The Double Layer Potential Operator
Through Functional Calculus

Michail Krimpogiannis
Abstract

Layer potential operators associated to elliptic partial differential equations have been an object of investigation for more than a century, due to their contribution in the solution of boundary value problems through integral equations.

In this Licentiate thesis we prove the boundedness of the double layer potential operator on the Hilbert space of square integrable functions on the boundary, associated to second order uniformly elliptic equations in divergence form in the upper half-space, with real, possibly non-symmetric, bounded measurable coefficients, that do not depend on the variable transversal to the boundary. This uses functional calculus of bisectorial operators and is done through a series of four steps.

The first step consists of reformulating the second order partial differential equation as an equivalent first order vector-valued ordinary differential equation in the upper half-space. This ordinary differential equation has a particularly simple form and it is here that the bisectorial operator corresponding to the original divergence form equation appears as an infinitesimal generator.

Solving this ordinary differential through functional calculus comprises the second step. This is done with the help of the holomorphic semigroup associated to the restriction of the bisectorial operator to an appropriate spectral subspace; the restriction of the operator is a sectorial operator and the holomorphic semigroup is well-defined on the spectral subspace.

The third step is the construction of the fundamental solution to the original divergence form equation. The behaviour of this fundamental solution is analogous to the behaviour of the fundamental solution to the classical Laplace equation and its conormal gradient of the adjoint fundamental solution is used as the kernel of the double layer potential operator. This third step is of a different nature than the others, insofar as it does not involve tools from functional calculus.

In the last step Green’s formula for solutions of the divergence form partial differential equation is used to give a concrete integral representation of the solutions to the divergence form equation. Identifying this Green’s formula with the abstract formula derived by functional calculus yields the sought-after boundedness of the double layer potential operator, for coefficients of the particular form mentioned above.
Acknowledgments

First of all I would like to thank my main supervisor Andreas Rosén. Secondly, I would like to thank my assistant supervisor Bengt Ove Turesson. I would also like to thank the Department of Mathematics as a whole, as well as the sovereign state of Sweden.

To paraphrase some very famous liner notes, if I had to thank all the people who have been cool to me in the 3 years I spent in Linköping, there wouldn’t be room for the thesis (sic). One or more raised glasses to: fellow students; the (2- & 4-legged) Edgar family; friends, family (including pets) & stray dogs back home; last but not least the Metal Hammer & Heavy Metal magazine (8♣ - hire Dr. D!).

Μιχαήλ Κριμπογιάννης
Michail Krimpogianannis
September 2012, Linköping
## Contents

1 Introduction ........................................................................................................................................... 1

2 The Functional Calculus .......................................................................................................................... 9
   2.1 The functional calculus for bisectorial operators ................................................................................. 9
   2.2 Quadratic estimates .......................................................................................................................... 19
   2.3 The equation $\text{div} A \nabla u = 0$ ........................................................................................................ 31
       2.3.1 Construction of solutions to equation (2.19) using $BD$ ......................................................... 44

3 The Fundamental Solution ...................................................................................................................... 49
   3.1 Construction and interior estimates ................................................................................................. 49
       3.1.1 The averaged fundamental solution ..................................................................................... 53
       3.1.2 Proof of Theorem 3.1.1 and Corollary 3.1.2 ........................................................................ 61
   3.2 Estimates on the boundary ........................................................................................................... 63

4 The Boundedness of the Double Layer Potential for Real Non-Symmetric Coefficients ............... 69
   4.1 Green’s formula ............................................................................................................................. 69
   4.2 The main result ............................................................................................................................. 76
In this section we formulate the Dirichlet boundary value problem for the Laplace operator in a bounded domain in $\mathbb{R}^{1+n}$, which we then solve with the help of the double layer potential operator. This should illuminate the importance of the double layer potential in the theory of boundary value problems for partial differential equations. The computations that follow are formal. For a more systematic treatment and rigorous proofs, we refer to [34, Chapter 1], [33, Section 3.3] and [40, Chapter 6].

