PDE-SVD BASED AUDIO DENOISING

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ABSTRACT

In this paper we present a new method for denoising audio signals. The method is based on the Singular Value Decomposition (SVD) of the frame matrix representing the signal in the Overlap Add decomposition. Denoising is performed by modifying both the singular values, using a tapering model, and the singular vectors of the representation, using a nonlinear PDE method. The performance of the method is evaluated and compared with denoising obtained by filtering.

Index Terms— Denoising, Audio Restoration, Speech Enhancement, Singular Value Decomposition, Partial Differential Equations

1. INTRODUCTION

Denoising is an important component of audio restoration systems that aims at recovering sound documents recorded in the past. Noise is introduced by the original analog recording systems, by transfer to other storage media and by the ageing of the media. Denoising is relevant for speech communication in noisy environments to improve intelligibility and enhance quality.

Several approaches have been proposed for audio and speech denoising in time, frequency and time-frequency domains [1, 2, 3, 4]. In this paper, we present an approach based on a time domain (TD) matrix representation of the signal. By means of a sliding window, the noisy signal is decomposed into fixed length partially overlapping frames that are collected in a matrix. The signal matrix is exactly represented by means of the Singular Value Decomposition (SVD) [5]. An approach to directly denoising the signal by retaining the principal components (lower rank from highest singular values) in the SVD proved unfruitful since the signal appears highly degraded even though some noise reduction occurs.

In the effort to reduce the contribution of the noise energy without destroying the signal, the singular values are tapered according to a threshold and a model for the energy decay of the high order singular values, which are less relevant for the signal.

The singular vectors, which contain traces of the time domain contribution of the noise, are denoised by means of a Partial Differential Equation (PDE) method, which is inspired by a denoising technique previously applied by the authors to image signals [6]. The PDE stems from a minimization problem attempting to obtain singular vectors closest to the unknown signal singular vectors, with a penalty term ensuring smooth gradient. Both linear and nonlinear solution methods are considered, with the nonlinear case providing best results at an increased computational cost.

The results show good performance of the algorithm in terms of objective and subjective measures. In our preliminary tests, an increase of SNR is achieved when the signal is highly corrupted, i.e. when the original SNR is in the 0 dB or negative range, which makes the method suitable for denoising highly degraded signals.

The paper is organized as follows. In Section 2 we review PDE based methods for denoising, in Section 3 we describe the SVD denoising approach, in Section 4 we show the results of some of our tests. Finally, in Section 5 we draw our conclusion and discuss further improvements.

2. DENOISING WITH PDE

It is by now well known that Partial Differential Equations (PDE) can be used to denoise signals. For methods for denoising sounds using PDE see, e.g., [7], [8], [2] and [9]. For techniques for denoising images see [10], [11], [12] and [13]. Recently, Baravdish and Svensson [6], have proposed a method to denoise images based on nonlinear PDE.

In this paper we are going to use methods similar to those in [6] to denoise two dimensional data derived from sound, as briefly described in Section 2.

The PDEs considered in our method are special cases of evolution equations. For a background on this topic see [14].

2.1. PDE Methods

Assume that u is a bidimensional clean original data and u_0 is a noisy version of u such that $u_0 = u + n$ where n is the noise. To denoise the data u_0 one minimizes the functional

$$J(u) = \int_{\Omega} \left(\frac{1}{2} (u - u_0)^2 + \lambda \Phi(|\nabla u|) \right) d\mathbf{x}, \quad \lambda > 0$$

*Thanks to the Scholar Rescue Fund agency for funding and Linköping University for generous hospitality

where Φ is a strictly convex function. In the energy functional J , the first term measures the fidelity to the noisy data and the second term imposes a smoothness condition on the clean data u . The Euler-Lagrange equation associated with the energy functional J is the partial differential equation (PDE) $\frac{\partial J}{\partial u}$ with Neumann boundary condition zero on the boundary of Ω .

