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On the Hadamard formula for second order systems in non-smooth domains

Vladimir Kozlov^a and Sergey Nazarov^b

^a*Department of Mathematics, Linköping University,
S-581 83 Linköping, Sweden*

^b*Institute for Problems in Mechanical Engineering,
Russian Academy of Sciences,
V.O., Bol'shoy pr. 61, St Petersburg 199178, RF*

Abstract. Perturbations of eigenvalues of the Dirichlet problem for a second order elliptic system in a bounded domain Ω in $\mathbb{R}^n$ are studied under variations of the domain Ω . We investigate the case when the perturbed domain is located in a d -neighborhood of the reference Lipschitz domain. A new asymptotic formula is derived; it contains terms that are absent in the classical formula of Hadamard. The latter is valid only for smooth domains and smooth perturbations. We give conditions that guarantee the validity of the Hadamard formula. The general asymptotic formula is applied when Ω is perturbed by small curvilinear and circular cuts. Most of the results are new even for the Laplace operator.

1 Introduction

One of the classical shape optimization problems deals with minimization of eigenvalues of a boundary value problem for elliptic operator in a bounded domain. In this problem, the optimization parameter is the domain, possibly subject to certain constraints. A standard approach to such a problem is to apply asymptotic formulae for eigenvalues when the domain is varied, and one of the first examples of such an asymptotic formula is the Hadamard formula for the first eigenvalue of the Dirichlet Laplacian (see [6]):

$$\lambda(h_t(\Omega)) \sim \lambda(\Omega) - t \int_{\partial\Omega} |\partial_\nu u|^2 \nu \cdot \nu \, d\Gamma.$$

Here the domain Ω is perturbed by the family of smooth mappings $h_t : \mathbb{R}^n \mapsto \mathbb{R}^n$ such that h_0 is the identical mapping; $v = dh_t/dt|_{t=0}$, ν is the outward unit normal on $\partial\Omega$; $d\Gamma$ is the standard measure on the boundary; $\lambda(\Omega)$ denotes the first eigenvalue of the Dirichlet Laplacian in a bounded domain $\Omega \subset \mathbb{R}^n$, and u is the eigenfunction corresponding to $\lambda(\Omega)$ and normalized in the Lebesgue space

$L^2(\Omega)$. Subsequently, this formula was generalized in order to make it applicable to an arbitrary eigenvalue of a second-order system, but the requirement that perturbations of the reference domain are described by a family of smooth isomorphisms remained unchanged (see [14], [4], [7], [2] and references therein). On the other hand, there are many problems in which more general classes of perturbations are involved, namely, non-smooth perturbations and perturbations that cannot be described by a class of isomorphisms. An example of such a problem is the shape optimization problem with perturbations subject to certain constraints.

In [10] and [11], a new abstract approach was proposed that allows to treat such problems. In particular, an example of the perturbation of the first eigenvalue of the Dirichlet Laplacian was used in [10] to demonstrate that the abstract Theorem 1 proved in this paper is applicable to non-smooth reference domains as well as to non-smooth perturbations.

In the present paper, we discuss perturbations of an arbitrary eigenvalue of the second-order matrix elliptic operator which is formally self-adjoint and is acting on vector functions subject to the homogeneous Dirichlet condition on the boundary of a domain satisfying minimal smoothness and geometrical assumptions. Minimal assumptions are imposed on the perturbations as well; in particular, they are not supposed to be given by a family of isomorphisms.

Let Ω_1 and Ω_2 be bounded domains with nonempty intersection and with boundaries Γ_1 and Γ_2 , respectively. We introduce the following sesquilinear forms:

$$(u, v)_k \equiv (u, v)_{\Omega_k} = \sum_{i,j=1}^n \int_{\Omega_k} (A_{ij} \partial_{x_j} u, \partial_{x_i} v) dx, \quad k = 1, 2. \quad (1)$$

Here $(\cdot, \cdot)$ is the standard inner product in $\mathbb{C}^M$, the constant $M \times M$ -matrices A_{ij} are such that $A_{ij} = A_{ji}^*$, where A_{ji}^* stands for the adjoint matrix. By

$$\mathcal{A}(\partial_x) = - \sum_{i,j=1}^n A_{ij} \partial_{x_i} \partial_{x_j}$$

we denote the corresponding differential operator; the matrix $-\mathcal{A}(\xi)$ is supposed to be positive definite for all $\xi \in \mathbb{R}^n \setminus \mathcal{O}$.

Let us consider the following spectral problems formulated in the variational form:

$$(\varphi, v)_1 = \lambda \langle \varphi, v \rangle_1 \quad \text{for all } v \in (\dot{W}^{1,2}(\Omega_1))^M \quad (2)$$

and

$$(U, V)_2 = \mu \langle U, V \rangle_2 \quad \text{for all } V \in (\dot{W}^{1,2}(\Omega_2))^M. \quad (3)$$

Here $\langle \cdot, \cdot \rangle_k$ is the inner product in $(L^2(\Omega_k))^M$, $\varphi \in (\dot{W}^{1,2}(\Omega_1))^M$ and $U \in (\dot{W}^{1,2}(\Omega_2))^M$. The space $\dot{W}^{1,2}(\Omega_j)$ is, as usual, the closure of $C_0^\infty(\Omega_j)$ in the norm of $W^{1,2}(\Omega_j)$.

Assumption I. We suppose that Ω_1 is a bounded Lipschitz domain and Ω_2 is an arbitrary domain such that $\Omega_2 \setminus \Omega_1$ and $\Omega_1 \setminus \Omega_2$ belong to the d -neighborhood of Γ_1 with a small $d > 0$.

Note that the perturbed domain Ω_2 can be arbitrary (and not assumed to be Lipschitz).

Let λ_m be the m th eigenvalue of problem (2), by X_m we denote the corresponding eigenspace, $\dim X_m = J_m$. According to [10], if d is sufficiently small, then there exist exactly J_m eigenvalues of problem (3) within the interval $((\lambda_{m-1} + \lambda_m)/2, (\lambda_m + \lambda_{m+1})/2)$; these eigenvalues are denoted by $\mu_1, \dots, \mu_{J_m}$. One of our main results is the following

Theorem 1 *If Assumption I holds, then the following asymptotic formula is valid:*

$$\frac{1}{\mu_j} = \frac{1}{\lambda_m} + \tau_j + O(d^{1+\delta}), \quad j = 1, \dots, J_m. \quad (4)$$

Here δ is a positive number depending only on Ω_1 and the matrices A_{ij} ; $\{\tau_j\}_{j=1}^{J_m}$ is the set of eigenvalues of the following finite-dimensional eigenvalue problem:

$$\frac{1}{\lambda_m} \left((\Psi_\varphi, \Psi_\psi)_2 - (\Phi_\varphi, \Phi_\psi)_2 - (\varphi, \psi)_{\Omega_1 \setminus \Omega_2} - (\Psi_\varphi, \psi)_2 - (\Psi_\psi, \varphi)_2 \right) = \tau(\varphi, \psi)_1 \quad (5)$$

for all $\psi \in X_m$, where the function $\Phi = \Phi_\varphi$ depends on $\varphi \in X_m$ and is such that the equality

$$(\Phi, w)_2 = 0 \quad \text{holds for all } w \in (\dot{W}^{1,2}(\Omega_2))^M$$

and $\Phi + \varphi \in (\dot{W}^{1,2}(\Omega_2))^M$. Furthermore, the function $\Psi = \Psi_\varphi \in (\dot{W}^{1,2}(\Omega_2))^M$ is a solution of

$$(\Psi, w)_2 = (\varphi, w)_2 - \lambda_m \langle \varphi, w \rangle_2 = \int_{\Gamma_1} (N\varphi, w) d\Gamma \quad \forall w \in (\dot{W}^{1,2}(\Omega_2))^M, \quad (6)$$

where

$$N\varphi = \sum_{i,j=1}^n A_{ij} \nu_j \partial_{x_i} \varphi \quad (7)$$

is the co-normal derivative and ν is the outward unit normal to Γ_1 .

Similar assertion was proved in [10] for the Laplace operator under the additional assumption that the boundary of perturbed domain Ω_2 is also Lipschitz. Thus, Theorem 1 covers a larger set of perturbed domains even for the Laplacian.

The next theorem deals with the remainder in the asymptotic formula (4).

Theorem 2 *Suppose that the boundary Γ_1 is of class C^1 . If the following positivity assumption*

$$\sum_{i,j=1}^n (A_{ij} \xi_i, \xi_j) \geq 0 \quad \text{for all } \xi_i \in \mathbb{C}^M, i = 1, \dots, n \quad (8)$$

holds for the operator $\mathcal{A}$, then formula (4) is valid for an arbitrary $\delta \in (0, 1)$.

In the next theorem we give conditions on the reference and perturbed domains that guarantee the validity of the Hadamard formula for perturbed eigenvalues. Below we use the following coordinates near the boundary Γ_1 : τ is the signed distance from x to Γ_1 (it is negative for $x \in \Omega_1$) and $z \in \Gamma_1$.

Theorem 3 *Let Ω_1 be of class $C^{1,\alpha}$, $\alpha \in (0,1)$, and let $\tau = h(z,d)$ be the equation of Γ_2 , where h satisfies*

$$\int_{\Gamma_1} |\nabla_z h(z,d)|^2 d\Gamma = o(1) \quad \text{as } d \rightarrow 0. \quad (9)$$

Then the following formula holds:

$$\frac{1}{\mu_j} = \frac{1}{\lambda_m} + \theta_j + o(d), \quad (10)$$

where $\{\theta_j\}_{j=1}^{J_m}$ are eigenvalues of the problem

$$\frac{1}{\lambda_m} \int_{\Gamma_1} h(z,d) \sum_{i,j=1}^n (A_{ij} \nu_j \nu_i \partial_\nu \varphi, \partial_\nu \psi) d\Gamma = \theta(\varphi, \psi)_1 \quad \text{for all } \psi \in X_m. \quad (11)$$

The last formula extends the classical Hadamard formula to second-order matrix operators and is new even for the Laplacian as well.

If $\Omega_2 \subset \Omega_1$, then $\Psi = 0$ in Theorem 1. Let the boundary of Ω_2 be also Lipschitz. By [5, Theorem 4.2],

$$\int_{\Omega_2} |\nabla \Phi_\varphi| dx \leq C \int_{\Gamma_2} |\nabla_\tau \varphi|^2 d\Gamma, \quad (12)$$

where ∇_τ is the tangent gradient on Γ_2 . The constant C depends on the number of fixed-size cylinders covering Γ_2 and on the local Lipschitz constant of functions that give local representations of the domain as an upper graph of Lipschitz functions. Using (12), one can write the asymptotic formula (4) as follows:

$$\frac{1}{\mu_j} = \frac{1}{\lambda_m} - \frac{\kappa_j}{\lambda_m^2} + O(d^{1+\delta}) + O(q). \quad (13)$$

Here

$$q = \max \int_{\Gamma_2} |\nabla_\tau \varphi|^2 d\Gamma,$$

the maximum taken over all $\varphi \in X_m$ such that $(\varphi, \varphi)_1 = 1$; κ_j , $j = 1, \dots, J_m$, are eigenvalues of the problem

$$\sum_{i,j=1}^n \int_{\Omega_1 \setminus \Omega_2} (A_{ij} \partial_{x_j} \varphi, \partial_{x_i} \psi) dx = \kappa \int_{\Omega_1} (\varphi, \psi) dx \quad \text{for all } \psi \in X_m, \quad (14)$$

where $\varphi \in X_m$. If Γ_1 is smooth and $q = o(d)$, then these formulae imply (10). In Sect. 3.6, we consider a two-dimensional domain whose boundary has

Figure 1: Perturbations by small circular and curvilinear cuts

corner points. In this case one cannot represent the boundary of perturbed domain through the normal shift and therefore the Hadamard formula cannot be applied. However, formula

$$\frac{1}{\mu_j} = \frac{1}{\lambda_m} - \frac{\kappa_j}{\lambda_m^2} + O(d). \quad (15)$$

still holds in this case; here κ_j , $j = 1, \dots, J_m$, are eigenvalues of (14).

