STRUCTURALLY PASSIVE SCATTERING ELEMENT FOR MODELING GUITAR PLUCK ACTION

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ABSTRACT

In this paper we propose new models for the plucking interaction of the player with the string for use with digital waveguide simulation of guitar. Unlike the previously presented models, the new models are based on structurally passive scattering junctions, which have the main advantage of being properly scaled for use in fixed-point waveguide implementations and of guaranteeing stability independently of the plucking excitation.

In a first model we start from the Cuzzucoli-Lombardo equations [1], within the Evangelista-Eckerholm [2] propagation formulation, in order to derive the passive scattering junction by means of bilinear transformation. In a second model we start from equations properly modeling the finger compliance by means of a spring. In a third model we formalize the interaction in terms of driving impedances. The model is also extended using nonlinear (feathering) compliance models.

1. INTRODUCTION

Physical models of the interaction of the player with the string during plucking were introduced in [1], for use with digital waveguide (DW) simulations [3]. In recent works, the first author together with F. Eckerholm introduced a more consistent model for simulating the plucking of the string by means of a finger [2, 4]. In this model, the finger is modeled as a linear spring-mass system coming into contact with the string during the plucking action. The action of the finger system builds up a traveling perturbation of the string displacement in a short time interval, lasting until the finger is completely detached and the string is released into free motion. Based on the equations of dynamics, the interface of the finger with the string can be described by means of a scattering matrix $\mathbf{S}$ [5] linking the two rails of the waveguide, together with a coupling term converting the force exerted by the player to string displacement. The scattering matrix, which is a function of frequency, also depends on the physical parameters of finger and string, such as tension, mass, damping and stiffness coefficients. The force exerted on the string by the player is converted into wave variables and injected to the two rails of the waveguide in equal amounts. In our model, the preferred choice of wave variables is displacement, in view of the fact that string-fret collisions are easier to detect and compute in this representation [4], albeit other choices are possible.

In [4] the discrete-time plucking model was derived by replacing derivatives with central differences and led to a scattering matrix $\mathbf{S}(z)$ that is not structurally passive, i.e., for some values of the physical parameters, and for some frequencies, the magnitude of the determinant of $\mathbf{S}(e^{j\omega})$, which is the power gain of the scattering junction, can grow larger than 1.

In this paper, we present new discrete-time models for the plucking scattering matrix that are derived from the Laplace domain counterpart of the PDE of the coupled finger-string system or directly from load impedances. The system is solved for the Laplace transform of the wave variables. The discrete-time form is obtained by means of the bilinear transformation, which preserves stability.

In Section 2 we review the Cuzzucoli-Lombardo pluck model, introduce a special form for the scattering matrix and formulate the corresponding pluck scattering junction. In Section 3 we introduce a structurally passive discrete-time scattering junction derived from the pluck model via bilinear transformation. We also provide a lattice-ladder implementation for the scattering filter, which helps preventing critical pole-zero cancellation at the offset of the pluck excitation. In Section 4 we introduce a more accurate model for the pluck, in which finger compliance is modeled by a spring. The model is revisited and extended by means of load impedance formulation in Section 5. In Section 6 we draw our conclusions.

2. MODELING THE FINGER-STRING INTERACTION

In this section we review the damped mass-spring model for the finger pluck introduced in [1, 2], together with its previous realization as scattering junction in a DW [4]. First-order nonlinear effects due to string pulling are disregarded since they can be reintroduced through suitable modulation of the string tension [6] and by modeling the collisions of the string with the neck or frets [4]. We also assume that the string is ideally flexible, i.e., dispersive propagation phenomena are disregarded.

A finger plucking the string is shown in Fig. 1. There, the finger comes in contact with a segment of the string of length Δ centered at coordinate point x_p along the string axis (at rest). During a pluck, the finger exerts a time-varying force $\vec{f}_0(t)$.

In the general case, the direction of the force changes with time and is contained in a plane orthogonal to the string rest line. However, for simplicity, here we assume that the player's force is not changing direction. Then we can consider only the projection $f_0(t)$ of the force in the vertical direction with respect to the soundboard. Projection onto the horizontal direction leads to a similar system. Oscillations in these two directions are coupled at the bridge (e.g., see [7] and references therein).

Two DW structures are needed to capture the two polarization modes of the string in planes orthogonal to the string rest axis. The force input is distributed among these two waveguides, according to the plucking direction. Once this structure is put together, changing the player's force direction in time is only a matter of dynamically changing the projection angle.

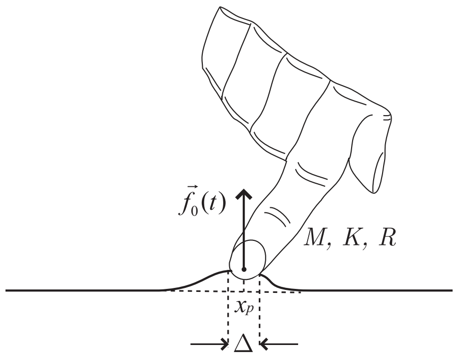

Figure 1: *A finger plucking the string.*

During a pluck, the wave equation for the string holds for coordinate points not in contact with the finger. For a string of length L_s we have

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}; \qquad x \in \left]0, x_p - \tfrac{\Delta}{2}\right[\cup \left]x_p + \tfrac{\Delta}{2}, L_s\right[, \quad (1)$$

where $c = \sqrt{K_0/\mu}$ is the propagation velocity, with K_0 the tension of the string, and μ the linear mass density, both assumed to be constant. Here we assume that all propagation losses along the string can be consolidated at one of the extremities and embedded in the bridge model [8]. The solution of (1) can be written in D'Alembert form as a superposition of a left-going u^- and a right-going u^+ wave:

$$u(x,t) = u^-(x,t) + u^+(x,t) = u_l(t+x/c) + u_r(t-x/c), \quad (2)$$

where $u_l(x/c) = u_r(x/c) = u(x,0)/2$ for a static initial condition.