By introducing a “time” variable, we instead solve a time dependent PDE $u'_t = -\frac{\partial J}{\partial u}$ with the noisy data u_0 as initial data and with zero Neumann boundary condition. Assume that $\Phi(|\nabla u|) = \frac{1}{p}|\nabla u|^p$, $1 \leq p < \infty$. In the case that the exponent $p = 2$ we get the heat equation and for $p = 1$ we get the total variation approach of Rudin, Osher and Fatemi [13]. Running the time dependent PDEs forward in time gives a cleaner data than the initial noisy version u_0 . Thus, we need a stopping time T to stop the process.

2.2. Inverse PDE

In the method in [6], it is assumed that the given degraded data u_0 is given at a later time T and the original data u is sought for at time $t = 0$. This is an inverse problem, which is backward in time. The solution is obtained by considering a sequence of well-posed problems. Each problem is a forward version where the initial data is chosen properly to reconstruct the original data u . The aim of this approach is twofold. By solving the PDE forward in time smoother data is obtained which is cleaner and the choice of the initial data is such that the data also get de-blurred.

Hence, instead of solving the forward problem, the following inverse problem is solved,

$$\begin{cases} \partial_t u - \operatorname{div}(|\nabla u|^{p-2} \nabla u) &= 0 & \Omega \times (0, T) \\ \partial_n u(\mathbf{x}, t) &= 0 & \partial\Omega \times (0, T), \\ u(\mathbf{x}, 0) &= \varphi(\mathbf{x}) & \mathbf{x} \in \Omega \end{cases} \quad (1)$$

where u and φ are unknown and an additional data

$$u(\mathbf{x}, T) = \psi \quad (2)$$

is given. The idea here is that the noisy data $u(\mathbf{x}, T) = \psi$ is given at a later time and we want to reconstruct the original data at an earlier time: $u(\mathbf{x}, 0) = \varphi$. There exists a nonlinear operator A such that the inverse problem in Eq. (1) and (2) can be reduced to a nonlinear operator equation

$$A(\varphi) = \psi. \quad (3)$$

There are by now several methods of solving the operator equation Eq. (3). One of them is the nonlinear Landweber method which starts with an arbitrary φ_0 :

$$\varphi_{k+1} = \varphi_k - A'^*(A\varphi_k - \psi) \quad (4)$$

where A'^* is the adjoint Frechet derivative of A . For the convergence rates and results on this method, see Engl, Hanke

and Neubauer [15]. To solve the problem in (1) and (2), the following iterative method was proposed in [6]. Let $\varphi_0 \in L^2(\Omega)$ be arbitrary. Assume that u_k has been constructed. Then we proceed to solve the linear adjoint problem

$$\begin{cases} \partial_t v_k + \operatorname{div}(L(|\nabla u_k|) \nabla v_k) &= 0, & \Omega \times (0, T) \\ \partial_n v_k(\mathbf{x}, t) &= 0, & \partial\Omega \times (0, T) \\ v_k(\mathbf{x}, T) &= u_k(\mathbf{x}, T) - \psi(\mathbf{x}), & \mathbf{x} \in \Omega \end{cases}$$

where

$$L_\varepsilon(|\nabla u|) = |\nabla u|^{p-2} I + (p-2)|\nabla u|^{p-4} \nabla u \nabla u^t.$$

For $k+1$ solve

$$\begin{cases} \partial_t u_{k+1} - \operatorname{div}(|\nabla u|^{p-2} \nabla u) &= 0, & \Omega \times (0, T) \\ \partial_n u_{k+1}(\mathbf{x}, t) &= 0, & \partial\Omega \times (0, T) \\ u_{k+1}(\mathbf{x}, 0) &= u_k(\mathbf{x}, 0) - v_k(\mathbf{x}, 0), & \mathbf{x} \in \Omega \end{cases}$$

Notice that for obvious numerical reasons, the $|\nabla u|$ is considered as a regularized version $\sqrt{|\nabla u|^2 + \delta^2}$, where δ is some fixed small number. This scheme is in analogue with the one given in Eq. (4).