In the last section, we apply the general formula to the case when a smooth bounded domain is perturbed either by curvilinear cuts, whose lengths are of the same order as the distances between them, or by circular cuts uniformly distributed along the boundary, see Fig.1. Of course, the Hadamard formula is not valid in these cases, because there are no smooth mappings from the unperturbed domains onto perturbed ones. Let us describe the first of these examples for the Laplace operator. Let Ω_1 be a 2D simply connected domain whose boundary has length 1. The perturbed domain is defined by

$$\Omega_2 = \Omega \setminus \cup_{j=1}^N \Upsilon_j^\varepsilon, \quad \Upsilon_j^\varepsilon = \{x \in \Omega \cap \mathcal{V} : z = j\varepsilon, \tau \in [-\varepsilon\ell(z), 0]\}, \quad (16)$$

where z is the arc length measured from a certain fixed point and $\ell(z)$ is a positive C^∞ function on Γ_1 . In this case, the function Ψ vanishes in (5) and the function Φ contributes to the asymptotics of μ . Assuming that μ is a simple eigenvalue located near the eigenvalue λ and the corresponding eigenfunction φ is normalized so that $\|\varphi\|_{L^2(\Omega_1)} = 1$, we get

$$\frac{1}{\mu} = \frac{1}{\lambda} - \varepsilon \frac{\tau}{\lambda^2} + O(\varepsilon^{1+\delta}). \quad (17)$$

Here δ is an arbitrary number from the interval $(0,1)$ and

$$\tau = \int_{\Gamma_1} \int_{-1/2}^{1/2} \int_{-\infty}^0 |\nabla_\xi w(\xi, z)|^2 d\xi dz,$$

where w is a bounded solution of the following problem:

$$\Delta_\xi w = 0 \quad \text{in } \Pi_z = \{\xi = (\xi_1, \xi_2) : \xi_1 < 0, \xi_2 \in (-1/2, 1/2)\} \setminus \mathcal{L}_z,$$

$\mathcal{L}_z = \{\xi = (\xi_1, 0) : \xi_1 \in (-\ell(z), 0)\}$. Moreover, w must satisfy the boundary conditions:

$$\begin{aligned} w(0, \xi_2, z) &= 0, \quad \xi_2 \in (-1/2, 1/2), \\ w(\xi_1, 0, z) &= -\xi_1 \partial_\nu \varphi(0, z), \quad \xi_1 > -\ell(z), \end{aligned}$$

and the following periodicity conditions

$$w(\xi_1, -1/2, z) = w(\xi_1, 1/2, z), \quad \partial_{\xi_2} w(\xi_1, -1/2, z) = \partial_{\xi_2} w(\xi_1, 1/2, z), \quad \xi_1 < 0.$$

Other examples showing importance of generalized Hadamard formula are presented in Sect.4, where we consider both types of perturbations shown in Fig. 1 in the case of second order systems.

Proofs of above asymptotic formulae are based on the abstract asymptotic formula obtained in [10] for eigenvalues of compact operators acting in various Hilbert spaces. However, further analysis differs essentially from that in [10]. Indeed, here we apply Hardy type inequalities instead of L^p -estimates valid for solutions of the Dirichlet problem in Lipschitz domains. The distance from the boundary of $2d$ -neighborhood of Ω_1 is taken as a weight function in the Hardy inequality. It does not depend on the perturbed domain which allows us to show in Lemma 3 that for domains close to Ω_1 there is always an interval for powers of the weight in which the Dirichlet problem in Ω_2 is solvable. Thus, we avoid imposing any smoothness assumption on Ω_2 . Another important tool is based on the fact that a weak solution in a Lipschitz domain has some additional smoothness. One more advantage of the method is that it does not require the use of an extension operator, see [10, Sect. 4.3]. This allows us to obtain all results under rather weak assumptions about smoothness of Ω_1 and Ω_2 , which are new even for the Dirichlet Laplacian.

2 General domains

In this section, we consider eigenvalue problems (2) and (3) assuming that Ω_1 and Ω_2 are bounded domains with non-empty intersection. The proximity of domains and operators is described by smallness of constants in some integral inequalities. The main result is obtained in Theorem 5, where an asymptotic representation is given for eigenvalues of problem (3) located near an eigenvalue of problem (2).

For convenience of the readers, we present in Sect. 2.1 an asymptotic result from [10] about eigenvalues of two compact operators acting in different Hilbert spaces. In Sect. 2.2, we apply this abstract theorem to problems (2) and (3).

2.1 Abstract theorem on asymptotics

Let K_j , $j = 1, 2$, be a compact, non-negative, self-adjoint operator in a Hilbert space H_j with the inner product $(\cdot, \cdot)_j$ and with the norm $\|\cdot\|_j$. By λ_k^{-1} , $k = 1, 2, \dots$, we denote the eigenvalues of K_1 numbered so that $0 < \lambda_1 < \lambda_2 < \dots$; the eigenspace corresponding to λ_k is denoted by X_k , and $J_k = \dim X_k$. The

notation λ_k^{-1} used for eigenvalues of a compact operator is applied because we reserve the notation λ_k for eigenvalues of the corresponding differential operator.

Our aim is to describe the set of those eigenvalues of the operator K_2 that are located near an eigenvalue λ_m^{-1} of the operator K_1 . In order to measure the proximity of these eigenvalues we introduce a bounded, linear operator $S : H_1 \rightarrow H_2$, a positive constant c_1 and an integer N such that $N \geq m$ and

$$\frac{c_1 \|S\|_{H_1 \rightarrow H_2}^2}{\lambda_{N+1}} \leq \frac{1}{\lambda_{m+1}}. \quad (18)$$

We set

$$\mathcal{X}_N = \bigoplus_{j=1}^N X_j$$

and denote by $\mathcal{Y}_N$ the orthogonal complement in H_2 to $S\mathcal{X}_N$. The proximity of the operators K_1 and K_2 is measured by the constant ε in the following three inequalities:

$$(K_2 w, w)_2 \leq c_1 (K_1 S^* w, S^* w)_1 + \varepsilon \|w\|_2^2 \quad \text{for all } w \in \mathcal{Y}_N, \quad (19)$$

where $S^* : H_2 \rightarrow H_1$ is the operator adjoint to S ;

$$| \|S\varphi\|_2^2 - \|\varphi\|_1^2 | \leq \varepsilon \|\varphi\|_1^2 \quad \text{for all } \varphi \in \mathcal{X}_N; \quad (20)$$

and

$$|(B\varphi, w)_2| \leq \varepsilon \|\varphi\|_1 \|w\|_2 \quad \text{for all } \varphi \in \mathcal{X}_N \text{ and } w \in H_2, \quad (21)$$

where

$$B = K_2 S - S K_1 : H_1 \rightarrow H_2. \quad (22)$$

Of course, all these inequalities hold provided ε is sufficiently large, but it occurs that if ε is small, then there are exactly J_m eigenvalues of the operator K_2 in a neighborhood of λ_m^{-1} (say in $(\frac{2}{\lambda_m + \lambda_{m+1}}, \frac{2}{\lambda_m + \lambda_{m-1}})$). Moreover, these eigenvalues belong to $(\lambda_m^{-1} - c\varepsilon, \lambda_m^{-1} + c\varepsilon)$, where c is a constant depending only on λ_{m-1} , λ_m and λ_{m+1} . We denote these eigenvalues of K_2 by $\mu_1^{-1}, \dots, \mu_{J_m}^{-1}$.

In order to describe the eigenvalues of the operator K_2 that are close to λ_m^{-1} more explicitly, we introduce the constants ρ_m and σ_m that appear in the inequalities

$$|(B\varphi, S\psi)_2| \leq \rho_m \|\varphi\|_1 \|S\psi\|_2 \quad \text{for all } \varphi, \psi \in X_m \quad (23)$$

and

$$|(B\varphi, w)_2| \leq \sigma_m \|\varphi\|_1 \|w\|_2 \quad \text{for all } \varphi \in X_m, w \in Y_m, \quad (24)$$

where Y_m is the orthogonal complement to SX_m in H_2 . It is clear that we can put $\rho_m = \sigma_m = \varepsilon$, but, in fact, these constants can be essentially smaller than ε . In what follows, we denote by P_m the orthogonal projector in H_2 onto SX_m , and so $Q_m = I - P_m$ is the orthogonal projector onto Y_m .

Theorem 4 ([10]) *There exists a constant ε_m depending only on λ_m and $\lambda_{m\pm 1}$ such that if (19) (20) and (21) hold for $\varepsilon \leq \varepsilon_0$, then for the eigenvalues μ_s^{-1} , $s = 1, \dots, J_m$, of the operator K_2 introduced above the following asymptotic representation holds:*

$$\frac{1}{\mu_j} = \frac{1}{\lambda_m} + \omega_j + O(\sigma_m^2(\rho_m + \sigma_m^2)), \quad (25)$$

where $\{\omega_j\}_{j=1}^{J_m}$ are eigenvalues of the problem

$$(B\varphi, S\psi)_2 + (\mathcal{R}(\varphi), B\psi)_2 = \omega(S\varphi, S\psi)_2 \quad \text{for all } \psi \in X_m. \quad (26)$$

Here $\varphi \in X_m$, ρ_m and σ_m are the constants in (23) and (24), respectively, and $\mathcal{R} = \mathcal{R}(\varphi) \in Y_m$ is a solution of the equation

$$\lambda_m^{-1}\mathcal{R} - Q_m K_2 \mathcal{R} = Q_m B\varphi. \quad (27)$$

This solution satisfies the estimate

$$\|\mathcal{R}\|_2 \leq C\sigma_m \|\varphi\|_1. \quad (28)$$

(The factor $\|\varphi\|_1$ is lost in the corresponding formula (19) in [10, Theorem 1].)

2.2 Eigenvalues of problem (3) located near λ_m

In what follows, vector functions from $\dot{W}^{1,2}(\Omega_k)^M$ and $\dot{W}^{1,2}(\Omega_1 \cap \Omega_2)^M$ are extended by zero outside Ω_k and $\Omega_1 \cap \Omega_2$, respectively.

Let us reduce the spectral problems (2) and (3) to equivalent spectral problems for compact operators. The operators K_1 and K_2 are defined by the equality

$$(K_s u, v)_s = \langle u, v \rangle_s \quad \text{that holds for all } u, v \in \dot{W}^{1,2}(\Omega_s)^M.$$

The operator K_s is compact and problems (2) and (3) take the form

$$K_1 \varphi = \lambda^{-1} \varphi \quad \varphi \in \dot{W}^{1,2}(\Omega_1)^M, \quad (29)$$

and

$$K_2 U = \mu^{-1} U \quad U \in \dot{W}^{1,2}(\Omega_2)^M, \quad (30)$$

respectively. In order to compare eigenvalues of these problems we apply Theorem 4. We put $H_k = \dot{W}^{1,2}(\Omega_k)^M$ and supply this space with the inner product $(\cdot, \cdot)_k$, the corresponding norm is denoted by $\|\cdot\|_k$. We define the operator $S : H_1 \rightarrow H_2$ as follows. Let $\varphi \in H_1$, then $u = S\varphi \in H_2$ is a unique solution of the problem

$$(u, w)_2 = (\varphi, w)_2 \quad \text{for all } w \in H_2. \quad (31)$$

The adjoint operator $S^* : H_2 \rightarrow H_1$ is defined by

$$(\varphi, S^* w)_1 = (\varphi, w)_1 \quad (32)$$

for all $\varphi \in H_1$. Clearly,

$$\|S\|_{H_1 \rightarrow H_2} = \|S^*\|_{H_2 \rightarrow H_1} = 1.$$

Moreover, if $w \in H_1 \cap H_2$, then $Sw = w$ and $S^*w = w$. If $w_k \in H_k$ and w_k is orthogonal to $H_1 \cap H_2$, then the orthogonality is also valid for Sw_1 and S^*w_2 .

Let $0 < \lambda_1 < \lambda_2 < \dots$ be eigenvalues of problem (2) or, equivalently, λ_j^{-1} are eigenvalues of problem (29), and let J_j be the multiplicity of λ_j . By X_j we denote the space of eigenfunctions corresponding to the eigenvalue λ_j .

The results in this section are valid without Assumption I made in Introduction. It is sufficient to assume that Ω_1 and Ω_2 are two bounded domains with a non-empty intersection and the following inequality

$$\int_{\Omega_k} |w|^2 dx \leq \sigma \|w\|_k^2 \quad (33)$$

is valid with a small constant σ for all $w \in H_k$, $k = 1, 2$, orthogonal to $H_1 \cap H_2$. The latter constant σ describes the proximity of Ω_1 and Ω_2 , cf. [10].

We introduce ρ_1 as the best constant in the inequality

$$|(B\varphi, S\psi)_2| \leq \rho_1 \|\varphi\|_1 \|S\psi\|_2 \quad (34)$$

that holds for B given by (22) and for all $\varphi, \psi \in X_m$. Let Y_m be the orthogonal complement of SX_m in H_2 , let also $Q_m = I - P_m$, where P_m denotes the orthogonal projector on SX_m in H_2 . By $(u, v)_\Omega$ we denote the sesquilinear form (1) defined on Ω , and $\langle u, v \rangle_\Omega = \int_\Omega (u, v) dx$. In particular, $(u, v)_{\Omega_k} = (u, v)_k$ and $\langle u, v \rangle_{\Omega_k} = \langle u, v \rangle_k$.