In the first part of this paper, we consider the Cuzzucoli-Lombardo (C-L) model for the string-finger interaction. Although this model is extremely simplified and not so well justified from a physical point of view, it provides good acoustic results for the synthesis of the pluck. In Sections 4 and 5 the C-L model is replaced by a more accurate model including finger compliance, as considered in [9] and [10].

According to the C-L model, on the string-finger contact segment, the equilibrium equation of the string with the damped spring-mass system modeling the finger is enforced:

$$(M + \mu\Delta)\, \frac{\partial^2 u}{\partial t^2} = -R\frac{\partial u}{\partial t} - Ku + f(t) + f_0(t) \qquad x \in \left]x_p - \tfrac{\Delta}{2}, x_p + \tfrac{\Delta}{2}\right[, \quad (3)$$

where M, K, and R are respectively the mass, stiffness, and damping parameters of the finger [1]. The force $f(t)$ is the resultant of the transversal component of the tensile force of the string acting at the extreme points of the plucking segment. For small deformations we have:

$$f(t) = K_0 \left(\left.\frac{\partial u}{\partial x}\right|_{x = x_p + \frac{\Delta}{2}} - \left.\frac{\partial u}{\partial x}\right|_{x = x_p - \frac{\Delta}{2}} \right). \quad (4)$$

Finally, at the interface points $x = x_p - \frac{\Delta}{2}$ and $x = x_p + \frac{\Delta}{2}$ between the string and string-finger systems, the continuity of the solution is enforced.

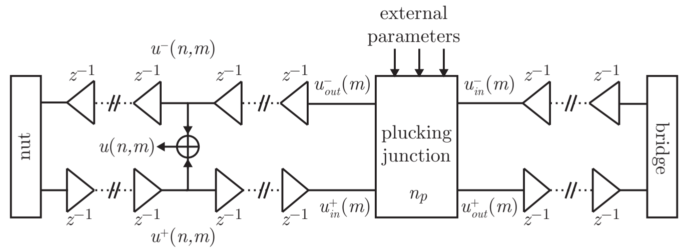

Figure 2: *Diagram of two DWs linked by a scattering junction modeling the pluck interaction. Nut and bridge terminations are also visible at the extremities.*

2.1. Scattering in discrete-time

In discrete-time, free wave propagation can be efficiently simulated by means of two DWs, one for each string segment on either side of the plucking zone, as shown in Fig. 2. The plucking interaction is suitably modeled by means of a scattering junction described, in a linear model, by means of a scattering matrix $\mathbf{S}(z)$ and a force coupling transfer function $G(z)$, linking the variables according to the following update equation:

$$\begin{bmatrix} U^-_{out}(z) \\ U^+_{out}(z) \end{bmatrix} = \mathbf{S}(z) \begin{bmatrix} U^-_{in}(z) \\ U^+_{in}(z) \end{bmatrix} + \frac{G(z)F_0(z)}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (5)$$

Here, $U^{\pm}_{in/out}$ are the z-transforms of the input and output signals on either side of the scattering junction and $F_0(z)$ is the z-transform of the discrete-time signal $f_0(n)$ representing the time-varying force exerted by the player on the string. For an interaction at a single point, by a string continuity argument, the total displacements on each side of the junction must be identical. This gives us the condition:

$$U^-_{out}(z) + U^+_{in}(z) = U^-_{in}(z) + U^+_{out}(z). \quad (6)$$

Since the two directions of propagation are physically equivalent, a two-port representation of the scattering junction should be *reciprocal*, i.e., it should look the same from either port. It is well known that the scattering matrix of a reciprocal two-port is symmetric. In other words, the changes $U^+_{in} \rightleftarrows U^-_{in}$ and $U^+_{out} \rightleftarrows U^-_{out}$ leaves the result unchanged. The most general stable scattering matrix satisfying these requirements has the following structure [7]:

$$\mathbf{S}(z) = \frac{1}{2} \begin{bmatrix} Q(z)+1 & Q(z)-1 \\ Q(z)-1 & Q(z)+1 \end{bmatrix}, \quad (7)$$

where $Q(z)$ is the transfer function of a stable filter.

When evaluated on the unit circle, the modulus of the determinant of (7) provides the power gain of the scattering junction:

$$P_g(e^{j\omega}) = \left|\det \mathbf{S}(e^{j\omega})\right| = \left|Q(e^{j\omega})\right|. \quad (8)$$

Both the scattering matrix and force coupling factor can be derived from a discrete model of the differential equations governing the plucking action. The procedure to derive the discrete model is, however, not unique.