3. OPTIMIZED PDE-SVD DENOISING

The proposed technique aims to reduce additive random noise present in the signal. It is based on the singular value decomposition (SVD) of the matrix representing the noisy signal in the time domain (TD). In our approach the noisy signal is transformed into a matrix of partially overlapping regularly spaced windowed signal segments.

By thresholding, the singular value matrix can be partitioned into two blocks, pertaining to two subspaces, the signal subspace and the noise subspace. Clearly, the connotation of signal and noise subspace is not strict, the signal subspace is the one that mostly contains the signal and the noisy subspace is the one that mostly contains noise. Since the singular values are intrinsic properties of the matrices, part of the denoising strategy is to reobtain the singular values of the original “clean” signal. The aim is to subtract from the singular values of the noisy signal the estimated singular values of the noise. We perform this operation on higher index singular values where the contribution of the signal is smaller.

The singular values pertaining to the noise subspace are tapered using an algorithm given in this section. Denoising is refined by smoothing the matrices of the singular vectors in the SVD representation using the inverse PDE method in [6]. The denoised signal is reconstructed using the smoothed singular vectors and the altered singular values.

Here below we describe the method in further detail.

3.1. Framing the noisy signal into a matrix

We divide the signal into M frames each of length N with half-length overlap. To each of these frames we apply a time

window, e.g. the von Hann or the rectangular window. The $M \times N$ matrix A_n is the collection of the windowed signal frames each arranged in a different row.

3.2. Singular value decomposition (SVD)

The SVD of the $M \times N$ matrix A_n is given by

$$A_n = U \Sigma V^T \quad (5)$$

where U , an $M \times M$ matrix, and V , an $N \times N$ matrix, are orthogonal matrices, and Σ is an $M \times N$ diagonal matrix of singular values (SVs) ($\Sigma_{ij} = 0$, if $i \neq j$ and $\Sigma_{11} \geq \Sigma_{22} \geq \dots \geq 0$). We will denote Σ_{ii} by σ_i .

The columns of the orthogonal matrices U and V are called the left and right singular vectors, respectively. The singular values $\{\sigma_i\}$ represent the importance of individual singular vectors in the composition of the matrix. In other words, larger singular values have more information about the structure of patterns embedded in the matrix than the smaller singular values.

3.3. Splitting Σ by thresholding

Using a threshold index τ the singular value matrix can be partitioned into two blocks, Σ_s and Σ_n as follows:

$$\Sigma = \begin{pmatrix} \Sigma_s & 0 \\ 0 & \Sigma_n \end{pmatrix}. \quad (6)$$

The threshold index τ can be computed as the center of mass for the singular values, defined by

$$\tau = \frac{\sum k \sigma_k}{\sum \sigma_k}.$$

The block Σ_n contains the singular values σ_k such that $k \geq M' = \lfloor \tau \rfloor$.

In Figure 1(a) a plot of the clean and noisy singular values is shown, together with the result of thresholded constant tapering. There, the solid line represents the singular values of the clean signal, the broken line the singular values of the noisy signal and the dotted line the tapered singular values used for denoising. The vertical segment shows the position of the center of mass threshold.

The center of mass choice for the tapering threshold proved to be too conservative in our tests. A certain quantity of noisy singular values still persists after tapering. For this reason we resorted to a threshold value computed as a fraction of the center of mass. In Figure 1(b), the effect of selecting the tapering threshold as 10% of the center of mass is shown, with the same line notation as in Figure 1(a). It is clear that the tapered values in this case better follow the clean singular values curve.

(a)

(b)

Fig. 1. Examples of thresholding and constant tapering of the singular values: (a) Center of mass choice for the threshold; (b) 10% of the center of mass choice for the threshold.