Theorem 5 *Let Ω_1 and Ω_2 be two domains with non-empty intersection, and let the constant σ in (33) be small. Then there are exactly J_m eigenvalues of problem (3) in the interval $((\lambda_{m-1} + \lambda_m)/2, (\lambda_m + \lambda_{m+1})/2)$, for which the following asymptotic formula holds:*

$$\frac{1}{\mu_j} = \frac{1}{\lambda_m} + \omega_j + O(\sigma(\rho_1 + \sigma)). \quad (35)$$

Here $\{\omega_j\}_{j=1}^{J_m}$ are the eigenvalues of problem (26), where $\varphi \in X_m$ and $\mathcal{R} = \mathcal{R}(\varphi) \in Y_m$ is a solution of equation (27).

Proof. In order to apply Theorem 4 we have to verify conditions (19), (20) and (21).

Let us prove inequality (19) with $c_1 = 1$ and $\varepsilon = 2\sqrt{2\sigma c_0}$, where c_0 is the constant in

$$\int_{\Omega_2} |\varphi|^2 dx \leq c_0 (\varphi, \varphi)_2, \quad \varphi \in H_2. \quad (36)$$

This inequality follows from

$$(K_2 w, w)_2 \leq (K_1 S^* w, S^* w)_1 + \varepsilon \|w\|_2^2 \quad \text{valid for all } w \in H_2.$$

Using the definition of K_1 and K_2 , we write the last inequality as follows:

$$\int_{\Omega_2} |w(x)|^2 dx \leq \int_{\Omega_1} |S^* w(x)|^2 dx + \varepsilon(w, w)_2 \quad \text{for all } w \in H_2. \quad (37)$$

In order to prove (37), we split $w \in H_2$ into the sum

$$w = w_0 + w_1, \quad \text{where } w_0 \in H_1 \cap H_2 \text{ and } w_1 \text{ is orthogonal to } H_1 \cap H_2. \quad (38)$$

Then inequality (37) takes the form

$$\int_{\Omega_2} |w_0 + w_1|^2 dx \leq \int_{\Omega_1 \cap \Omega_2} |w_0 + S^* w_1|^2 dx + \varepsilon(w_0, w_0)_2 + \varepsilon(w_1, w_1)_2. \quad (39)$$

Using (33) for $S^* w_1$, we obtain

$$\int_{\Omega_1} |S^* w_1|^2 dx \leq \sigma(S^* w_1, S^* w_1)_1 \leq \sigma(w_1, w_1)_2. \quad (40)$$

Applying (33) to w_1 , we get that

$$\int_{\Omega_2} |w_1|^2 dx \leq \sigma(w_1, w_1)_2. \quad (41)$$

Since

$$\begin{aligned} \int_{\Omega_2} |w_0 + w_1|^2 dx &= \int_{\Omega_2} (|w_0|^2 + |w_1|^2 + 2\Re(w_0, w_1)) dx = \int_{\Omega_1} |w_0 + S^* w_1|^2 dx \\ &+ \int_{\Omega_2} (2\Re(w_0, w_1) - 2\Re(w_0, S^* w_1) + |w_1|^2 - |S^* w_1(x)|^2) dx, \end{aligned}$$

we have that

$$\int_{\Omega_2} |w_0 + w_1|^2 dx \leq \int_{\Omega_2} (|w_0 + S^* w_1|^2 + 2\delta|w_0|^2) dx + \frac{2\sigma(1 + \delta)}{\delta} (w_1, w_1)_2.$$

Here (40) and (41) are also taken into account. Using (36) and choosing $\varepsilon = 2c_0\delta$ and $\delta = \sqrt{2\sigma c_0^{-1}}$, we arrive at (39), and so (19) is valid.

In particular, $c_1 = 1$ in our case, and so we can take $N = m$ in (18).

Let us verify (20) for $\varphi \in \mathcal{X}_m = X_1 \oplus \cdots \oplus X_m$. Clearly, we have that $\|S\varphi\|_2 \leq \|\varphi\|_1$. Let us estimate $\|S\varphi\|_2$ from below. We represent φ as the sum

$$\varphi = \varphi_0 + \varphi_1, \quad \text{where } \varphi_0 \in H_1 \cap H_2 \text{ and } \varphi_1 \text{ is orthogonal to } H_1 \cap H_2. \quad (42)$$

Since $S\varphi = \varphi_0 + S\varphi_1$ and $\|S\varphi\|_2^2 = \|\varphi_0\|_1^2 + \|S\varphi_1\|_2^2$, we have $\|S\varphi\|_2 \geq \|\varphi_0\|_1$. Furthermore, (33) implies that

$$\int_{\Omega_1} |\varphi_1|^2 dx \leq \sigma(\varphi_1, \varphi_1)_1. \quad (43)$$

Let $\varphi = \Phi_1 + \dots + \Phi_m$, where $\Phi_j \in X_j$. From $(\varphi_1, \varphi_1)_1 = (\varphi, \varphi_1)_1$ it follows that

$$(\varphi_1, \varphi_1)_1 = \sum_{j=1}^m (\Phi_j, \varphi_1)_1 = \sum_{j=1}^m \lambda_j \langle \Phi_j, \varphi_1 \rangle_1 \leq \left(\sum_{j=1}^m \lambda_j^2 \|\Phi_j\|_{L^2(\Omega_1)}^2 \right)^{1/2} \|\varphi_1\|_{L^2(\Omega_1)}.$$

Since Φ_j is an eigenvector corresponding to λ_j , we get from (43) that

$$(\varphi_1, \varphi_1)_1 \leq \lambda_m^{1/2} \sigma^{1/2} a_1(\varphi, \varphi).$$

Therefore,

$$\|S\varphi\|_2^2 \geq \|\varphi_0\|_1^2 = \|\varphi\|_1^2 - \|\varphi_1\|_1^2 \geq (1 - \lambda_m^{1/2} \sigma^{1/2}) \|\varphi\|_1^2, \quad (44)$$

and so (20) holds with $\varepsilon = \lambda_m^{1/2} \sigma^{1/2}$.

Finally, let us show that (21) holds with $\varepsilon = c\sigma^{1/2}$, or, equivalently, we have that

$$\|B\varphi\|_2 \leq c\sigma^{1/2} \|\varphi\|_1 \quad \text{for all } \varphi \in \mathcal{X}_m. \quad (45)$$

Since

$$(B\varphi, w)_2 = (K_2 S\varphi, w)_2 - (SK_1 \varphi, w)_2 = \langle S\varphi, w \rangle_2 - \langle \varphi, S^* w \rangle_1 \quad (46)$$

for $\varphi \in \mathcal{X}_m$ and $w \in H_2$, the representations (38) and (42) yield that

$$(B\varphi, w)_2 = \langle \varphi_0, w_1 \rangle_2 - \langle \varphi_1, w_0 \rangle_1 + \langle S\varphi_1, w \rangle_2 - \langle \varphi, S^* w_1 \rangle_1. \quad (47)$$

Now, applying (33) to the functions w_1 , $S\varphi_1$, φ_1 and $S^* w_1$, we arrive at

$$|(B\varphi, w)_2| \leq c\sigma^{1/2} \|\varphi\|_1 \|w\|_2 \quad \text{for } \varphi \in X_1 \text{ and } w \in H_2, \quad (48)$$

where c is a constant depending on the constant c_0 in (36). Thus, (45), and consequently (21) are satisfied. Using results obtained in Sect.2.1, we conclude that there are exactly J_m eigenvalues of problem (30) in the interval $(2/(\lambda_{m+1} + \lambda_m), 2/(\lambda_m + \lambda_{m-1}))$ and they are located in $(\lambda_m^{-1} - c\varepsilon, \lambda_m^{-1} + c\varepsilon)$, where $\varepsilon = c\sigma^{1/2}$ and the constant c depends on λ_{m-1} , λ_m and λ_{m+1} . This gives the first assertion of lemma.

In order to apply Theorem 4, we put $\sigma_m = c\sigma^{1/2}$ and take ρ_m equal to the constant ρ_1 in (34). This choice of constants yields formula (25) in the form (35). The proof of Lemma is complete.

Remark 1 *It follows from (48) that the constant σ_m is estimated by $c\sigma^{1/2}$. Using this along with (48) and (28), we obtain*

$$|(\mathcal{R}(\varphi), B\psi)_2| \leq c\sigma \|\varphi\|_1 \|\psi\|_1 \quad \text{for } \varphi, \psi \in X_m. \quad (49)$$

In conclusion of this section, we obtain a useful representation of $(B\varphi, S\psi)_2$ involved in (26). Let $\varphi, \psi \in X_m$, then we use the representations

$$S\varphi = \varphi + \Phi_\varphi \quad \text{and} \quad S\psi = \psi + \Phi_\psi, \quad (50)$$

where the vector function $\Phi = \Phi_\varphi$ satisfies the integral identity $(\Phi, w)_2 = 0$ for all $w \in H_2$, $\Phi + \varphi \in H_2$ and the same is true for Φ_ψ . Therefore,

$$(B\varphi, S\psi)_2 = \langle \varphi + \Phi_\varphi, \psi + \Phi_\psi \rangle_2 - \lambda_m^{-1}(\varphi + \Phi_\varphi, \psi + \Phi_\psi)_2.$$

From the definition of Φ_φ and Φ_ψ , it follows that

$$(\Phi_\varphi, \psi + \Phi_\psi)_2 = (\varphi + \Phi_\varphi, \Phi_\psi)_2 = 0, \quad (51)$$

which implies

$$(B\varphi, S\psi)_2 = \langle \varphi + \Phi_\varphi, \psi + \Phi_\psi \rangle_2 + \lambda_m^{-1}((\Phi_\varphi, \Phi_\psi)_2 - (\varphi, \psi)_2). \quad (52)$$

Using the relation

$$(\Psi_\phi, S\psi)_2 = (\Psi_\phi, \psi)_2 = (\varphi, \Phi_\psi + \psi)_2 - \lambda_m \langle \varphi, \Phi_\psi + \psi \rangle_2$$

and similar equality with exchanged φ and ψ , we obtain

$$\begin{aligned} \langle \varphi, \Phi_\psi \rangle_2 &= \frac{1}{\lambda_m}((\varphi, \psi + \Phi_\psi)_2 - (\Psi_\varphi, \psi)_2) - \langle \varphi, \psi \rangle_2 \\ &= \frac{1}{\lambda_m}((\varphi, \psi)_2 - (\Phi_\varphi, \Phi_\psi)_2 - (\Psi_\varphi, \psi)_2) - \langle \varphi, \psi \rangle_2, \end{aligned}$$

where we used (51). Applying this relation (and corresponding relation with exchanged φ and ψ) to (52), we get

$$(B\varphi, S\psi)_2 = -\langle \varphi, \psi \rangle_2 + \langle \Phi_\varphi, \Phi_\psi \rangle_2 - \lambda_m^{-1}((\Phi_\varphi, \Phi_\psi)_2 - (\varphi, \psi)_2 + (\Psi_\varphi, \psi)_2 + (\Psi_\psi, \varphi)_2).$$

Now using that $(\varphi, \psi)_1 = \lambda \langle \varphi, \psi \rangle_1$, we arrive at

$$\begin{aligned} (B\varphi, S\psi)_2 &= \langle \varphi, \psi \rangle_{\Omega_1 \setminus \Omega_2} + \langle \Phi_\varphi, \Phi_\psi \rangle_2 \\ &\quad - \lambda_m^{-1}((\Phi_\varphi, \Phi_\psi)_2 + (\varphi, \psi)_{\Omega_1 \setminus \Omega_2} + (\Psi_\varphi, \psi)_2 + (\Psi_\psi, \varphi)_2). \end{aligned} \quad (53)$$

3 Domains satisfying Assumption I

In Sect. 3.1 and 3.2, we present estimates used for proving the main asymptotic formulae. In Sect. 3.3, we derive the asymptotic formula (4). In Sect. 3.4, we show that if the boundary of Ω_1 is of class C^1 and $\mathcal{A}$ satisfies assumption (8), then δ in (4) is an arbitrary number belonging to the interval $(0, 1)$. In Sect. 3.5, we give sufficient conditions that guarantee that the classical Hadamard formula is valid, and in Sect. 3.6, we consider perturbations of a plane domain with angular points on the boundary.