2.2. The CLEE scattering junction

In [1, 2, 4], equation (3) was discretized by replacing partial derivatives with central differences and by sampling all signals. Under the simplifying assumption that the width of the finger-string contact is exactly one space sampling interval we have $\Delta = X$, where $X = cT$ is the string spatial sampling interval, and T is time sampling interval. In this case, in [4], a scattering junction in the form (7) was derived, where $Q(z) = 1/A(z)$ is an allpole filter with

$$A(z) = \frac{M}{\mu X}\left(1 - z^{-1}\right)^2 + \rho\left(1 - z^{-2}\right) + \kappa z^{-1} + 1 \qquad (9)$$

and

$$\rho = \frac{R}{2\sqrt{\mu K_0}}, \qquad \kappa = \frac{KX}{K_0} \qquad (10)$$

are dimensionless parameters respectively proportional to the damping coefficient R and to the stiffness constant K of the finger. The force coupling filter for the same discrete model is

$$G(z) = \frac{Xz^{-1}}{K_0 A(z)}. \qquad (11)$$

We refer to the above model as the CLEE scattering junction, where CLEE stands for Cuzzucoli-Lombardo scheme as revised by Evangelista-Eckerholm.

Notice that, since $Q(z) = 1/A(z)$ is a generic 2nd order allpole filter, then P_g in (8) is not constrained to guarantee that the CLEE scattering matrix (7) is passive for all values of physical parameters.

In order to simulate the effect of variable contact of the finger with the string during the preliminary and final phases of plucking, together with the force, the finger parameters M, K and R are considered as time varying signals that are identically zero when the finger is away from the string.

Since the plucking transient has typically short duration, so that the scattering matrix is different from the unit matrix only for a finite time interval, there is no overall DW stability concern as long as the filters in (7) and (11) are stable. This remains true even in the time-varying case [4]. However, in order to maintain a fixed output level range, or in order to prevent overflow in fixed point applications, suitable scaling must be applied to the non-passive scattering junction, where the scaling gains must be estimated also depending on the duration of the plucking action.

In order to circumvent these problems and simplify the use of the plucking scattering junction in DWs simulating a guitar, it would be desirable that the scattering matrix be passive. The derivation of a structurally passive junction is the object of the next section.

3. A STRUCTURALLY PASSIVE PLUCKING JUNCTION

Following the general method outlined in [11], a structurally passive scattering junction for pluck synthesis can be derived by combining the Laplace transform version of the differential equation (3) with the Laplace domain rewriting of the traveling wave solution (2). A discrete-time passive scattering junction is then derived by means of bilinear transformation, which preserves passivity.

Taking the Laplace transform on both sides of (3) we obtain

$$\left[(M + \mu\Delta)\, s^2 + Rs + K\right] U(x, s) = F(s) + F_0(s), \qquad (12)$$

where

$$U(x, s) = \mathcal{L}[u(x, t)](s) = \int_0^\infty u(x, t) e^{-st} dt \qquad (13)$$

is the Laplace transform, with respect to time, of the solution $u(x, t)$, while $F_0(s)$ is the Laplace transform of the player's force signal and

$$F(s) = K_0 \left(\left.\frac{\partial U(x, s)}{\partial x}\right|_{x=x_p+\frac{\Delta}{2}} - \left.\frac{\partial U(x, s)}{\partial x}\right|_{x=x_p-\frac{\Delta}{2}} \right). \qquad (14)$$

is the Laplace domain counterpart of (4). On the other hand,

$$\begin{aligned} U^-(x, s) &= \mathcal{L}[u^-(x, t)](s) = \mathcal{L}[u_l(t + x/c)](s) = e^{+\frac{sx}{c}} U_l(s) \\ U^+(x, s) &= \mathcal{L}[u^+(x, t)](s) = \mathcal{L}[u_r(t - x/c)](s) = e^{-\frac{sx}{c}} U_r(s) \end{aligned} \qquad (15)$$

are the Laplace transforms of the traveling waves (2). At the interface points $x = x_p \pm \frac{\Delta}{2}$ the solution is continuous. Thus, in order to obtain the equations coupling the two systems, one can substitute (15) in (14). Knowing that $U(x, s) = U^-(x, s) + U^+(x, s)$, and that

$$\begin{aligned} \frac{\partial U^-(x, s)}{\partial x} &= +\frac{s}{c} e^{+\frac{sx}{c}} U_l(s) = +\frac{s}{c} U^-(x, s) \\ \frac{\partial U^+(x, s)}{\partial x} &= -\frac{s}{c} e^{-\frac{sx}{c}} U_r(s) = -\frac{s}{c} U^+(x, s), \end{aligned} \qquad (16)$$

we obtain

$$\begin{aligned} F(s) = \frac{K_0 s}{c} &\left[U^-(x_p + \tfrac{\Delta}{2}, s) - U^+(x_p + \tfrac{\Delta}{2}, s)\right. \\ &\left. - U^-(x_p - \tfrac{\Delta}{2}, s) + U^+(x_p - \tfrac{\Delta}{2}, s)\right]. \end{aligned} \qquad (17)$$