3.4. Modifying Σ_n

In this section we will try to reduce the contribution of noise to the singular values in Σ_n in (5). We denote by $\sigma_{n,1} \geq \sigma_{n,2} \geq \dots \geq 0$ the singular values in Σ_n .

Here we are interested in modifying the singular values in the following manner: $\sigma_k^* = \alpha \sigma_{n,k}$, where α is any positive function of k .

If we replace the noisy singular values in Σ_n by the denoised values σ_k^* , we obtain another diagonal matrix Σ_n^* . Substituting for this matrix in (5) yields the following estimate for the singular value matrix of the clean signal Σ_c :

$$\Sigma_d = \begin{pmatrix} \Sigma_s & 0 \\ 0 & \Sigma_n^* \end{pmatrix}. \quad (7)$$

The MSE measure for the SVs gives

$$\begin{aligned} \|\Sigma_c - \Sigma_d\|^2 &= \sum_1^{M'-1} |\sigma_k^c - \sigma_k^*|^2 + \sum_{M'}^M |\sigma_k^c - \sigma_k^*|^2 \\ &= \sum_1^{M'-1} |\sigma_k^c - \sigma_k^*|^2 + \sum_{M'}^M |\sigma_k^c - \alpha \sigma_{n,k}|^2. \end{aligned} \quad (8)$$

In the case of constant α , the right hand side in (8) is minimized by choosing

$$\alpha = \frac{\sum_{k=M'}^M \sigma_k^c \cdot \sigma_{n,k}}{\sum_{k=M'}^M (\sigma_{n,k})^2}. \quad (9)$$

The value of α in (9) yields a lower upper bound on the MSE in our proposed method for the case when σ_k^c is known.

In the case of unknown clean signal we proceed from (9) as follows. By using the Cauchy-Schwarz inequality we obtain the estimate

$$\alpha = \frac{\sum_{k=M'}^M \sigma_k^c \cdot \sigma_{n,k}}{\sum_{k=M'}^M (\sigma_{n,k})^2} \leq \frac{\left(\sum_{k=M'}^M (\sigma_k^c)^2 \right)^{1/2} \left(\sum_{k=M'}^M (\sigma_{n,k})^2 \right)^{1/2}}{\sum_{k=M'}^M (\sigma_{n,k})^2}$$

Since the additive noise increases the energy then $\sum_{k=M'}^M (\sigma_k^c)^2 < \sum_{k=M'}^M (\sigma_{n,k})^2$. The choice of α is therefore limited to the interval $0 < \alpha < 1$.

3.5. Denoising the Orthogonal Matrices

We use the nonlinear inverse PDE method to reduce the noise present in the two orthogonal matrices U and V in (2.2), thus obtaining an estimate for the clean matrices U_d and V_d . These matrices are normalized in order to limit energy change due to the PDE denoising procedure. After denoising Σ , U and V , we get the enhanced TD matrix as follows:

$$\hat{A} = U_d \Sigma_d V_d^T \quad (10)$$

3.6. Summary of the method

The proposed noise reduction technique can be summarized as follows:

1. Use windowing in TD to transform the noisy signal s_n into a matrix A_n whose rows host the partially overlapping signal segments.
2. Represent the matrix A_n with the SVD, obtaining $A_n = U \Sigma V^T$.
3. Split Σ into signal and noise blocks according to the threshold derived in Section 3.3. Modify the noise block according to the method in Section 3.4.

4. Denoise U and V by the PDE method in the Section 2 and normalize matrices.
5. Compose the modified SVs and normalized denoised matrices to obtain the enhanced TD matrix $\hat{A}$ as in (10).
6. The noise reduced signal s_d is obtained from the matrix $\hat{A}$ by Overlap Add.

4. PERFORMANCE EVALUATION

In order to test our method we performed tests on a variety of sounds from speech and music, corrupting them with additive gaussian noise. We use the sampling rate 16 kHz for speech and 44.1 kHz for music.