3.1 Corollaries of Assumption I and an estimate for σ

Let $D_d^+ = \{x : \text{dist}(x, \Omega_1) < d\}$ and $D_d^- = \{x \in \Omega_1 : \text{dist}(x, \Gamma_1) > d\}$, and so $\Omega_2 \subset D_d^+$ and $D_d^- \subset \Omega_1 \cap \Omega_2$ according to our assumptions. Furthermore, we put $r = r(x) = \text{dist}(x, \partial D_{2d}^+)$. If $\beta < 1$, then the following Hardy inequality

$$\int_{D_{2d}^+} r^{\beta-2} |v|^2 dx \leq C_\beta \int_{D_{2d}^+} r^\beta |\nabla v|^2 dx \quad (54)$$

is valid for functions belonging to $C_0^\infty(D_{2d}^+)$. This estimate is a consequence of the one-dimensional Hardy inequality combined with a local change of variables flattening the boundary.

In what follows, we fix a small positive number d_0 so that the domains D_{2d}^+ and D_{2d}^- are Lipschitz provided $0 \leq d \leq d_0$; moreover, they can be defined locally by functions with uniformly bounded Lipschitz constants using the same local coordinates.

Lemma 1 *Let $w \in (\dot{W}^{1,2}(D_d^+))^M$, then*

$$\int_{D_d^+ \setminus D_d^-} |w|^2 dx \leq cd^2 \int_{D_d^+ \setminus D_d^-} |\nabla w|^2 dx \quad (55)$$

and

$$\int_{\partial D_d^-} |w|^2 dx \leq Cd \int_{D_d^+ \setminus D_d^-} |\nabla w|^2 dx. \quad (56)$$

Here C is a positive constant that may depend on Ω_1 , d_0 and A_{ij} .

Proof. First we prove (55). The domain $D_d^+ \setminus D_d^-$ is locally described by $\xi_d^-(y') < y_n < \xi_d^+(y')$, where $y_n = \xi_d^-(y')$ and $y_n = \xi_d^+(y')$ are the local equations of the boundaries of D_d^- and D_d^+ , respectively. Using the Newton–Leibniz formula with respect to y_n , one directly verifies that

$$\int_{|y'| < r} \int_{\xi_d^-(y')}^{\xi_d^+(y')} |w|^2 dy_n dy' \leq Cd^2 \int_{|y'| < r} \int_{\xi_d^-(y')}^{\xi_d^+(y')} |\nabla w|^2 dy_n dy'$$

and

$$\int_{|y'| < r, y_n = \xi_d^-(y')} |w|^2 dy' \leq Cd \int_{|y'| < r} \int_{\xi_d^-(y')}^{\xi_d^+(y')} |\nabla w|^2 dy_n dy'.$$

These inequalities yield (55) and (56), respectively.

Corollary 1 (i) *Let $w \in (\dot{W}^{1,2}(D_d^+))^M$, and let w be orthogonal to $(\dot{W}^{1,2}(D_d^-))^M$ with respect to the inner product $(\cdot, \cdot)_{D_d^+}$. Then*

$$\int_{D_d^+} |w|^2 dx \leq Cd \int_{D_d^+ \setminus D_d^-} |\nabla w|^2 dx. \quad (57)$$

(ii) *The number σ in (33) can be taken as cd .*

Proof. Since the function w satisfies $\mathcal{A}w = 0$ in D_d^- , it follows from [5, Theorem 2.1] that

$$\int_{D_d^-} |w|^2 dx \leq C \int_{\partial D_d^-} |w|^2 d\Gamma.$$

This together with (55) and (56) leads to (57).

Assertion (ii) follows from (i).

Lemma 2 *Let $\varphi \in X_m$ and $\|\varphi\|_2 = 1$, then*

(i)

$$\int_{\Omega_1 \setminus D_{2d}^-} |\varphi|^2 dx \leq cd^3, \quad \int_{\Omega_1 \setminus D_{2d}^-} |\nabla \varphi|^2 dx \leq cd \quad \text{and} \quad \int_{\Gamma_1} |\nabla \varphi|^2 d\Gamma \leq c. \quad (58)$$

(ii) *Let Φ be the same as in (50), then*

$$\int_{\Omega_2} |\Phi|^2 dx \leq cd^2 \quad \text{and} \quad \int_{\Omega_2} |\nabla \Phi|^2 dx \leq cd. \quad (59)$$

Proof. (i) Let us write $\varphi = \Phi_1 + \Phi_2$, where $\Phi_1 \in (W^{3,2}(\mathbb{R}^n))^M$ solves the problem $-\mathcal{A}\Phi_1 = \lambda\varphi$ in $\mathbb{R}^n$, whereas the function Φ_2 solves the boundary value problem $\mathcal{A}\Phi_2 = 0$ in Ω_1 and $\Phi_2 = -\Phi_1$ on Γ_1 .

Let $\gamma_\tau = \{x \in \Omega_1 : \text{dist}(x, \Gamma_1) = \tau\}$, then the trace theorem yields

$$\int_{\gamma_\tau} (|\Phi_1|^2 + |\nabla \Phi_1|^2) d\gamma \leq c.$$

Using this estimate with $\tau = 0$ and [5, Theorem 4.2] for the function Φ_2 , we obtain that

$$\int_{\gamma_\tau} |\nabla \Phi_2|^2 d\gamma \leq c.$$

These inequalities with $\tau = 0$ imply the third estimate (58). Integrating the above inequalities with respect to τ , we arrive at the second inequality (58). The first inequality (58) follows from the second one combined with (55).

(ii) Let τ be the normal variable and let $\zeta = \zeta(\tau)$ be a smooth function vanishing for $\tau \leq 1$ and equal to 1 for $\tau \geq 2$. Putting $\zeta_d(\tau) = \zeta(\tau/d)$, we seek Φ in the form $\zeta_d \varphi + \Psi$. Then Ψ must satisfy the boundary value problem $\mathcal{A}\Psi = -\mathcal{A}(\zeta_d \varphi)$ in Ω_2 and $\Psi = 0$ on Γ_2 . Hence we have

$$(\Psi, \Psi)_2 = \int_{\Omega_2} \left(- (A_{ij} \partial_j \zeta_d \varphi, \partial_i \Psi) + (A_{ij} \partial_j \zeta_d \partial_i \varphi, \Psi) - \lambda_m (\zeta_d \varphi, \Psi) \right) dx,$$

and so

$$\begin{aligned} (\Psi, \Psi)_2 &\leq c(d^{-1} \|\varphi\|_{L^2(\Omega_1 \setminus D_{2d}^-)} \|\nabla \Psi\|_{L^2(\Omega_2 \setminus D_{2d}^-)} \\ &\quad + d^{-1} \|\nabla \varphi\|_{L^2(\Omega_1 \setminus D_{2d}^-)} \|\Psi\|_{L^2(\Omega_2 \setminus D_{2d}^-)} + \|\varphi\|_{L^2(\Omega_1 \setminus D_{2d}^-)} \|\Psi\|_{L^2(\Omega_2 \setminus D_{2d}^-)}). \end{aligned}$$

Since estimate (55) gives

$$(\Psi, \Psi)_2 \leq c \|\nabla \varphi\|_{L^2(\Omega_1 \setminus D_{2d}^-)}^2,$$

the second inequality (59) follows from (58). Furthermore, in view of (55), (56) and the second inequality (59) we obtain

$$\int_{\Omega_2 \setminus D_{2d}^-} |\Psi|^2 dx \leq cd^3 \quad (60)$$

and

$$\int_{\partial D_{2d}^-} |\Psi|^2 dx \leq cd^2. \quad (61)$$

Applying [5, Theorem 2.1] to the function Ψ in D_{2d}^- and using (61), we arrive at

$$\int_{D_{2d}^-} |\Psi|^2 dx \leq cd^2,$$

which together with (60) leads to the first inequality (59).

The first estimates (58) and (59) combined with (53) immediately give

Corollary 2 *The following relation holds:*

$$(B\varphi, S\psi)_2 = \frac{-1}{\lambda_m} \left((\Phi_\varphi, \Phi_\psi)_2 + (\varphi, \psi)_{\Omega_1 \setminus \Omega_2} + (\Psi_\varphi, \psi)_2 + (\Psi_\psi, \varphi)_2 \right) + O(d^2). \quad (62)$$

3.2 Weighted estimates

In a domain Ω subject to $D_d^- \subset \Omega \subset D_d^+$, we consider the problem

$$(u, w)_\Omega = \sum_{i=1}^n \langle f_k, \partial_k w \rangle_\Omega + \langle f_0, w \rangle_\Omega \quad \text{for all } w \in (\mathring{W}^{1,2}(\Omega))^M, \quad (63)$$

where $f_0, f_1, \dots, f_n \in (L^2(\Omega))^M$ and $u \in (\mathring{W}^{1,2}(\Omega))^M$.

Lemma 3 *There exist a constant $\beta_0 \in (0, 1)$ such that if $u \in (\mathring{W}^{1,2}(\Omega))^M$ is a solution of (63), then the following inequality*

$$\int_{\Omega} r^{\beta-2} |u|^2 dx + \int_{\Omega} r^{\beta} |\nabla u|^2 dx \leq C \int_{\Omega} \left(r^{\beta} \sum_{k=1}^n |f_k|^2 + r^{\beta+2} |f_0|^2 \right) dx, \quad (64)$$

holds for $|\beta| \leq \beta_0$; here C is a positive constant depending only on the constant β_0 , the domain Ω and the coefficients A_{ij} .

Proof. Taking $w = r^\beta u$ in (63) and setting $v = r^{\beta/2}u$, we obtain

$$(r^{-\beta/2}v, r^{\beta/2}v)_\Omega = \sum_{k=1}^n \langle f_k, \partial_k r^{\beta/2}v \rangle_\Omega + \langle f_0, r^{\beta/2}v \rangle_\Omega.$$

Therefore, we have

$$\begin{aligned} (v, v)_\Omega - \frac{\beta^2}{4} \int_\Omega r^{-2} \sum_{i,j=1}^n (A_{ij} \partial_i r \partial_j r v, v) dx &= \sum_{i=1}^n \langle r^{\beta/2} f_k, \partial_k v \rangle_\Omega \\ &+ \frac{\beta}{2} \sum_{i=1}^n \langle r^{\beta/2} f_k, v r^{-1} \partial_k r \rangle_\Omega + \langle r^{\beta/2+1} f_0, r^{-1} v \rangle_\Omega. \end{aligned} \quad (65)$$

Using Hölder inequality together with (54), we arrive at

$$\int_\Omega |\nabla v|^2 dx \leq c_1 C \int_\Omega (r^\beta |\nabla F|^2 + r^{\beta+2} |f|^2) dx,$$

provided $|\beta|$ is small. This together with (54) gives (64).

Applying Lemma 3 to the operators S , S^* and to the projection operator onto $H_1 \cap H_2$, one obtains the following

Corollary 3 *There exists a constant $\beta_0 \in (0, 1)$ such that the following estimates are valid for $|\beta| \leq \beta_0$.*

(i) *Let $v \in H_2$, then*

$$\|S^* v\|_{\Omega_1, \beta} \leq C \|v\|_{\Omega_2, \beta}.$$

(ii) *Let $v \in H_1$, then*

$$\|S v\|_{\Omega_2, \beta} \leq C \|v\|_{\Omega_1, \beta}.$$

(iii) *Let P be the orthogonal projector (with respect to the inner product $(\cdot, \cdot)_{\Omega_1 \cup \Omega_2}$) from $\dot{W}^{1,2}(\Omega_1 \cup \Omega_2)$ onto $\dot{W}^{1,2}(\Omega_1 \cap \Omega_2)$. If $v \in W^{1,2}(\Omega_1 \cup \Omega_2)$, then*

$$\|P v\|_{\Omega_1 \cap \Omega_2, \beta} \leq C \|v\|_{\Omega_1 \cup \Omega_2, \beta}.$$

Proof. The operators appearing in (i)–(iii) can be written in the form (63) with $f = 0$, see (31) and (32) for the operators S and S^* , respectively, and the operator $u = P v$ is defined by

$$(u, w)_{\Omega_1 \cap \Omega_2} = (v, w)_{\Omega_1 \cap \Omega_2}.$$

Therefore, assertions (i)–(iii) follow from Lemma 3.