Moreover, one can consider (12) at the interface points, again using the substitutions (15). This yields the following system:

$$\begin{cases} \left[U^-(x_p - \frac{\Delta}{2}, s) + U^+(x_p - \frac{\Delta}{2}, s)\right] E(s) - F(s) = F_0(s) \\ \left[U^-(x_p + \frac{\Delta}{2}, s) + U^+(x_p + \frac{\Delta}{2}, s)\right] E(s) - F(s) = F_0(s), \end{cases} \qquad (18)$$

where

$$E(s) = (M + \mu\Delta)\, s^2 + Rs + K. \qquad (19)$$

Substituting (17) in (18) and solving for $U^-(x_p - \frac{\Delta}{2}, s)$ and $U^+(x_p + \frac{\Delta}{2}, s)$ in terms of the other variables obtains

$$\begin{aligned} \begin{bmatrix} U^-(x_p - \frac{\Delta}{2}, s) \\ U^+(x_p + \frac{\Delta}{2}, s) \end{bmatrix} = \tilde{\mathbf{S}}(s) &\begin{bmatrix} U^-(x_p + \frac{\Delta}{2}, s) \\ U^+(x_p - \frac{\Delta}{2}, s) \end{bmatrix} \\ &+ \frac{\tilde{G}(s) F_0(s)}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \end{aligned} \qquad (20)$$

where the matrix

$$\tilde{\mathbf{S}}(s) = \frac{1}{2} \begin{bmatrix} \tilde{Q}(s) + 1 & \tilde{Q}(s) - 1 \\ \tilde{Q}(s) - 1 & \tilde{Q}(s) + 1 \end{bmatrix}, \qquad (21)$$

has the same structure as (7), with

$$\tilde{Q}(s) = -\frac{cE(s) - 2\, sK_0}{cE(s) + 2\, sK_0} \qquad (22)$$

and the force coupling factor is

$$\tilde{G}(s) = \frac{2c}{cE(s) + 2\, sK_0}. \qquad (23)$$

The denominator $\tilde{D}(s)$ and the numerator $\tilde{N}(s)$ of the transfer function $\tilde{Q}(s) = -\tilde{N}(s)/\tilde{D}(s)$, respectively, are

$$\begin{aligned} \tilde{D}(s) &= s^2(M+\mu\,\Delta) + \left(\frac{2K_0}{c} + R\right)\, s + K \\ \tilde{N}(s) &= s^2(M+\mu\,\Delta) - \left(\frac{2K_0}{c} - R\right)\, s + K, \end{aligned} \tag{24}$$

where the parameters M, K, R, K_0, μ and Δ are all nonnegative physical quantities. Also notice that $\tilde{G}(s) = 2c/\tilde{D}(s)$. The transfer functions $\tilde{Q}(s)$ and $\tilde{G}(s)$ are both stable since the coefficients of the denominator $\tilde{D}(s)$ all have the same sign (positive). In fact, this is a necessary and sufficient condition for the second order polynomial $\tilde{D}(s)$ to be Hurwitz, i.e., its roots all lie in the left hand semiplane in the Laplace domain. When the damping coefficient $R = 0$ then $\tilde{Q}(s) = -\tilde{D}(-s)/\tilde{D}(s)$ has the form of a 2nd-order allpass filter, the coefficients of the numerator being the same as those of the denominator but alternating in sign. In this case, the system (20) is lossless. Moreover, it is easy to show that in the general case where $R \geq 0$ we have $|\tilde{Q}(j\omega)| \leq 1$, which means that the system (20) is passive.

3.1. Passive scattering in discrete time

In order to derive a scattering junction for use with the discrete-time DWs in Fig. 2, one can re-interpret (20) so as to "shrink" the finger-string system to a single computational node (in-between two delays of the DW), without changing its physical length Δ. In other words, we concentrate the plucking system to a point x_p belonging to the spatial sampling grid $x_p = n_p X$ where n_p is an integer. This can be interpreted as an infinite sound-speed across the plucking system, as if it were rigid.

To obtain a discrete-time structurally passive junction from its continuous-time counterpart, it suffices to apply the bilinear transformation

$$s \leftrightarrow \frac{2}{T}\frac{z-1}{z+1} \tag{25}$$

to the system (20), with sampling interval T. This transformation has the property of preserving both stability and passivity when mapping from continuous time to discrete time. The main ingredients of (20), i.e., the transfer functions $\tilde{Q}(s)$ and $\tilde{G}(s)$, respectively, transform as follows:

$$\begin{aligned} Q(z) &= \tilde{Q}\left(\frac{2}{T}\frac{z-1}{z+1}\right) = -\frac{N(z)}{D(z)} \\ G(z) &= \tilde{G}\left(\frac{2}{T}\frac{z-1}{z+1}\right) = \frac{2cT^2(z+1)^2}{D(z)} \end{aligned} \tag{26}$$

where

$$\begin{aligned} D(z) &= (V + 2cRT)\, z^2 - 2Y z + W - 2cRT \\ N(z) &= (W + 2cRT)\, z^2 - 2Y z + V - 2cRT \end{aligned} \tag{27}$$

and we have defined

$$\begin{aligned} V &= 4\,c(\mu\,\Delta + M) + cKT^2 + 4\,K_0\,T \\ W &= 4\,c(\mu\,\Delta + M) + cKT^2 - 4\,K_0\,T \\ Y &= 4\,c(\mu\,\Delta + M) - cKT^2. \end{aligned} \tag{28}$$

The discrete-time update equation is in the form (5). The discrete-time scattering matrix $\mathbf{S}(z)$ is obtained from $Q(z)$ in (26) using the same matrix structure as in (7).