The time domain signal frame matrix is obtained by sliding a length 512 samples rectangular window over the signal, using half length overlap. After denoising, the signal is reconstructed by overlap adding the PDE-SVD denoised frames using the von Hann window.

For the parameters in the PDE we used times T in the order of 10^{-2} and a value $p = 1.6$ for the exponent (see Section 2). These values were selected after some testing on performance of the PDE denoising procedure.

As quality measures, we use the Mean Square Error (MSE), defined as

$$MSE = \frac{1}{N} \sum_{i=1}^N (s_c(i) - s_d(i))^2$$

and the Signal to Noise Ratio (SNR)

$$SNR = 20 \log_{10} \left(\frac{\|s_d\|}{\|s_d - s_c\|} \right)$$

where s_c is the original signal and s_d is the denoised signal, with $\|s\|^2 = \sum_i s(i)^2$. We also considered the PEAQ measure [16], however it did not seem to provide sufficient information.

In Table 1 an excerpt of our tests is shown, where three signals are considered: two speech signals “thanks” and “welcome” and a rock music sample “music”. We compare our PDE-SVD method with the Savitzky-Golay (SG) filtering [17] (last column to the right) for several values of the SNR ranging from 10 dB down to -5 dB (highly corrupted signal), as reported in the first column from the left. For the PDE-SVD method we consider different variants for the singular values matrix: center of mass thresholding, with constant tapering α , shown in the second column from the left (CM), a less conservative thresholding calculated as 10% of the center of mass shown in the third column (10% CM) and, as a reference, the case where in the PDE-SVD method we assume the clean singular values to be known, i.e. where only the singular vectors are denoised, shown in fourth column from the left. The results concerning the MSE are not reported in the

table since they lead to similar conclusions as those for the SNR.

SG filtering performed better than the empirical Wiener filter in both objective and subjective tests; for this reason it was retained as a benchmark in the table. The results show that the PDE-SVD method outperforms the SG filtering in most cases, with near equivalence in the very low SNR case. Additionally, as the known singular values case shows, the PDE-SVD method is still subject to improvement deriving from the application of a more refined strategy in the estimation of the clean singular values from the noisy ones. Moreover, extensive optimization of the PDE denoising parameters and model is bound to improve performance, which will be the object of further work.

In our listening tests we concluded that the background noise was clearly reduced and the speech signal is more intelligible. The bandwidth of the denoised music signals is wide and details are better preserved compared to SG filtering.

Sound	SNR	CM	10% CM	Known SV	SG
speech "thanks"	10	12.2	12.4	13.5	11.9
	5	7.5	9.5	9.6	8.1
	0	3.7	6.1	7.2	4.5
	-5	1.4	2.4	4.1	2.6
speech "welcome"	10	11.7	11.9	12.7	9.9
	5	7.0	8.2	9.0	7.2
	0	3.4	4.4	5.8	4.2
	-5	1.3	1.7	3.1	1.9
music "rock"	10	11.9	12.7	12.8	12.9
	5	7.2	9.0	9.3	8.6
	0	3.5	5.1	6.3	4.8
	-5	1.3	2.0	3.9	2.2

Table 1. Results of the denoising tests on signals speech and music at different SNR (in dB).

5. CONCLUSIONS AND DISCUSSIONS

In this paper we proposed a new approach for reducing the noise from signal in which the signal is represented by the SVD of the frame matrix. We enhanced the singular values by tapering and the singular vectors of the noisy signal by means of a nonlinear inverse PDE method. We compared our method with Savitzky-Golay filter in terms of MSE and SNR. The results show that our method has good performance in signal noise reduction.

Our results can be further improved by considering a tighter approach to the estimation of the singular values of the clean signal from the singular values of the noisy signal. A probabilistic approach is currently under study. Further improvements will also derive from the adaptation and optimization of the PDE model using a database of sound for benchmarking.

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