3.3 Proof of Theorem 1

Let us consider the term $(\mathcal{R}(\varphi), B\psi)_2$ in (27). Let us write $\mathcal{R} = \lambda_m Q_m B\varphi + \mathcal{R}_1$, where $\mathcal{R}_1 = \mathcal{R}_1(\varphi)$ satisfies

$$\lambda_m^{-1} \mathcal{R}_1 - Q_m K_2 \mathcal{R}_1 = g, \quad g = \lambda_m Q_m K_2 Q_m B\varphi. \quad (66)$$

According to Remark 1, we have

$$\|\mathcal{R}_1\|_2 \leq C\|g\|_2. \quad (67)$$

Since the operators Q_m and K_2 are bounded in H_2 , we combine this fact and (45), thus obtaining

$$\|\mathcal{R}_1\|_2 \leq C d^{1/2} \|\varphi\|_1. \quad (68)$$

Next we rewrite equation (66) in the form

$$(\mathcal{R}_1, w)_2 = \lambda_m (K_2 \mathcal{R}_1, Q_m w)_2 + \lambda_m (g, w)_2 \quad \text{for all } w \in H_2. \quad (69)$$

It occurs that the right-hand side of this equation is as follows:

$$\lambda_m \langle \mathcal{R}_1, Q_m w \rangle_2 + \lambda_m^2 \langle Q_m B\varphi, Q_m w \rangle_2 - \lambda_m \langle (\mathcal{R}_1 + \lambda_m Q_m B\varphi), P_m w \rangle_2. \quad (70)$$

In order to represent it in the form $\langle f, w \rangle_2$ with some f , we fix an orthonormal basis $\{\Phi_k\}_{k=1}^{J_m}$ in SX_m and write

$$P_m \psi = \sum_{k=1}^{J_m} (\psi, \Phi_k)_2 \Phi_k. \quad (71)$$

Then we have

$$(P_m v, w)_2 = \sum_{k=1}^{J_m} (v, \Phi_k)_2 (\Phi_k, w)_2 = \lambda_m \sum_{k=1}^{J_m} (v, \Phi_k)_2 \langle \Phi_k, w \rangle_2,$$

and so (70) is equal to $\langle f, w \rangle_2$, where

$$f = \lambda_m \mathcal{R}_1 + \lambda_m^2 Q_m B\varphi - \lambda_m^2 \sum_{k=1}^{J_m} (\mathcal{R}_1 + \lambda_m B\varphi, \Phi_k)_2 \Phi_k. \quad (72)$$

It is clear that

$$\|f\|_{L^2(\Omega_2)} \leq c(\|\mathcal{R}_1\|_2 + \|B\varphi\|_2) \leq c d^{1/2} \|\varphi\|_1,$$

where estimates (45) and (70) are used. Now, applying Lemma 3 to equation (69), we get the inequality

$$\|\mathcal{R}_1\|_{W_\beta^{1,2}(\Omega_2)} \leq c d^{1/2} \|\varphi\|_1 \quad (73)$$

for $|\beta| \leq \beta_0$.

Lemma 1 yields the following estimate of expression (47) with φ replaced by ψ :

$$\begin{aligned} |(B\psi, w)_2| &\leq C \left(\|\psi_0\|_{L^2(\Omega_1)} \|w_1\|_{L^2(\Omega_2)} + \|S\psi_1\|_{L^2(\Omega_1)} \|w\|_{L^2(\Omega_2)} \right. \\ &\quad \left. + \|\psi_1\|_{L^2(\Omega_1)} \|w_0\|_{L^2(\Omega_2)} + \|\psi\|_{L^2(\Omega_1)} \|S^*w_1\|_{L^2(\Omega_2)} \right). \end{aligned}$$

Applying Corollary 1 (i) to the terms containing w_1 , ψ_1 , $S\psi_1$ and S^*w_1 , we obtain

$$\begin{aligned} |(B\psi, w)_2| &\leq Cd^{(1+\beta)/2} \left(\|\psi_0\|_{L^2(\Omega_1)} \|w_1\|_{W_{-\beta}^{1,2}(\Omega_2)} + \|S\psi_1\|_{W_{-\beta}^{1,2}(\Omega_1)} \|w\|_{L^2(\Omega_2)} \right. \\ &\quad \left. + \|\psi_1\|_{W_{-\beta}^{1,2}(\Omega_1)} \|w_0\|_{L^2(\Omega_2)} + \|\psi\|_{L^2(\Omega_1)} \|S^*w_1\|_{W_{-\beta}^{1,2}(\Omega_2)} \right) \end{aligned}$$

for $|\beta| \leq \beta_0$. Using Corollary 3 and noting that $w_0 = Pw$ and $\psi_0 = P\psi$, we arrive at

$$|(B\psi, w)_2| \leq Cd^{(1+\beta)/2} (\|\psi\|_1 \|w\|_{W_{-\beta}^{1,2}(\Omega_2)} + \|\psi\|_{W_{-\beta}^{1,2}(\Omega_1)} \|w\|_2). \quad (74)$$

Since Ω_1 is Lipschitz and $\psi \in X_m$, we have that $\psi \in W^{1,p}(\Omega_1)$ for some $p > 2$ (see, for example, [1] or [5]), and so

$$\int_{\Omega_1} r^{-\beta} |\nabla \psi|^2 dx \leq C \|\psi\|_1^2$$

with some positive β , which together with (73) leads to the inequalities

$$|(\mathcal{R}_1(\varphi), B\psi)_2| \leq Cd^{1+\beta/2} \|\varphi\|_1 \|\psi\|_1 \quad (75)$$

and

$$|(B\varphi, S\psi)_2| \leq cd^{(1+\beta)/2} \|\varphi\|_1 \|\psi\|_1 \quad (76)$$

valid for $\varphi, \psi \in X_m$. This implies, in particular, that ρ_1 can be taken in the form $cd^{(1+\beta)/2}$ and that the asymptotic formula (35) can be written as follows:

$$\frac{1}{\mu_j} = \frac{1}{\lambda_m} + q_j + O(d^{1+\beta/2}), \quad (77)$$

where $\{q_j\}_{j=1}^{J_m}$ are eigenvalues of the problem

$$(B\varphi, S\psi)_2 + \lambda_m(Q_m B\varphi, B\psi)_2 = q(S\varphi, S\psi)_2 \quad \text{for all } \psi \in X_m. \quad (78)$$

Furthermore, using representation (50) for $\phi \in X_m$ and Lemma 2(ii), we have

$$\|S\varphi - \varphi\|_2^2 = \|\Phi\|_2^2 \leq cd\|\varphi\|_1^2.$$

This and the second inequality in (58) show that

$$|(S\varphi, S\psi)_2 - (\varphi, \psi)_1| \leq cd\|\varphi\|_1 \|\psi\|_1 \quad \text{for } \varphi, \psi \in X_m.$$

Moreover, it follows from (76) and (45) that $q_j = O(d^{(1+\beta)/2})$. Now using (71), we obtain

$$(P_m B\varphi, B\psi)_2 = \sum_{j=1}^{J_m} (B\varphi, S\varphi_j)_2 (S\varphi_j, B\psi)_2,$$

where $S\varphi_j = \Phi_j$, $\varphi_j \in X_m$. Furthermore, applying (76), we get

$$(P_m B\varphi, B\psi)_2 \leq cd^{1+\beta} \|\varphi\|_1 \|\psi\|_1.$$

These observations lead to the following simplification of formula (77)

$$\frac{1}{\mu_j} = \frac{1}{\lambda_m} + \tau_j + O(d^{1+\beta/2}), \quad (79)$$

where $\{\tau_j\}_{j=1}^{J_m}$ are eigenvalues of the problem

$$(B\varphi, S\psi)_2 + \lambda_m (B\varphi, B\psi)_2 = \tau(\varphi, \psi)_1 \quad \text{for all } \psi \in X_m. \quad (80)$$

In order to simplify the expression $(B\varphi, B\psi)_2$, we write

$$K_2\varphi = \frac{1}{\lambda_m} S\varphi - \frac{1}{\lambda_m} \Psi, \quad (81)$$

where $\Psi = \Psi_\varphi \in H_2$. Using the definitions of the operators K_2 and S , one can verify that Ψ solves the problem (6). Clearly, Ψ is orthogonal to $H_1 \cap H_2$.

According to (50) and (81), we have $B\varphi = K_2\Phi_\varphi - \lambda_m^{-1}\Psi_\varphi$, and so

$$(B\varphi, B\psi)_2 = \frac{1}{\lambda_m^2} (\Psi_\varphi, \Psi_\psi)_2 + (K_2\Phi_\varphi, K_2\Phi_\psi)_2 - \frac{1}{\lambda_m} \left((K_2\Phi_\varphi, \Psi_\psi)_2 + (\Psi_\varphi, K_2\Phi_\psi)_2 \right). \quad (82)$$

Using the third inequality (58) combined with the definition of Ψ , we get

$$(\Psi, \Psi)_2 \leq c \left(\int_{\Gamma_1} |\nabla\varphi|^2 d\Gamma \right)^{1/2} \left(\int_{\Gamma_1} |\Psi|^2 dx \right)^{1/2} \leq c \left(\int_{\Gamma_1} |\Psi|^2 dx \right)^{1/2}. \quad (83)$$

Reasoning as in the proof of Lemma 1, we derive the estimate

$$\int_{\Gamma_1} |\Psi|^2 dx \leq cd \int_{D_d^+ \setminus D_d^-} |\nabla\Psi|^2,$$

cf. (56). Estimating the right-hand side in (83) by the right-hand side in the last inequality, we obtain

$$(\Psi, \Psi)_2 \leq cd,$$

which together with (55) and (56) yields

$$\int_{D_d^+ \setminus D_d^-} |\Psi|^2 dx \leq cd^2 \int_{D_d^+ \setminus D_d^-} |\nabla\Psi|^2 dx \quad (84)$$

and

$$\int_{\partial D_d^-} |\Psi|^2 d\Gamma \leq cd \int_{D_d^+ \setminus D_d^-} |\nabla \Psi|^2 dx.$$

Applying again [5, Theorem 2.1] to the function Ψ in D_d^- , we obtain

$$\int_{D_d^-} |\Psi|^2 d\Gamma \leq cd \int_{D_d^+ \setminus D_d^-} |\nabla \Psi|^2 dx,$$

which implies that

$$\int_{D_d^-} |\Psi|^2 dx = O(d^2),$$

and so (84) yields

$$\int_{\Omega_2} |\Psi|^2 dx \leq Cd^2. \quad (85)$$

The last estimate together with (50) and (82) leads to

$$(B\varphi, B\psi)_2 = \frac{1}{\lambda_m^2} (\Psi_\varphi, \Psi_\psi)_2 + O(d^2),$$

which along with (80) and (62) gives (5).

Thus we the proof of Theorem 1 is complete.

We conclude this section by the following representation:

$$\Psi_\varphi = \varphi - T_\varphi, \quad (86)$$

where $T = T_\varphi \in H_2$ solves the problem $(T, w)_2 = \lambda_m \langle \varphi, w \rangle_2$ for all $w \in H_2$.

3.4 The case when Γ_1 is C^1 -smooth. Proof of Theorem 2

In this section, we suppose that Γ_1 belongs to the class C^1 , and the operator $\mathcal{A}$ satisfies (8).

Here we use a regularized distance $\rho = \rho(x)$ to the boundary ∂D_{2d}^+ . One can define ρ so that it is smooth in D_{2d}^+ , $c_1 r \leq \rho \leq c_2 r$ and $|\partial^\alpha \rho| \leq c_\alpha r^{1-|\alpha|}$. Since Γ_1 is C^1 -smooth, ρ can be chosen that it also satisfies the estimate

$$|\partial_{x_j} \partial_{x_i} \rho| \leq \varepsilon \rho^{-1} + C_\varepsilon \rho, \quad (87)$$

where ε is an arbitrary positive number. Let us explain how such a function can constructed locally. Consider the Lipschitz graph domain

$$G = \{x = (x', x_n) \in \mathbb{R}^n : x_n > h(x')\},$$

where h is a Lipschitz function. Let us extend h onto $\mathbb{R}_+^n$ as follows:

$$\Phi(\xi) = \int_{\mathbb{R}^{n-1}} \omega_\mu(\tau) h(\tau \xi_n + \xi') d\tau, \quad (88)$$

where $\omega_\mu(\tau) = \mu^{1-n}\omega(\tau/\mu)$ and $\omega(\tau) = c \exp((|\tau|^2 - 1)^{-1})$ for $|\tau| \leq 1$, and $\omega(\tau) = 0$ for $|\tau| > 1$. The constant c is chosen so that

$$\int_{\mathbb{R}^{n-1}} \omega(\tau) d\tau = 1.$$

It is clear that Φ is continuous on $\overline{\mathbb{R}_+^n}$ and $\Phi(\xi', 0) = h(\xi')$. It was shown in [9] that the mapping $x' = \xi'$, $x_n = \xi_n + \Phi(\xi)$ is a bi-Lipschitz isomorphism of $\mathbb{R}_+^n$ onto G , provided μ in (88) is sufficiently small. Taking, for example, $\rho(x) = \xi_n(x)$, one verifies directly that all required properties are fulfilled.