Let $D \subset \mathbb{R}^{1+n}$ be a bounded domain with smooth boundary $\partial D$. The (interior) Dirichlet boundary value problem is the following:

$\text{(DIR)}$ Given a $u_0 : \partial D \to \mathbb{R}$, find a function $u : \mathcal{D} \to \mathbb{R}$, such that

\[
\begin{align*}
\triangle u &= 0, \quad \text{in} \quad D \\
u &= u_0, \quad \text{on} \quad \partial D,
\end{align*}
\]

where $\triangle = \sum_{j=0}^{n} \partial^2_{x_j}$ is the Laplace operator in $1+n$ variables.

\[\text{Figure 1.1: The Dirichlet boundary value problem.}\]
Recall that the fundamental solution to $\triangle$ in $\mathbb{R}^{1+n}$ is given by

$$
\Gamma(y;x) := \begin{cases} 
-\frac{1}{(n-1)\sigma_n |y-x|^{n-1}}, & \text{for } n \geq 2, \\
\frac{1}{2\pi \ln|y-x|}, & \text{for } n = 1,
\end{cases}
$$

where $\sigma_n$ stands for the area of the unit sphere $S^n$ in $\mathbb{R}^{1+n}$. For fixed $x \in \mathbb{R}^{1+n}$

$$
\nabla_y \Gamma(y;x) = \frac{y-x}{\sigma_n |y-x|^{1+n}} \quad \text{and} \quad \nabla_y \cdot \Gamma(y;x) = \text{div}_y \nabla_y \Gamma(y;x) = \delta_y(x),
$$

where $\delta_y(\cdot)$ denotes the Dirac delta distribution with pole at $x$. Thus, $\Gamma(\cdot;x)$ is a harmonic function in $\mathbb{R}^{1+n} \setminus \{x\}$.

For sufficiently smooth functions $u, w$ defined on $\overline{D}$, Green’s second identity holds, namely

$$
\int_D (u(y) \nabla_y w(y) - w(y) \nabla_y u(y)) \, dy = \int_{\partial D} (u(y) \partial_\nu w(y) - w(y) \partial_\nu u(y)) \, d\sigma(y), \quad (1.1)
$$

where $\nu(y)$ is the unit normal vector at the point $y$ on the boundary $\partial D$, directed into the exterior of $D$, $\partial_\nu$ is the outward normal derivative and $d\sigma$ is the surface measure on the boundary. Thus, assuming that $u$ is harmonic in $D$, equal to $u_0$ on $\partial D$ and using $\Gamma$ as $w$, (1.1) becomes

$$
\int_D (u(y) \delta_y(y) - 0) \, dy = \int_{\partial D} (u_0(y) \nu(y) \cdot \nabla_y \Gamma(y;x) - \Gamma(y;x) \partial_\nu u_0(y)) \, d\sigma(y),
$$

where $x \in D$ and "\cdot" stands for the ordinary Euclidean inner product in $\mathbb{R}^{1+n}$. Thus, using the properties of Dirac’s delta, the function $u$ can be written as

$$
u(x) = \int_{\partial D} (u_0(y) \nu(y) \cdot \nabla_y \Gamma(y;x) - \Gamma(y;x) \partial_\nu u_0(y)) \, d\sigma(y), \quad x \in D. \quad (1.2)
$$

This representation formula (also known as Green’s formula, see [40, Theorem 6.5]), is
the analogue for harmonic functions of Cauchy-Pompeiu’s integral formula encountered in the theory of analytic functions, see [3, Chapter 1]. It shows that it is possible to reconstruct a harmonic function using the fundamental solution and the Dirichlet and Neumann boundary data, i.e. $u = u_0$ on $\partial D$ and $\partial_\nu u$ equal to some other given function on $\partial D$, respectively.

Since we are only interested in the Dirichlet problem, it is possible, by omitting the term involving the normal derivative of $u$ in (1.2), to use the ansatz

$$
u(x) := \int_{\partial D} \nu(y) \cdot \nabla_y \Gamma(y;x) h(y) \, d\sigma(y), \quad x \notin \partial D, \quad (1.3)
$$

for the solution, where $h : \partial D \to \mathbb{R}$ is some auxiliary function belonging to a suitable function space on the boundary. This function is called the double layer potential of $h$. The function $h$ is referred to as the density of the double layer potential. This gives rise
to a linear integral operator acting on suitable function spaces $X(\partial D)$ on the boundary of the domain $D$

$$K = K_t : X(\partial D) \longrightarrow X(\partial D) : h \longmapsto Kh,$$

(1.4)

where $Kh$ is given by the right-hand-side of (1.3).