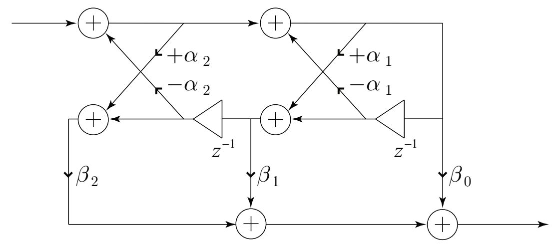

Figure 3: *Lattice-ladder implementation of the scattering filter* $Q(z)$.

When the damping coefficient R is zero, we can check that the discrete-time system becomes lossless since, in that case, the determinant of the scattering matrix $Q(z) = -z^2 D(z^{-1})/D(z)$ has the form of a second-order digital allpass filter, so it is lossless.

When the finger is detached from the string, all the finger parameters M, K, R, and Δ become zero, together with the player's force. In practical uses of the plucking junction, one would like to continuously transition from "finger touching the string" to "finger away from the string" cases. However, when all the finger parameters are zero, from (27) and (28) it is easy to see that the transfer function $Q(z)$ becomes equal to 1 only through second order pole-zero cancellation, with two poles on the unit circle at $z = \pm 1$:

$$Q_{M,K,R,\Delta \to 0}(z) = \frac{z^2-1}{z^2-1} = 1. \tag{29}$$

This is a critical feature that is not so relevant when the parameters are exactly zero since one can switch off the scattering matrix filters in that case. However, when the parameters are assigned time-varying values gradually approaching zero as the result of the loosening of the finger-string contact, the system may transition through scattering matrices in which imperfect pole-zero cancellation occurs with poles very close to the unit circle. This could be the source of numerical instabilities especially in fixed point implementations. This will be addressed further below.

Notice that even if one leaves a non-zero damping term for last, the poles are still on the unit circle, but retaining some mass M, stiffness K or simply Δ will do.

An improvement over the direct implementation of the IIR scattering filters can be achieved if the filter $Q(z)$ is implemented in lattice-ladder form, shown in Fig. 3. In this case, one finds the values for the reflection coefficients

$$\begin{aligned} \alpha_1 &= \frac{-2Y}{V+W} \\ \alpha_2 &= \frac{W-2cRT}{V+2cRT}, \end{aligned} \tag{30}$$

which are both not larger than 1 (stability). For the ladder coefficients β one has

$$\begin{aligned} \beta_0 &= \frac{4\,cRT\left((W+V)^2 - 4\,Y^2\right)}{(V+W)\,(V+2\,cRT)^2} \\ \beta_1 &= -\frac{8\,cRTY}{(V+2\,cRT)^2} \\ \beta_2 &= \frac{V-2\,cRT}{V+2\,cRT}. \end{aligned} \tag{31}$$

Both reflection coefficients α tend to -1 when all the finger parameters become 0. The lattice-ladder implementation ensures that

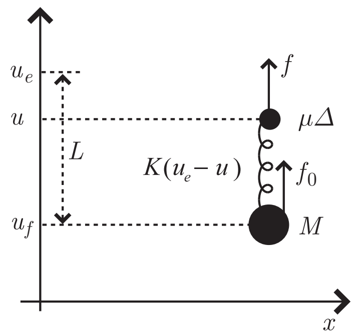

Figure 4: *A complete spring-mass system modeling the pluck interaction where the finger pulls the string (upwards in the figure).*

when the parameters are not all zero the filter is robustly stable with respect to coefficient rounding or operation round off error in fixed point implementation. When the finger parameters all become 0, a switch is necessary to revert the scattering matrix to the identity matrix, thus avoiding the exact pole-zero cancellation problem.

4. MODELING FINGER COMPLIANCE

In the Cuzzucoli-Lombardo model (3) the finger-spring K appears to be always attached to the string, with its elongation computed from the string rest position. The masses of finger and string segment are always in contact during pluck so that they sum. In this section and in Section 5, we present more realistic models in which finger compliance, i.e., compression of the finger flesh, is introduced by means of a spring whose end-points are attached, respectively, to the finger and to the string during a pluck, as diagrammed in Fig. 4. There, the finger of equivalent mass M interacts with a segment of length Δ of the string of linear mass density μ by means of a spring of elastic constant K and elongation at rest L. If u denotes the displacement of the string and u_f the vertical coordinate of the finger, both with respect to rest position, then the elastic force "felt" by the string is $-K(u-u_f-L) = K(u_e-u)$, where u_e is the "end at rest" of the spring K. More rigorously, the spring should be considered as one-sided, i.e., the elastic force should be present only if the finger is in contact with the string, which happens when $u - u_f \leq L$, which is $u \leq u_e$. We will not consider this complication until Section 5.1; here we assume that the finger is always in contact with the string during the pluck action. In the figure, all forces directed upward are considered to be positive. The finger exerts a force $f_0(t)$ on one end of the spring connecting the mass M to the string. The string feels the vertical resultant of the tension $f(t)$ at the two sides of the plucking segment, as given in (4). In addition to the above forces, a damping factor $-R\dot{u}$ is introduced. The overall system modelling the finger-string interaction is described by the following set of equations:

$$\begin{cases} \mu\Delta\ddot{u} = f(t) - K(u-u_e) - R\dot{u} \\ M\ddot{u}_f = K(u-u_e) + f_0(t), \end{cases} \tag{32}$$

where dots over symbols denote time derivatives.