Lemma 4 *Let $v \in \dot{W}^{1,2}(D_d^+)$ and let the operator $\mathcal{A}$ satisfy (8). Then for every $\varepsilon > 0$ there exists a constant C_ε such that*

$$J \leq \frac{4}{(1-\varepsilon)^2} (v, v)_{D_d^+} + C_\varepsilon \int_{D_d^+} |v|^2 dx, \quad (89)$$

where

$$J = \int_{\Omega} \rho^{-2} \sum_{i,j=1}^n (A_{ij} \partial_i \rho \partial_j \rho v, v) dx. \quad (90)$$

Proof. We have that

$$\begin{aligned} J &= - \int_{\Omega} \sum_{i,j=1}^n (A_{ij} \partial_i (\rho^{-1}) \partial_j \rho v, v) dx \\ &= \int_{\Omega} \rho^{-1} \sum_{i,j=1}^n \left((A_{ij} \partial_i v, \partial_j \rho v) + (A_{ij} \partial_j \rho v, \partial_i v) + (A_{ij} (\partial_i \partial_j \rho) v, v) \right) dx, \end{aligned} \quad (91)$$

which together with (8) and (87) implies

$$J \leq 2(v, v)_{D_d^+}^{1/2} J^{1/2} + \varepsilon J + C_\varepsilon \int_{D_d^+} \rho^{-1} |v|^2 dx.$$

The last inequality yields (89).

Lemma 5 *Assume that the boundary Γ_1 is of class C^1 and that the operator $\mathcal{A}$ satisfies (8). Then for $\beta \in (-1, 1)$ the following assertions are valid:*

- (i) *estimate (64) holds with a positive constant C which may depend on β ;*
- (ii) *all assertions of Corollary 3 hold.*

Proof. Putting $v = \rho^{\beta/2} w$, we have that

$$(w, \rho^\beta w)_\Omega = (v, v)_\Omega - \frac{\beta^2}{4} J,$$

where J is given by (90). Then (89) yields

$$(w, \rho^\beta w)_\Omega \geq \left(1 - \frac{\beta^2}{(1-\varepsilon)^2}\right) (v, v)_\Omega - C_\varepsilon \int_{\Omega} |v|^2 dx. \quad (92)$$

Using this inequality with $|\beta| \leq 1 - \varepsilon$ for estimating the second term in (65), we obtain

$$(v, v)_\Omega \leq C_\varepsilon \left(\sum_{i=1}^n \langle r^{\beta/2} f_k, \partial_k v \rangle_\Omega + \frac{\beta}{2} \sum_{i=1}^n \langle r^{\beta/2} f_k, v r^{-1} \partial_k r \rangle_\Omega \right. \\ \left. + \langle r^{\beta/2+1} f_0, r^{-1} v \rangle_\Omega + \int_\Omega |v|^2 dx \right),$$

which together with the Hardy inequality (54) gives

$$(v, v)_\Omega \leq C_\varepsilon \int_\Omega (r^\beta \sum_{k=1}^n |f_k|^2 + r^{\beta+2} |f_0|^2) dx + C_\varepsilon (u, u)_\Omega. \quad (93)$$

Estimating $(u, u)_\Omega$ with the help of (64), where $\beta = 0$, we get (64) with $\beta \leq 0$.

Now, let $\beta \in (0, 1)$. In (63), we choose a solution of the problem

$$(w, V)_2 = \langle u, V \rangle_2 \quad \text{for all } V \in H$$

as w . Then (63) takes the form

$$\langle u, u \rangle_2 = \sum_{k=1}^n \langle f_k, \partial_k w \rangle_2 + \langle f_0, w \rangle_2.$$

Furthermore, applying (64) with negative β to w , we obtain

$$\int_\Omega (r^{-\beta} |\nabla w|^2 + r^{-\beta-2} |\nabla w|^2) \leq C_\beta \int_\Omega r^{-\beta+2} |u|^2 dx.$$

Therefore, we have

$$\langle u, u \rangle_2 \leq \sum_{k=1}^n \left(\int_\Omega r^\beta |f_k|^2 dx \right)^{1/2} \left(\int_\Omega r^{-\beta} |\nabla w|^2 dx \right)^{1/2} \\ + \left(\int_\Omega r^{2+\beta} |f_0|^2 dx \right)^{1/2} \left(\int_\Omega r^{-2-\beta} |w|^2 dx \right)^{1/2},$$

which implies

$$\langle u, u \rangle_2 \leq C \left(\sum_{k=1}^n \int_\Omega r^\beta |f_k|^2 dx + \int_\Omega r^{2+\beta} |f_0|^2 dx \right)^{1/2} \left(\int_\Omega r^{-\beta+2} |u|^2 dx \right)^{1/2}.$$

Since $2 - \beta > 0$, the last inequality yields that

$$\langle u, u \rangle_2 \leq C \sum_{k=1}^n \int_\Omega r^\beta |f_k|^2 dx + \int_\Omega r^{2+\beta} |f_0|^2 dx.$$

Finally, using this inequality for estimating of the last term in (93) we get (64) for $\beta \in (0, 1)$. This completes the proof of assertion (i).

Assertion (ii) follows from (i) in the same way as Corollary 3 follows from Lemma 3.

In the next lemma weighted estimates for Bw are obtained.

Lemma 6 Let $\beta \in (0, 1)$, then we have

$$\int_{\Omega_2} r^\beta |\nabla B\varphi|^2 dx \leq C_\beta d^{2\beta} \|\varphi\|_1^2. \quad (94)$$

for $\varphi \in X_m$.

Proof. We apply the representation $S^*w = \zeta_d(x)w + W$, where $\zeta_d = 1$ in D_{2d}^- and $\zeta_d = 0$ outside D_d^- . The function ζ_d can be chosen so that $|\partial_x^\alpha \zeta_d| \leq c_\alpha r^{-|\alpha|}$. Then the function W satisfies the equation

$$(W, v)_1 = ((1 - \zeta_d)w, v)_1 = \sum_{i,j=1}^n \int_{\Omega_1} (A_{ij}((1 - \zeta_d)\partial_i w - \partial_i \zeta_d w, \partial_j v)) dx.$$

In view of Lemma 5(i), we get

$$\int_{\Omega_1} r^\beta |\nabla W|^2 dx \leq C \int_{\Omega_1} r^\beta (|1 - \zeta_d|^2 |\nabla w|^2 + |\nabla \zeta_d|^2 |w|^2) dx.$$

Using (55), we obtain

$$\int_{\Omega_1} r^\beta |\nabla W|^2 dx \leq C \int_{\Omega_1 \setminus D_{2d}^-} r^\beta |\nabla w|^2 dx.$$

It follows from (46) that

$$(B\varphi, w)_2 = \langle \Phi_\varphi, w \rangle_2 - \langle \varphi, (\zeta_d - 1)w + W \rangle_1.$$

Since $\varphi \in X_m$, we have $(\varphi, W)_1 = \lambda_m \langle \varphi, W \rangle_1$, and so

$$(B\varphi, w)_2 = \langle \Phi_\varphi, w \rangle_2 - \langle \varphi, (\zeta_d - 1)w \rangle_1 - \lambda_m^{-1} \langle \varphi, (1 - \zeta_d)w \rangle_1.$$

Using again Lemma 5(i) together with (55), we get

$$\int_{\Omega_2} r^\beta |\nabla(B\varphi)|^2 dx \leq C_\beta \left(\int_{\Omega_2 \setminus D_{2d}^-} r^\beta |\nabla \varphi|^2 dx + \int_{\Omega_2} r^{2+\beta} |\Phi_\varphi|^2 dx \right).$$

This inequality and Lemma 2(ii) give

$$\int_{\Omega_2} r^\beta |\nabla B\varphi|^2 dx \leq C_\beta \left(\int_{\Omega_2 \setminus D_{2d}^-} r^\beta |\nabla \varphi|^2 dx + d^2 \|\varphi\|_1^2 \right). \quad (95)$$

It follows from (95) that

$$\int_{\Omega_2} r^\beta |\nabla B\varphi|^2 dx \leq C_\beta d^{2\beta} \left(\int_{\Omega_2 \setminus D_{2d}^-} r^{-\beta} |\nabla \varphi|^2 dx + \|\varphi\|_1^2 \right) \quad (96)$$

for $\beta \in (0, 1)$. Since the function φ belongs to $W^{1,p}(\Omega)$ for all $p > 1$, the integral in the left-hand side is bounded by $\|\varphi\|_1^2$ and we obtain (94). The proof is complete.

Let us consider the right-hand side g in (66). We have

$$(g, w)_2 = \lambda_m \langle Q_m B\varphi, Q_m w \rangle_2 = \lambda_m \langle B\varphi, Q_m w \rangle_2 - \lambda_m \langle P_m B\varphi, Q_m w \rangle_2. \quad (97)$$

Using the Hardy inequality (54), we estimate the first term in the right-hand side as follows:

$$\begin{aligned} |\lambda_m \langle B\varphi, Q_m w \rangle_2| &\leq \lambda_m \|B\varphi\|_{L^2(\Omega_2)} \|Q_m w\|_{L^2(\Omega_2)} \\ &\leq C_\beta \left(\int_{\Omega_2} r^\beta |\nabla B\varphi|^2 dx \right)^{1/2} \|Q_m w\|_2, \end{aligned}$$

where $\beta \in (0, 1)$. Applying (94), we get

$$|\lambda_m \langle B\varphi, Q_m w \rangle_2| \leq C_\beta d^\beta \|\varphi\|_1 \|w\|_2. \quad (98)$$

For estimating the second term in the right-hand side of (97), we note that according to (71) with $\Phi_k = S\varphi_k$, $\varphi_k \in X_m$, we have

$$\langle P_m B\varphi, Q_m w \rangle_2 = \sum_{k=1}^{J_m} \langle B\varphi, S\varphi_k \rangle_2 \langle \Phi_k, Q_m w \rangle_2.$$

Using estimate (76), in which the exponent β can be taken arbitrary close to 1 because we deal with the C^1 -case, we obtain

$$|\langle P_m B\varphi, Q_m w \rangle_2| \leq C_\beta d^{(1+\beta)/2} \|\varphi\|_1 \|w\|_2. \quad (99)$$

Applying inequalities (98) and (99) for estimating the right-hand side in (97), we arrive at

$$|(g, w)_2| \leq C_\beta d^\beta \|\varphi\|_1 \|w\|_2.$$

This leads to the estimate

$$\|g\|_2 \leq C_\beta d^\beta \|\varphi\|_1,$$

which combined with (67) gives

$$\|\mathcal{R}_1\|_2 \leq C_\beta d^\beta \|\varphi\|_1. \quad (100)$$

Now let us turn again to equation (69), in which the right-hand side can be written as $\langle f, w \rangle_2$ with f given by (72). Using Lemma 5, we obtain

$$\int_{\Omega_2} r^{-\beta} |\nabla \mathcal{R}_1|^2 dx \leq C \int_{\Omega_2} r^{2-\beta} |f|^2 dx,$$

where $\beta \in (0, 1)$. One can check that

$$\int_{\Omega_2} r^{2-\beta} |f|^2 dx \leq C \int_{\Omega_2} r^\beta (|\nabla \mathcal{R}_1|^2 + |\nabla B\varphi|^2) dx.$$

By (100) and (94), the right-hand side does not exceed $cd^{2\beta}||\varphi||_1$. Therefore,

$$\int_{\Omega_2} r^{-\beta} |\nabla \mathcal{R}_1|^2 dx \leq Cd^\beta ||\varphi||_1.$$

Applying this for estimating the right-hand side in (74), we get

$$|(\mathcal{R}_1(\varphi), B\psi)_2| \leq Cd^{(1+3\beta)/2} ||\varphi||_1 ||\psi||_1.$$

This leads to

$$\frac{1}{\mu_j} = \frac{1}{\lambda_m} + q_j + O(d^{2\beta}),$$

where $\{q_j\}_{j=1}^{J_m}$ are eigenvalues of problem (78). Repeating the reasoning in Sect. 3.3 that goes after formula (78), we obtain

$$\frac{1}{\mu_j} = \frac{1}{\lambda_m} + \tau_j + O(d^{2\beta}),$$

where $\{\tau_j\}_{j=1}^{J_m}$ are eigenvalues of problem (80).

Thus, Theorem 2 is proved.

3.5 On the validity of the Hadamard formula.

Proof of Theorem 3.