The function

$$D \ni x \longmapsto \nabla \Gamma(y; x) \cdot \nu(y) \in \mathbb{R},$$

is harmonic for each fixed $y$. Thus, after differentiating under the integral sign, it is seen that $\Delta u(x) = 0$, for all $x \in D$. Therefore, (1.3) will be a solution to the Dirichlet problem, provided the auxiliary function $h$ is chosen in a way such that $u|_{\partial D} = u_0$. In order to do this, it is necessary to study the behavior of the double layer potential on the boundary. Let $x_0 \in \partial D$ and let $h$ be sufficiently smooth at $x_0$, so that the function

$$\partial D \ni y \longmapsto \nabla \Gamma(y; x_0) \cdot \nu(y) (h(y) - h(x_0)),$$

that has a singularity at $y = x_0$ is integrable on $\partial D$. Let $x \in D$, such that $x = x_0 + r\nu(x_0)$, for some parameter $r \in \mathbb{R}$. Then, by Lebesgue’s dominated convergence theorem, it follows that

$$\int_{\partial D} \nabla \Gamma(y; x_0 + r\nu(x_0)) \cdot \nu(y) (h(y) - h(x_0)) \, d\sigma(y) \to \int_{\partial D} \nabla \Gamma(y; x_0) \cdot \nu(y) (h(y) - h(x_0)) \, d\sigma(y),$$

as $r \to 0$. Write

$$\int_{\partial D} \nabla \Gamma(y; x_0) \cdot \nu(y) (h(y) - h(x_0)) \, d\sigma(y) = \lim_{\varepsilon \to 0} \int_{\partial D \setminus B(x_0; \varepsilon)} \nabla \Gamma(y; x_0) \cdot \nu(y) h(y) \, d\sigma(y)$$

$$- \lim_{\varepsilon \to 0} \int_{\partial D \setminus B(x_0, \varepsilon)} \nabla \Gamma(y; x_0) \cdot \nu(y) h(x_0) \, d\sigma(y)$$

$$+ \lim_{\varepsilon \to 0} \int_{\partial D \setminus B(x_0, \varepsilon) \setminus D} \nabla \Gamma(y; x_0) \cdot \nu(y) h(x_0) \, d\sigma(y)$$

$$= I_1 + I_2 + I_3.$$

The integral $I_1$ is nothing other than a Cauchy principal value integral (see, for example, [53, Chapter II]), written

$$\text{p.v.} \int_{\partial D} \nabla \Gamma(y; x_0) \cdot \nu(y) h(y) \, d\sigma(y).$$

The divergence theorem for the vector field $y \mapsto \nabla \Gamma(y; x_0) h(x_0)$ and the fact that $\Gamma$ is the fundamental solution of the Laplacian, yield $I_2 = -h(x_0)$. Furthermore, $\nabla \Gamma(y; x_0) \cdot \nu(y) = 1/(\sigma_0 |y - x_0|^r)$, so for $y \in \partial B(x_0; \varepsilon)$, $\nabla \Gamma(y; x_0) \cdot \nu(y) = 1/(\sigma_0 \varepsilon^r) = 1/|\partial B(x_0; \varepsilon)|$. Accordingly

$$I_3 = \lim_{\varepsilon \to 0} \frac{1}{|\partial B(x_0, \varepsilon)|} \left| \frac{1}{\varepsilon^r} \int_{\partial B(x_0; \varepsilon)} h(x_0) \right| = \frac{1}{2} h(x_0).$$
since $\partial D$ has been assumed smooth. What is more, by (1.3) and the divergence theorem which yields $\int_{\partial D} \nabla \cdot \Gamma(y;x_0 + r\nu(x_0)) \cdot \nu(y) \, d\sigma(y) = 1$, it follows that

$$\int_{\partial D} \nabla \cdot \Gamma(y;x_0 + r\nu(x_0)) \cdot \nu(y) \left( h(y) - h(x_0) \right) \, d\sigma(y) = u(x) - h(x_0).$$

Thus, we end up with the following expression for the trace of the double layer potential on the boundary

$$\lim_{x \to \partial D} u(x) = \frac{1}{2} h(x_0) + \text{p.v.} \int_{\partial D} \nabla \cdot \Gamma(y;x_0) \cdot \nu(y) h(y) \, d\sigma(y).$$

The principal value double layer potential is the linear integral operator

$$K : X(\partial D) \longrightarrow X(\partial D) ; h \mapsto Kh,$$

where $Kh(x) := 2 \text{p.v.} \int_{\partial D} \nabla \cdot \Gamma(y;x_0) \cdot \nu(y) h(y) \, d\sigma(y)$, for $x \in \partial D$.