We remark that the model we employ here is simplified and does not include, e.g., the finger stick-slip behavior, which can be introduced as in [10].

Given the external force $f_0(t)$, simultaneous solution of the two equations in (32) will determine both string $u(x_p,t)$ and finger $u_f(t)$ trajectories at the plucking point x_p. However, computation can be simplified further for real-time implementations: If, instead of the force $f_0(t)$ the input of the system is directly the trajectory of the finger $u_f(t)$, then only the first equation in (32) needs to be considered. In this case, passing to the Laplace transform domain, one obtains

$$\mu\Delta s^2 U(s) = F(s) - K(U(s) - U_e(s)) - RsU(s), \tag{33}$$

where $F(s)$ is given in (14) and $U_e(s)$ is the Laplace transform of $u_e(t)$.

Reasoning as in Section 3, one can consider equation (33) at both sides of the plucking segment, replacing U with the sum of progressive and regressive waves at these points. Solving for $U^-(x_p - \frac{\Delta}{2}, s)$ and $U^+(x_p + \frac{\Delta}{2}, s)$ in terms of the other variables, one arrives at a scattering equation similar to (20)

$$\begin{bmatrix} U^-(x_p-\frac{\Delta}{2},s) \\ U^+(x_p+\frac{\Delta}{2},s) \end{bmatrix} = \tilde{\mathbf{S}}(s) \begin{bmatrix} U^-(x_p+\frac{\Delta}{2},s) \\ U^+(x_p-\frac{\Delta}{2},s) \end{bmatrix} + \tilde{G}(s)U_e(s)\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{34}$$

where the matrix $\tilde{\mathbf{S}}(s)$ has the same structure as in (21), where now

$$\tilde{Q}(s) = \frac{2K_0 s - (\mu\Delta s^2 + Rs + K)c}{2K_0 s + (\mu\Delta s^2 + Rs + K)c} \tag{35}$$

and the finger coordinate coupling factor is

$$\tilde{G}(s) = \frac{Kc}{2K_0 s + (\mu\Delta s^2 + Rs + K)c}. \tag{36}$$

The discrete counterpart of this plucking model can be obtained by applying the bilinear transform (25) to the system (34). The transfer functions $\tilde{Q}(s)$ and $\tilde{G}(s)$, respectively, transform as follows:

$$\begin{aligned} Q(z) &= \tilde{Q}\left(\frac{2}{T}\frac{z-1}{z+1}\right) = -\frac{N(z)}{D(z)} \\ G(z) &= \tilde{G}\left(\frac{2}{T}\frac{z-1}{z+1}\right) = \frac{KcT^2(z+1)^2}{D(z)} \end{aligned} \tag{37}$$

where $D(z)$ and $N(z)$ are given as in (27) but with the following different definitions for V, W and Y:

$$\begin{aligned} V &= 4\,c\mu\,\Delta + cKT^2 + 4\,K_0\,T \\ W &= 4\,c\mu\,\Delta + cKT^2 - 4\,K_0\,T \\ Y &= 4\,c\mu\,\Delta - cKT^2. \end{aligned} \tag{38}$$

We notice that here too, as in Section 3.1, the scattering filter $Q(z)$ reduces to singular pole-zero cancellation when all the finger parameters go to 0 as a result of detachment. In order to prevent critical round-off effects, a lattice-ladder implementation is considered. With the new definitions (38), the equations for the reflection and ladder coefficients are formally the same as in (30) and (31), respectively.

5. IMPEDANCE SCATTERING FORMULATION

From the point of view of traveling waves in the string, the plucking system can be formulated as a "load impedance" at the junction of two identical waveguides (strings) [5, p. 124], [12].[1] Referring to Fig. 4, letting $\Delta = R = 0$ (R will be re-introduced in Section 5.2 below) and assuming the finger position u_f is approximately constant (relative to vibrations on the string), and that the string is in contact with the spring, then the Laplace-domain impedance of the plucking finger is that of the spring K:[2]

$$\tilde{R}(s) = \frac{K}{s} \tag{39}$$

(The subscript "a" means "analog" as opposed to "digital".) Denoting the wave impedance of the string by $r = \sqrt{K_0 \mu}$, the reflectance of the finger-impedance $\tilde{R}(s)$ on the string for force waves is given by[3]

$$\tilde{\rho}(s) = \frac{[\tilde{R}(s) + r] - r}{[\tilde{R}(s) + r] + r} = \frac{\frac{K}{2r}}{s + \frac{K}{2r}}$$

and the transmittance for force waves is

$$\tilde{\tau}(s) = 1 + \tilde{\rho}(s).$$

For velocity and displacement waves, the reflectance and transmittance are given by $-\tilde{\rho}(s)$ and $1 - \tilde{\rho}(s)$, respectively. The scattering relations for "small-signal" displacement waves given a constant finger position (i.e., eliminating any static component) are

$$\begin{aligned} U^-_{out}(s) &= -\tilde{\rho}(s) U^+_{in}(s) + [1 - \tilde{\rho}(s)] U^-_{in}(s) \\ &= U^-_{in}(s) - \tilde{\rho}(s)[U^+_{in}(s) + U^-_{in}(s)] \end{aligned} \tag{40}$$

$$\begin{aligned} U^+_{out}(s) &= -\tilde{\rho}(s) U^-_{in}(s) + [1 - \tilde{\rho}(s)] U^+_{in}(s) \\ &= U^+_{in}(s) - \tilde{\rho}(s)[U^+_{in}(s) + U^-_{in}(s)] \end{aligned} \tag{41}$$