We assume that Ω_1 is of class $C^{1,\alpha}$, $\alpha \in (0, 1)$. This implies, in particular, that the function φ belongs to $C^{1,\alpha}(\Omega_1)^M$. We will use the following coordinates in a neighborhood of Γ_1 : if x belongs to this neighborhood, then $z \in \Gamma_1$ is the nearest to x point on Γ_1 and τ is such that $x = z + \tau\nu(z)$, where $\nu(z)$ is the unit outward normal to Γ_1 at z . Then Γ_2 is given by $\tau = h(z)$. We suppose that $|\nabla_z h(z)|$ is bounded by a constant independent of d , and h satisfies (9). Our immediate aim is to prove formula (10) with θ_j given by (11).

Let Ω_d be the domain whose boundary Γ_d is given by $\tau = d$. We split T_φ into the sum $T_\varphi^{(1)} + T_\varphi^{(2)}$, where $T_\varphi^{(1)}$ solves the problem $-\mathcal{A}T_\varphi^{(1)} = \lambda_m \varphi$ in Ω_d and $T_\varphi^{(1)} = 0$ on Γ_d . The function $T_\varphi^{(2)}$ is a solution of the following problem: $\mathcal{A}T_\varphi^{(2)} = 0$ in Ω_2 and $T_\varphi^{(2)} = -T_\varphi^{(1)}$ on Γ_2 .

Since $T_\varphi^{(1)} \in C^{1,\alpha}(\Omega_d)^M$ and $T_\varphi^{(1)} = 0$ on Γ_d , we obtain that

$$|T_\varphi^{(1)}(x)| \leq c|h(z) - d| \quad \text{and} \quad |\nabla_z T_\varphi^{(1)}(x)| \leq c|h(z) - d|^\alpha \quad \text{on } \Gamma_2. \quad (101)$$

This together with (9) implies

$$\int_{\Gamma_2} |T_\varphi^{(1)}|^2 dz = O(d^2) \quad \text{and} \quad \int_{\Gamma_2} |\nabla_z T_\varphi^{(1)}(z, h(z))|^2 dz = o(1).$$

Applying Theorems 2.1 and 4.2 from [5] to the solution $T_\varphi^{(2)}$ in Ω_2 , we obtain

$$\int_{\Gamma_2} |T_\varphi^{(2)}|^2 d\Gamma \leq c \int_{\Gamma_2} |T_\varphi^{(1)}(z)|^2 d\Gamma = O(d^2)$$

and

$$\int_{\Gamma_2} |\nabla T_\varphi^{(2)}|^2 d\Gamma \leq c \int_{\Gamma_2} |\nabla_z T_\varphi^{(1)}(z, h(z))|^2 d\Gamma = o(1).$$

Hence we have that

$$(T_\varphi^{(2)}, T_\psi^{(2)})_{\Omega_2} = \int_{\Gamma_2} (T_\varphi^{(2)}, NT_\psi^{(2)}) dx = o(d),$$

and so

$$(\Psi_\varphi, \Psi_\psi)_2 = (\varphi - T_\varphi^{(1)}, \psi - T_\psi^{(1)})_2 + o(d).$$

Let us consider the function $V = \varphi - T_\varphi^{(1)}$ in $\Omega_1 \cap \Omega_2$. This function satisfies $\mathcal{A}V = 0$ in $\Omega_1 \cap \Omega_2$ and $V = \varphi - T_\varphi^{(1)}$ on the boundary of $\Omega_1 \cap \Omega_2$, say, Γ_{12} . Since Γ_{12} is given by $\tau = h_1(z)$, $h_1(z) = \min(h(z), 0)$, we obtain

$$\begin{aligned} & \int_{\Gamma_{12}} |\nabla_z T_\varphi^{(1)}(z, h_1(z))|^2 d\Gamma \\ & \leq \int_{\Gamma_{12}} (|T_{\varphi,z}^{(1)}(z, h_1(z))| + |T_{\varphi,\tau}^{(1)}(z, h_1(z))| |\nabla_z h_1(z)|)^2 d\Gamma = o(1), \end{aligned} \quad (102)$$

where relations (9) and (101) are taken into account, and $T_{\varphi,z}^{(1)}$ and $T_{\varphi,\tau}^{(1)}$ stand for the derivatives with respect to z and τ , respectively. Furthermore, we have

$$|\varphi(x)| \leq c|\tau|, \quad |\nabla_z \varphi(x)| \leq c|\tau|^\alpha \quad \text{and} \quad |\partial_\tau \varphi(x)| \leq c \quad (103)$$

because $\varphi \in C^{1,\alpha}(\Omega_1)^M$. This leads to the following relations:

$$\int_{\Gamma_{12}} |\varphi|^2 dz = O(d^2) \quad \text{and} \quad \int_{\Gamma_{12}} |\nabla_z \varphi(z, h(z))|^2 dz = o(1). \quad (104)$$

Using the first inequalities (101) and (103) combined with [5, Theorem 2.1], we obtain

$$\int_{\Gamma_{12}} |V|^2 d\Gamma \leq c \int_{\Gamma_{12}} |\varphi - T_\varphi^{(1)}|^2 d\Gamma = O(d^2).$$

In view of (102), (104) and (9), we apply [5, Theorem 4.2], thus getting

$$\int_{\Gamma_{12}} |\nabla V|^2 d\Gamma = o(1), \quad (105)$$

and so

$$(V, V)_{\Omega_1 \cap \Omega_2} = \int_{\Gamma_{12}} (V, NV) dx = o(d).$$

This and (86) lead to

$$(\Psi_\varphi, \Psi_\varphi)_2 = (T_\varphi^{(1)}(\varphi), T_\psi^{(1)}(\psi))_{\Omega_2 \setminus \Omega_1} + o(d). \quad (106)$$

Since $T_\varphi^{(1)} \in C^{1,\alpha}$, we obtain

$$|\nabla T_\varphi^{(1)}(z, \tau) - \nabla T_\varphi^{(1)}(z, 0)| \leq c|\tau|^\alpha.$$

This and (105) imply

$$(T_\varphi^{(1)}, T_\psi^{(1)})_{\Omega_2 \setminus \Omega_1} = \int_{\Gamma_0} h(z)(A_{ij}\nu_j\nu_i\partial_\nu\varphi, \partial_\nu\psi)dz + O(d^{1+\alpha}), \quad (107)$$

where $\Gamma_0 = \Gamma_1 \cap \Omega_2$.

Using (103) and (9), we obtain

$$\int_{\Gamma_2} |\varphi(z + h(z)\nu)|^2 dz = O(d^2) \quad \text{and} \quad \int_{\Gamma_2} |\nabla_z \varphi(z + h(z)\nu)|^2 dz = o(1).$$

Applying Theorems 4.2 and Theorem 2.1 [5] to the problem for Φ_φ , we obtain

$$\int_{\Gamma_2} |\nabla \Phi|^2 d\Gamma = o(1) \quad \text{and} \quad \int_{\Gamma_2} |\Phi|^2 d\Gamma = O(d^2),$$

which leads to

$$(\Phi_\varphi, \Phi_\psi)_2 = \int_{\Gamma_2} (\Phi_\varphi, N\Phi_\psi) dx = o(d). \quad (108)$$

Reasoning in the same way as in the case of the function $T_\varphi^{(1)}$, we get

$$(\varphi, \psi)_{\Omega_1 \setminus \Omega_2} = \int_{\Gamma_1 \setminus \Gamma_0} |h(z)|(A_{ij}\nu_j\nu_i\partial_\nu\varphi, \partial_\nu\psi)dz + o(d). \quad (109)$$

Finally, let us consider the terms $(\Psi_\varphi, \psi)_2$ and $(\varphi, \Psi_\psi)_2$. Denote by P_{12} the orthogonal projector onto $H_1 \cap H_2$ and represent ψ as $P_{12}\psi - \Phi_\psi^*$. Then Φ_ψ^* is harmonic in $\Omega_1 \cap \Omega_2$ and $\Phi_\psi^* + \psi \in H_1 \cap H_2$. Similar to (108) one derives the estimate $(\Phi_\psi^*, \Phi_\psi^*)_2 = o(d)$. This relation together with (106) implies

$$(\Psi_\varphi, \psi)_2 = o(d) \quad (110)$$

and similarly $(\varphi, \Psi_\psi)_2 = o(d)$

Now these formulae together with (108), (109), (106), (107) along with (4), (5) give (10) and (11). Thus, the following theorem is proved.

Theorem 6 *Let Ω_1 be a domain with $C^{1,\alpha}$ boundary with some $\alpha \in (0, 1)$. Let the boundary of Ω_2 be given by $\tau = h(z)$, where h is a continuous function such that $h < d$ and condition (9) holds for it. Then the asymptotic formula (10) is valid, in which θ_j are eigenvalues of the spectral problem (11).*

3.6 Plane domains with corner points; proof of (15) and (14)

In this section, we consider the 2D case. We assume that the boundary of Ω_1 contains corner points $p_1, \dots, p_l$, outside of which the boundary belongs to the class $C^{1,\delta}$, $\delta \in (0, 1)$. We denote by α_s the interior angle at p_s . It is clear that $\alpha_s < 2\pi$. In view of known results about the behavior of solutions of the

Dirichlet problem for second order systems near corner points (see [8, Sect. 8.6.3 and 8.7.1]), there exist numbers $\lambda_s > 1/2$ such that

$$|\varphi(x)| + r|\nabla\varphi(x)| + \frac{|r(x)^{1+\delta}\nabla\varphi(x) - r^{1+\delta}(y)\nabla\varphi(y)|}{|x-y|^\delta} \leq CR, \quad (111)$$

where $r = r_1 r_2 \cdots r_l$, $R = \prod_{s=1}^l r_s^{\lambda_s}$ and $r_s(x) = |x - p_s|$. Here and in what follows, by $c, C, c_k \dots$ we denote various constants independent of d .

We assume that $\Omega_2 \subset \Omega_1$, and so the asymptotic formula (4) is valid, but the functions Ψ_φ and Ψ_ψ in (5) are equal to zero in the present case. Therefore, the numbers $\{\tau_j\}_{j=1}^{J_m}$ in (4) are eigenvalues of the problem

$$\frac{1}{\lambda_m} \left(-(\Phi_\varphi, \Phi_\psi)_2 - (\varphi, \psi)_{\Omega_1 \setminus \Omega_2} \right) = \tau(\varphi, \psi)_2 \quad \text{for all } \psi \in X_m. \quad (112)$$

Let $\Gamma' = \{z \in \Gamma_1 : |z - p_s| > cd\}$. In a neighborhood of Γ' , we use the coordinates z and τ , the first of which is measured along Γ' , whereas τ is the normal coordinate. The part of Γ_2 lying outside Ω_1 is given by $\tau = h(z, d)$. We denote this part by Γ'_2 , and assume that $h \leq d$, $|\partial_z h(z, d)| \leq C$ and

$$\int_{\Gamma'} |h'(z, d)|^2 R^2 r^{-2} dz = o(1). \quad (113)$$

We also suppose that the remaining part of Γ_2 is located in $B(p_s, c_2 d) \setminus B(p_s, c_1 d)$, where $B(p, r)$ is the disc having its center at p and radius equal to r ; let each of these parts be Lipschitz with uniformly bounded Lipschitz constants.

Since $\varphi = 0$ on the boundary of Ω_1 , we use estimates (111) for obtaining

$$|\partial_z \varphi(z, \tau)| \leq C \tau^\delta R r^{-1-\delta},$$

and so

$$|\partial_z \varphi(z, h(z, d))| \leq C(\tau^\delta R r^{-1-\delta} + R r^{-1} |h'_z|).$$

Furthermore, it follows from (111) that

$$|\varphi(z, h(z, d))| \leq C h(z, d) R r^{-1} \quad \text{and} \quad |(\nabla \varphi)(z, h(z, d))| \leq C R r^{-1}, \quad (114)$$

which imply

$$\int_{\Gamma'} |\varphi(z, h(z, d))|^2 d\Gamma = O(d^2). \quad (115)$$

Assumption (113) together with (114) gives

$$\int_{\Gamma'} |\partial_z \varphi|^2 dz = o(1) \quad \text{as } d \rightarrow 0.$$

Moreover, it follows from (111) that

$$\int_{\Gamma_2 \setminus \Gamma'_2} |\nabla \varphi|^2 d\Gamma = C \max_s d^{2\lambda_s - 1}.$$

Applying [5, Theorem 4.2] to the function Φ , we obtain

$$\int_{\Gamma_2} |\partial_\nu \Phi|^2 d\Gamma = o(1). \quad (116)$$

Using (115) and (116), we arrive at

$$\int_{\Omega_2} |\nabla \Phi(x)|^2 dx = \int_{\Gamma_2} \varphi \partial_\nu \Phi d\Gamma = o(d).$$

This together with (112) leads to (15), where the numbers $\{\theta_j\}_{j=1}^{J_m}$ are eigenvalues of the problem (14). The reader can check the relation $\theta_j = O(d)$, for which purpose the above estimates of φ must be applied. If one of the angles, say α_s , is greater than π , then [8, Theorem 8.6.2] yields that there is an eigenfunction such that the norm of its gradient is estimated from below by $cr_s^{\lambda-1}$ with $\lambda \in (1/2, 1)$. Then the Hadamard formula (11) will give the inequality $\theta_j \geq cd^{2\lambda-1}$, which shows that the eigenvalue problem (14) is a right replacement of problem (11) for obtaining the asymptotic formula expressing the eigenvalue μ_j in the case of a non-smooth boundary when a corner point of opening greater than π is present.