Taking everything into consideration, we reach the following two-step “algorithm”, known as the boundary integral equation method, for solving the Dirichlet problem:

(i) First, solve the linear integral equation $(h + Kh)/2 = g$ for $g \in X(\partial D)$.
(ii) Then, the function defined by $u(x) = \int_{\partial D} \nabla \cdot \Gamma(y;x_0) \cdot \nu(y) h(y) \, d\sigma(y)$ is the solution to the Dirichlet problem.

For (i), it is required to establish the boundedness of the operator $K$ on $X(\partial D)$ and the invertibility of the operator $I + K$ on $X(\partial D)$. Note that not only the choice of the function space $X(\partial D)$ (for example $C(\partial D)$, $C^{1,1}(\partial D)$, or $L^2(\partial D)$), but also the degree of smoothness of the boundary affect the outcome of the necessary investigations pertaining to (i).

As the following discussion indicates, it is possible to formulate the Dirichlet boundary value problem for unbounded domains, or domains with rather “bad” (non-smooth) boundaries. Consider a domain in $\mathbb{R}^{1+n}$, $n \geq 2$, that lies above the graph of a Lipschitz function, i.e. a domain $D = \{(t,x) \in \mathbb{R}^{1+n} : t > g(x)\}$, where $g : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function. Note that such a domain can be viewed as a “building block” of a bounded Lipschitz domain, see [54, Section 0], [28, Section 1.2.1].

Using the change of variables, sometimes called a Lipschitz diffeomorphism (see [5, Section 2])

$$\phi : \mathbb{R}^{1+n} \longrightarrow D ; (t,x) \longmapsto (t + g(x),x),$$

it is seen that the unbounded domain $D$ gets “pulled-back” to the upper half-space, namely $\mathbb{R}^{1+n} := \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t > 0\}$. Moreover, we see that the equation $\Delta u = 0$ in $D$ corresponds to the equation $\text{div}_A \nabla \tilde{u} = 0$ in $\mathbb{R}^{1+n}$, where

$$A = \begin{bmatrix} 1 + |\nabla g|^2 & -\langle \nabla g, g \rangle^t \\ -\langle \nabla g, g \rangle & I \end{bmatrix}$$

and $\tilde{u} = u \circ \phi : \mathbb{R}^{1+n} \longrightarrow \mathbb{C}$, as an application of the chain rule shows. The boundary conditions carry over from $\partial D$ to $\partial \mathbb{R}^{1+n}$ in the appropriate way: $u = u_0$ on $\partial D$ corresponds to $\tilde{u} = u_0 \circ \phi_0$ on $\mathbb{R}^n$, where

$$\phi_0 : \mathbb{R}^n \longrightarrow \partial D ; x \longmapsto (g(x),x).$$
Observe that the coefficient matrix $A$ is independent of the transversal coordinate $t$ and that it is real and symmetric. Such coefficients are referred to as being of Jacobian type, see [7, Section 1]. The following figure illustrates the situation.

![Figure 1.2: Flattening of the domain via the Lipschitz diffeomorphism $\phi^{-1} : D \to \mathbb{R}^{1+n} ; (t,x) \mapsto (t - g(x), x)$. Notice the phenomenon where “an easy equation in a difficult domain” corresponds to “a difficult equation in an easy domain”.