Note that the expressions (40) and (41) indicate a one-filter scattering-junction implementation (dropping the common 's' argument for simplicity of notation):

$$\begin{aligned} U^+ &= U^+_{in} + U^-_{in} \\ U^-_{out} &= U^-_{in} - \tilde{\rho} U^+ \\ U^+_{out} &= U^+_{in} - \tilde{\rho} U^+ \end{aligned}$$

where $\tilde{\rho}(s) = (K/2r)/[s + (K/2r)]$. Here again, the scattering matrix has the form (21) with

$$\tilde{Q}(s) = 1 - 2\tilde{\rho}(s) = \frac{s - \frac{K}{2r}}{s + \frac{K}{2r}}. \tag{42}$$

It correspond to (35) when $\Delta = R = 0$. This one-filter scattering junction is diagrammed in Fig. 5. The filter $\tilde{\rho}(s)$ may now be digitized using the bilinear transform (25). However, before we do this, we should decide how the finger will drive the string.

[1] http://ccrma.stanford.edu/~jos/pasp/-Loaded_Waveguide_Junctions.html

[2] https://ccrma.stanford.edu/~jos/pasp/-Spring_Mass_System.html

[3] https://ccrma.stanford.edu/~jos/pasp/-Simplified_Impedance_Analysis.html

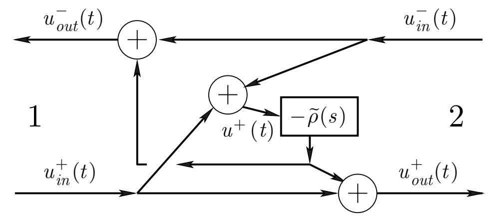

Figure 5: *Displacement-wave scattering model for a spring.*

5.1. Incorporating Finger Motion

The finger position u_f causes a force $f_K = K \cdot (u_e - u)$ to be exerted upward on the string, where $u_e = u_f + L$. The force f_K is applied given $u_e \geq u$ (spring is in contact with string) and given $f_K < f_{\text{max}}$ (the force at which the pluck releases). For $u_e < u$ or $f_K \geq f_{\text{max}}$ the applied force is zero and the entire plucking system disappears to leave $U^-_{out} = U^-_{in}$ and $U^+_{out} = U^+_{in}$, or equivalently, $\tilde{\rho} = 0$ above.

Let the subscripts 1 and 2 each denote one side of the scattering system. Thus, for example, $u_1 = u^-_{out} + u^+_{in}$ is the displacement of the string on the left (side 1) of plucking. Force equilibrium at the plucking point requires[4]

$$0 = f_1 + f_K - f_2$$

where $f_i = -K_0 \partial u_i / \partial x$. Expressing $f_i = f^+_i + f^-_i = r v^+_i - r v^-_i$ and solving for the velocity at the plucking point yields

$$v = v^+_{in} + v^-_{in} + \frac{1}{2r} f_K$$

or, for displacement waves,

$$u = u^+_{in} + u^-_{in} + \frac{1}{2r} \int_t f_K \tag{43}$$

Substituting $f_K = K \cdot (u_e - u)$ in (43), with $u_e = u_f + L$, taking the Laplace transform, and solving for $U(s)$ yields

$$\begin{aligned} U(s) &= [1 - \tilde{\rho}(s)] \left[U^+_{in}(s) + U^-_{in}(s)\right] + \tilde{\rho}(s) \left[U_f(s) + \frac{L}{s}\right] \\ &= U^+_{in}(s) + U^-_{in}(s) - \tilde{\rho}(s) \left[U^+_{in}(s) + U^-_{in}(s) - U_e(s)\right] \end{aligned}$$

so that we can formulate the one-filter form as

$$\begin{aligned} U^+_d &= U_e - \left(U^+_{in} + U^-_{in}\right) \\ U^-_{out} &= U^-_{in} + \tilde{\rho} U^+_d \\ U^+_{out} &= U^+_{in} + \tilde{\rho} U^+_d \end{aligned}$$

This system can be rewritten in a vector form similar to (20) where the scattering matrix is constructed as in (21), with and $\tilde{Q}(s)$ as in (42), and $\tilde{G}(s) = 2\tilde{\rho}(s)$ coupling the input $U_e(s)$ with the two rails of the DW.

The system is diagrammed in Fig. 6. The manipulation of the minus signs makes it convenient for restricting $u^+_d(t)$ to positive values only (as shown in the figure), corresponding to the

[4] https://ccrma.stanford.edu/~jos/pasp/-Mass_Termination_Model.html

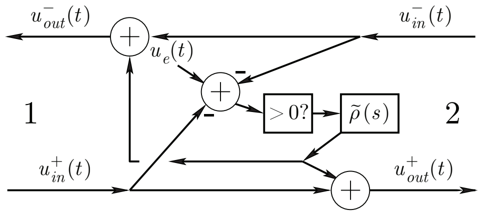

Figure 6: *Instantaneous spring displacement-wave scattering model driven by the spring edge $u_e(t) = u_f(t) + L$.*

finger/plectrum engaging the string. This uses the approximation $u_1(t) = u_2(t) \approx u^+_{in}(t) + u^-_{in}(t)$, which is exact when $\tilde{\rho} = 0$, i.e., when the finger/pick does not affect the string displacement at the current time. Similarly, $u^+_d(t) > f_{\max}/K$ can be used to trigger a release of the string from the finger/plectrum. After a release, a bit of state is needed to inhibit further engagement of the string and plectrum until plectrum "comes back around". For example, if only "down picks" are supported, then engagement can be suppressed after a release until $u_e(t)$ comes back down below the envelope of string vibration (e.g., $u_e(t) < -u_{\max}$). On the other hand, intermittent disengagements as a plucking cycle begins are normal; there is often an audible "buzzing" or "chattering" when plucking an already vibrating string.