4 Examples

In this section, we give applications of the general formula (4) to perturbations of the reference domain which involves removing small cuts or small discs.

Let Ω_1 be a $2D$ domain with a smooth boundary Γ_1 whose length is rescaled to unity. In a neighborhood $\mathcal{V}$ of Γ_1 , we introduce coordinates (z, τ) so that z is the arc length measured from a fixed point of Γ_1 , whereas τ is the oriented distance to the boundary ($\tau > 0$ outside Ω). We put

$$\Omega_2 = \Omega \setminus \bigcup_{j=1}^N \Upsilon_j^\varepsilon, \quad \Upsilon_j^\varepsilon = \{x \in \Omega \cap \mathcal{V} : z = j\varepsilon, \tau \in [-\varepsilon\ell(z), 0]\},$$

where ℓ is a positive $C^\infty(\Gamma)$ -function, $\varepsilon = 1/N$ and N is a large positive integer. Clearly the domains Ω_1 and Ω_2 satisfy Assumption I made in Introduction with $d = c\varepsilon$. Since the reference domain is smooth and $\Omega_2 \subset \Omega_1$ formulae (4) and (5) are valid with $\Psi = 0$ and an arbitrary $\delta \in (0, 1)$. Let us construct the function $\Phi_\varphi = \Phi$ that solves the problem $\mathcal{A}(\partial_x)\Phi = 0$ in Ω_2 and $\Phi = -\varphi$ on Γ_2 . In the variables $\xi = (\xi_1, \xi_2) = (\varepsilon^{-1}\tau, \varepsilon^{-1}z)$, Φ is sought in the form

$$\Phi(x) = \chi(x)\varepsilon w_\varphi(\xi, z) + \varepsilon R(x, \varepsilon), \quad (117)$$

where χ is a smooth function which has a compact support in $\mathcal{V}$ and is equal to 1 in a neighborhood of Γ_1 . Let us describe a family of boundary value problems for $w = w_\varphi$. Near every point $z_0 \in \Gamma_1$, the operator $\mathcal{A}$ has the following form in variables ξ :

$$\mathcal{A}(\partial_x) = \varepsilon^{-2}L_2(z_0, \varepsilon\xi, \partial_\xi) + \varepsilon^{-1}L_1(z_0, \varepsilon\xi, \partial_\xi) + L_0(z_0, \varepsilon\xi),$$

where $L_k(z, \xi, \partial_\xi)$ is a differential operator of k th order; its coefficients are smooth functions of z and ξ . In $\Pi_z = \Pi \setminus \mathcal{L}_z$, where

$$\Pi = \{\xi : \xi_1 < 0, \xi_2 \in (-1/2, 1/2)\} \quad \text{and} \quad \mathcal{L}_z = \{\xi = (\xi_1, 0) : \xi_1 \in (-\ell(z), 0)\},$$

the term w is described by the boundary value problem that consists of the following equation:

$$L_2(z, 0, \partial_\xi)w(\xi, z) = 0, \quad \xi \in \Pi_z,$$

and two types of conditions, namely, the periodicity conditions:

$$w(\xi_1, -1/2, z) = w(\xi_1, 1/2, z), \quad \partial_{\xi_2} w(\xi_1, -1/2, z) = \partial_{\xi_2} w(\xi_1, 1/2, z), \quad \xi_1 < 0,$$

and the Dirichlet boundary conditions imposed on a part of the boundary:

$$w(0, \xi_2, z) = 0, \quad \xi_2 \in (-1/2, 1/2),$$

$$w(\xi_1, 0, z) = -\xi_1 \partial_\nu \varphi(0, z), \quad \xi_1 > -\ell(z).$$

This problem has a unique solution in the class of vector functions such that the following norm is finite:

$$\left(\int_{\Pi} |\nabla_\xi u|^2 d\xi + \int_{-1/2}^{1/2} |u(\xi_1, 0)|^2 d\xi_1 \right)^{1/2}. \quad (118)$$

Moreover, this solution has the following asymptotic representation at the negative infinity:

$$w(\xi, z) = \mathbf{M}(z) + o(\exp(\delta \xi_1)), \quad \delta > 0, \quad \xi_1 \rightarrow -\infty, \quad (119)$$

where $\mathbf{M}(z)$ is a smooth vector-function of z , see [13].

In order to estimate the remainder R in (117), we proceed as follows. We begin with rewriting representation (117) in the form:

$$\begin{aligned} \Phi(x) &= \hat{\Phi}^\varepsilon(x) + \varepsilon R_1(x, \varepsilon), \\ \hat{\Phi}^\varepsilon(x) &= (-\mathcal{X}_\varepsilon(x)\varphi(x) + \mathcal{X}_\varepsilon(x)\partial_\nu \varphi(z, 0)\tau) + \chi(x)\varepsilon w(\xi, z). \end{aligned}$$

Here $\mathcal{X}_\varepsilon(x) = 0$ for $x \in \Omega \setminus \mathcal{V}$ and $\mathcal{X}_\varepsilon(x) = \chi_0(\varepsilon^{-1}\tau)$ for $x \in \Omega \cap \mathcal{V}$, whereas χ_0 is a smooth function such that $\chi_0(t) = 1$ for $|t| < 2\ell_0$ and $\chi_0(t) = 0$ for $|t| > 3\ell_0$ and $\ell_0 = \max \ell(s)$. One can check that

$$R = R_1 - \mathcal{X}_\varepsilon(x) \frac{\varphi(x) - \partial_\nu \varphi(z, 0)\tau}{\varepsilon}. \quad (120)$$

The main reason for introducing the cut-off function $\mathcal{X}_\varepsilon$ in the representation of Φ is to ensure that the remainder R_1 satisfies the homogeneous Dirichlet boundary condition on $\partial\Omega^\varepsilon$. The function Φ satisfies the following integral identity:

$$(\Phi, u)_2 = 0 \quad \text{for all } u \in H_2 \text{ and } \Phi + \varphi \in H_2.$$

If $u = R_1$ here, then we have that

$$(R_1, R_1)_2 = \langle \mathcal{A}(\partial_x) \hat{\Phi}^\varepsilon, R_1 \rangle_2. \quad (121)$$

Let us estimate the norm $\|\rho \mathcal{A}(\partial_x) \hat{\Phi}^\varepsilon\|_{L^2(\Omega^\varepsilon)}$, where ρ is the same as in Sect. 3.4. Since

$$\begin{aligned} \mathcal{A}(\partial_x) \hat{\Phi}^\varepsilon &= -\mathcal{X}_\varepsilon(x) \mathcal{A}(\partial_x)(\varphi(x) - \partial_\nu \varphi(z, 0)\tau) - [\mathcal{A}, \mathcal{X}_\varepsilon](\varphi(x) - \partial_\nu \varphi(z, 0)\tau) \\ &+ \varepsilon(\chi(x) \mathcal{A}(\partial_x)w + [\mathcal{A}, \chi]w) = I^\varepsilon(x), \end{aligned}$$

it is straightforward to verify that

$$\|\rho I^\varepsilon\|_{L^2(\Omega_2)} = O(\varepsilon^{3/2}).$$

Combining this relation and the Hardy inequality (54), we obtain that

$$(R_1, R_1)_2 \leq c\varepsilon^3. \quad (122)$$

This along with (120) leads to the inequality

$$(R, R)_2 \leq c\varepsilon^2.$$

Noting that

$$\begin{aligned} &(\chi(x)w_\varphi(\xi, z), \chi(x)w_\psi(\xi, z))_2 \\ &= \varepsilon^{-1} \int_{\Gamma_1} \int_{-1/2}^{1/2} \int_{-\infty}^0 \sum (A_{ij} \partial_{\xi_i} w_\varphi(\xi, z), \partial_{\xi_j} w_\psi(\xi, z)) d\xi dz + O(1), \end{aligned}$$

we obtain

$$\frac{1}{\mu_j} = \frac{1}{\lambda_m} - \varepsilon \frac{\tau_j}{\lambda_m} + O(\varepsilon^{1+\delta}), \quad (123)$$

where δ is an arbitrary positive number from the interval $(0, 1)$ and $\{\tau_j\}_{j=1}^{J_m}$ are the eigenvalues of the problem

$$\int_{\Gamma_1} \int_{-1/2}^{1/2} \int_{-\infty}^0 \sum (A_{ij} \partial_{\xi_i} w_\varphi(\xi, z), \partial_{\xi_j} w_\psi(\xi, z)) d\xi dz = \tau(\varphi, \psi)_1 \quad \text{for all } \psi \in X_m.$$

The "Hadamard term" here is equal to zero, since $\Omega_1 \setminus \Omega_2$ consists of a finite number of cuts.

The second example is as follows. Let Ω_1 be the same as above, and let

$$\Omega_2 = \Omega \setminus \cup_{j=1}^N \mathcal{Q}_j^\varepsilon, \quad \mathcal{Q}_j^\varepsilon = \{x \in \Omega \cap \mathcal{V} : (z - j\varepsilon)^2 + (\tau - q(z)\varepsilon)^2 < \ell^2(z)\varepsilon^2\},$$

where q and ℓ are smooth functions on Γ_1 such that $0 < \ell(z) < q(z)$ and $\ell(z) < 1/2$. In the same way as in the previous example we introduce the variables ξ and put

$$\Pi_z = \{\xi : \xi_1 < 0, \xi_2 \in (-1/2, 1/2) \text{ and } (\xi_1 - q(z))^2 + \xi_2^2 < \ell(z)^2\}.$$

Similarly to the previous case, formulae (4) and (5) are valid with $\Psi = 0$ and $\delta \in (0, 1)$. Let us construct the function $\Phi_\varphi = \Phi$. We seek Φ in the form

$$\Phi(x) = \chi(x)\varepsilon w(x, z) + \varepsilon R(x, \varepsilon), \quad (124)$$

where the function $w = w_\varphi$ satisfies the boundary value problem that consists of the following equation:

$$L_2(z, 0, \partial_\xi)w(\xi, z) = 0, \quad \xi \in \Pi,$$

and two types of conditions, namely, the periodicity conditions:

$$w(\xi_1, -1/2, z) = w(\xi_1, 1/2, z), \quad \partial_2 w(\xi_1, -1/2, z) = \partial_2 w(\xi_1, 1/2, z), \quad \xi_1 < 0,$$

and the Dirichlet boundary conditions on a part of the boundary:

$$w(0, \xi_2, z) = 0, \quad \xi_2 \in (-1/2, 1/2),$$

$$w(\xi, z) = -\xi_1 \partial_\nu \varphi(0, z), \quad \text{for } (\xi_1 - q(z))^2 + \xi_2^2 = \ell(z)^2.$$

This problem has a unique solution in the class of vector-functions such that the norm (118) is finite. Moreover, this solution admits the asymptotic representation (119) at the negative infinity, where $\mathbf{M}(z)$ depends smoothly on z . Reasoning as above, one can show that the remainder in (124) satisfies (122) and the asymptotic formula (123) is valid, where $\delta \in (0, 1)$ is arbitrary, whereas $\{\tau_j\}_{j=1}^{J_m}$ are the eigenvalues of the problem

$$\begin{aligned} & \left(\int_{\Gamma_1} \int_{-1/2}^{1/2} \int_{-\infty}^0 \sum (A_{ij} \partial_{\xi_i} w_\varphi(\xi, z), \partial_{\xi_j} w_\psi(\xi, z)) d\xi dz \right. \\ & \left. + \sum_{j=1}^N \int_{S_j} \xi_1^2 (A_\nu \partial_\nu \varphi(j/N, 0), \partial_\nu \psi(j/N, 0)) d\xi \right) = \tau(\varphi, \psi)_1 \quad \text{for all } \psi \in X_m. \end{aligned}$$

Here $A_\nu = \sum A_{ij} \nu_i \nu_j$ and $S_j = \{\xi : (\xi_1 - q(j/N))^2 + \xi_2^2 < \ell(j/N)^2\}$.

Despite the perturbation of the eigenvalue is of the same order as in the Hadamard formula, the coefficient is different here.

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