The main result

In this thesis we consider second order uniformly elliptic equations in divergence form, in the upper half-space, with $t$-independent and pointwise strictly accretive coefficients; i.e. equations of the form

$$\text{div}_{t,x} A(x) \nabla_{t,x} u(t,x) = 0, \quad (t,x) \in \mathbb{R}^{1+n}_{+},$$

where $n \geq 2$ (or, as is the case in Chapter 1, $n \geq 1$), and for almost every $x \in \mathbb{R}^n$, $A(x) = (A_{ij}(x))_{i,j=0}^n$ is a $(1+n) \times (1+n)$ accretive matrix with complex entries; see (2.19) in Section 2.3. We stress that apart from the $t$-independence, there are no smoothness or symmetry assumptions on the coefficients $A \in L^\infty(\mathbb{R}^n; M_{(1+n)}(\mathbb{C}))$. Thus, even when the coefficients are real-valued (as will be the case from Chapter 3 onwards), there is no guarantee that they will be of Jacobian type. For coefficients of Jacobian type, the $L^2(\partial D)$-solvability of the Dirichlet problem, without the use of layer potentials, was obtained in [18]. For general (i.e. not of Jacobian type) $t$-independent, real and symmetric coefficients, $L^2(\mathbb{R}^n)$-solvability of the Dirichlet problem was obtained in [35], without the use of layer potentials.

In [2] solvability of the Dirichlet problem for the upper half-space with square integrable boundary data was obtained through the use of double layer potentials, for $t$-independent, real and symmetric coefficients and small $L^\infty$-perturbations; see [2, Theorems 1.11 and 1.12]. The main result of this thesis, included in Theorem 4.2.1 and Corollary 4.2.2, pertains to the $L^2$-boundedness of the operator which generalises the double layer potential operator for non-symmetric coefficients. It reads as follows:

Let $n \geq 2$ and let $A \in L^\infty(\mathbb{R}^n; M_{(1+n)}(\mathbb{R}))$ be pointwise strictly accretive. Then there exists a positive constant $C$, depending on the ellipticity constant, the $L^\infty$-norm of $A$ and the dimension $n$, such that for all $h \in L^2(\mathbb{R}^n)$

$$\sup_{t>0} \left\| \int_{\mathbb{R}^n} (A^T(y) \nabla_{t,y} \Gamma^T(0,y,t,x) \cdot e_0) h(y) \, dy \right\|_{L^2(\mathbb{R}^n)} \leq C \|h\|_{L^2(\mathbb{R}^n)},$$
with strong convergence as \( t \to 0 \). In other words, there exists a linear operator

\[
L^2(\mathbb{R}^n) \ni h \mapsto \lim_{t \to 0} \int_{\mathbb{R}^n} (A^T(y)\nabla_{y,t} \Gamma^T(0,y,t,x) \cdot e_0) h(y) \, dy \in L^2(\mathbb{R}^n),
\]

which is bounded on \( L^2(\mathbb{R}^n) \). Here \( e_0 = (1,0,\ldots,0) \in \mathbb{R}^{1+n} \) stands for the unit vector pointing into the upper half-space, \( \Gamma^T(\cdot,\cdot;t,x) \) denotes the fundamental solution of \( \text{div}_{x,y} A^T(y)\nabla_{x,y} u(s,y) = 0 \) in \( \mathbb{R}^{1+n}_+ \), with pole at \((t,x)\), \( \cdot \cdot \cdot \) is the Euclidean inner product in \( \mathbb{R}^{1+n} \) and \( A^T \) stands for the adjoint of \( A \).

This result generalises [2, Theorem 1.12] insofar as the \( L^2 \)-boundedness of the double layer potential operator is concerned.

We remark that the theory of Chapters 2 and 3 goes through in the case of uniformly elliptic divergence form systems of equations as well (single equations with complex coefficients being a particular case of this situation), with mild modifications as far as the formalism is concerned but with an important extra hypothesis in Chapter 3. It is necessary to assume that both the operator \( L = -\text{div}A\nabla \) and its adjoint \( L^T = -\text{div}A^T \nabla \) satisfy certain De Giorgi-Nash-Moser estimates (see Section 3.1 and [31]), in order to construct the fundamental solution (called the fundamental matrix in [31]). This way it is possible to obtain the boundedness of the double layer potential operator for general systems for which De Giorgi-Nash estimates hold. In particular, this holds true for small \( L^\omega \)-perturbations of real scalar equations.

We mention that as this thesis was being written, Hofmann et alia showed in [32] that the double layer potentials associated to any \( t \)-independent operator \( L = -\text{div}A\nabla \) with real coefficients acting on scalar functions, and to its complex perturbations, are \( L^2 \)-bounded, see [32, Corollary 1.25]. Even though our result is subsumed in theirs, their methods of proof revolve around the harmonic measure, whereas ours use functional calculus. As already mentioned, our proof generalizes to cover the case of uniformly elliptic divergence form systems, unlike the one given in [32] which is limited to scalar equations and their small complex perturbations.