5.2. Finger Damping

To add damping R to the finger-flesh model, the load impedance (39) becomes instead

$$\tilde{R}(s) = \frac{K}{s} + R.$$

That is, the spring K and its damping R are formally in "series" because they share a common velocity, so that their impedances sum. The corresponding force reflectance is then

$$\tilde{\rho}(s) = \frac{[\tilde{R}(s)+r]-r}{[\tilde{R}(s)+r]+r} = \frac{Rs+K}{(R+2r)s+K} = \frac{R}{R+2r}\,\frac{s+\frac{K}{R}}{s+\frac{K}{R+2r}}. \tag{44}$$

Thus, in addition to a single real pole at $s = -K/(R+2r)$, which is more damped than the previous pole at $s = K/(2r)$, we now have a zero at $s = -K/R$, farther from the frequency axis than the pole, and formerly at infinity.

The scattering matrix has the form (21) with

$$\tilde{Q}(s) = 1 - 2\tilde{\rho}(s) = \frac{(-R+2r)s-K}{(R+2r)s+K} = \frac{2r-R}{2r+R}\,\frac{s+\frac{K}{(2r-R)}}{s+\frac{K}{2r+R}}, \tag{45}$$

which corresponds to (35) when $\Delta = 0$ (using $K_0/c = r$).

In addition to being a more realistic model, spring damping prevents the reflection coefficient from reaching magnitude 1 at any frequency. That means the string segments are never completely isolated from each other, which has led to discontinuity problems in prior work.

5.3. Digitization

Applying the bilinear transformation (25) to the reflectance (44) $\tilde{\rho}(s)$ (including damping) yields the following first-order digital reflectance filter:

$$\rho(z) = \frac{R}{R+2r}\,\frac{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + \frac{K}{R}}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + \frac{K}{R+2r}} = g\frac{1-\zeta z^{-1}}{1-pz^{-1}}$$

where

$$p = \frac{1-\frac{KT}{2(R+2r)}}{1+\frac{KT}{2(R+2r)}} \quad \text{(digital pole)} \tag{46}$$

$$\zeta = \frac{1-\frac{KT}{2R}}{1+\frac{KT}{2R}} \quad \text{(digital zero)} \tag{47}$$

$$g = \frac{1-p}{1-\zeta} \quad \text{(gain term)} \tag{48}$$

5.4. Feathering

Since the pluck model is linear, the parameters are not signal-dependent. As a result, when the string and spring separate, there is a discontinuous change in the reflection and transmission coefficients. In practice, it is useful to "feather" the switch-over from one model to the next [13]. In this instance, one appealing choice is to introduce a *nonlinear spring*, as is commonly used for piano-hammer models [14].[5] In such models, the layer of felt surrounding the wooden hammer-head is represented as a nonlinear spring with a compression equation of the form

$$f_K(u_d) = K u_d^p$$

where $p = 1$ for linear behavior, and generally $2 < p < 3$ for pianos.

The linearized spring constant is

$$K(u_d) = f'_K(u_d) = pKu_d^{p-1}$$

which, for $p > 1$, approaches zero as $u_d \to 0$. We see from (44) above that this reduces the reflectance to a frequency-independent reflection coefficient $\tilde{\rho} = R/(R+2r)$ resulting from the damping R that remains in the spring model. As a result, there is still a discontinuity when the spring disengages from the string.

The foregoing suggests a nonlinear tapering of the damping R as well as the stiffness K as the spring compression approaches zero. A natural choice would be

$$R(u_d) = pRu_d^{p-1}$$

so that $R(u_d)$ approaches zero at the same rate as $K(u_d)$. It would be interesting to estimate p for the spring and damper from measured data. In the absence of such data, $p = 2$ is easy to compute (requiring a single multiplication). More generally, an interpolated lookup of u_d^p values can be used.

In summary, the engagement and disengagement of the plucking system can be "feathered" by a nonlinear spring and damper in the finger-flesh/plectrum model.

[5] http://paws.kettering.edu/~drussell/Piano/NonlinearHammer.html

6. CONCLUSION

This paper introduces structurally passive models of the plucking action in guitar playing, where the player's finger (or plectrum) is modeled as a damped mass-spring system. A passive version of a previously presented non-passive model for the pluck interaction is provided. The model was further extended, both in PDE and impedance formulations, to allow for the introduction of finger compliance, which is further generalized to a nonlinear system. The passive structure has the advantage of not requiring signal dependent scaling for its use in limited-level-range or fixed-point applications.

Sound examples for the techniques illustrated can be found at `http://staffwww.itn.liu.se/~giaev/soundexamples.html`. Future work will include parameter estimation and evaluation of the presented models relative to real-life mechanical plucking.

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