmallGroups(11*k) do
  for H in MaximalSubgroups(G) do
```

```
   if Order(H)=11 then
    Display([IdSmallGroup(G),StructureDescription(G)]);
   fi;
  od;
 od;
od;

>[ [ 22, 1 ], "D22" ]
>[ [ 22, 2 ], "C22" ]
>[ [ 33, 1 ], "C33" ]
>[ [ 55, 1 ], "C11 : C5" ]
>[ [ 55, 2 ], "C55" ]
>[ [ 77, 1 ], "C77" ]
```