**History and known results**

The reduction of (elliptic) boundary value problems to boundary integral equations and the analysis of the latter have been studied intensively since the nineteenth century, see [34, Section 1.3.1], [38, Chapter XI]. A variety of methods have been developed and established alongside the effort to deal with (i); the theory of singular integrals (in order to make sense of principal value integrals, see [53]) and Fredholm theory (in order to recourse to the Fredholm alternative, see [34, Section 5.3], [40, Theorem 4.15]) being but two of them. For example, if we assume that \( D \) is a domain in \( \mathbb{R}^n \), with \( C^2 \)-boundary, then (DIR) can be solved for \( f \in C(\partial D) \) using Fredholm theory as in [19, Chapter 0]; this is because the kernel \( v(y) \cdot \nabla \Gamma(y;x) \) in (1.3) is weakly singular and the equation appearing in (i) is a Fredholm integral equation of the second kind, see [34, Chapter 1], [40, Chapter 7]. Note that the same technique can be applied when the boundary of the domain belongs to the Hölder class \( C^{1+\alpha} \), where \( \alpha > 0 \); however, the situation changes drastically for domains with less regular boundaries – even \( C^1 \), let alone Lipschitz; see [19, Chapter 0].
In [23] the Dirichlet (and Neumann) problem was treated for a bounded domain $D$ in $\mathbb{R}^n$, $n \geq 3$, with a $C^1$-boundary and with boundary datum in $L^p(\partial D)$ (with respect to the surface measure), $1 < p < \infty$. There the operator $K$ was shown to be compact on $L^p(\partial D)$, the operator $(I + K)/2$ was shown to be invertible on $L^p(\partial D)$ and the Dirichlet problem was shown to have a solution in the form of the double layer potential (see [23, Theorems 1.6, 2.1, 2.3]), essentially covering the steps (i) and (ii) in the aforementioned algorithm.

In [54] the operator $(I + K)/2$ was shown to be invertible on $L^2(\partial D)$, where $D$ is now a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$, and the Dirichlet problem was shown to have a solution in the form of the double layer potential for square integrable boundary datum (see [54, Theorem 3.1, Corollary 3.2]), so (i) and (ii) go through in this case as well. This happens in spite of the fact that, unlike on $C^1$-domains, the operator $K$ is not compact in this case, something which renders the Fredholm theory inapplicable. We should mention that both for Lipschitz and for $C^1$-domains, the boundary values of the harmonic function are attained as non-tangential limits almost everywhere. What is more, these developments are intimately related to A. P. Calderón’s result on the boundedness of the Cauchy integral on Lipschitz curves in the plane, with small Lipschitz constant, see [13]. This condition was removed in [14], where it was shown that the Cauchy integral is indeed an $L^p$-bounded operator for any Lipschitz domain, for $1 < p < \infty$. In [23] and [54] these deep results were used in an essential way. For more on boundary value problems on Lipschitz domains, see [19, Appendix 1].

It is worth mentioning that double layer potentials play an important rôle in the realm of mathematical physics. For example, they are strongly related to charge distributions on surfaces, see [17, Chapter IV], [38, Chapter VII]. In fact, the electric field at any point in space generated by electric charge distributions of opposite sign on two parallel surfaces is the vector field given by the gradient of the double layer potential. Double layer potentials also appear in the study of the direct obstacle scattering problem for acoustic waves, see [15]. Therefore, the study of double layer potentials is of interest from an applied viewpoint as well.

**Outline of the thesis**

In Chapter 2 we develop parts of the theory of the functional calculus of bisectorial operators in Hilbert spaces (Sections 2.1, 2.2). We then proceed to reformulate the second order equation (1.7) as a first order system and then as an evolution equation in the $t$-variable, involving a certain bisectorial operator $DB$, where $B$ is a bounded operator, related to $A$ via an explicit algebraic formula, and $D$ is a closed self-adjoint, but not positive, homogeneous first order differential operator with constant coefficients in the Hilbert space $L^2(\mathbb{R}^n; C^{1+n})$. Subsequently, the evolution equation is solved via functional calculus; semigroup theory in particular (Section 2.3). In addition we obtain estimates for these solutions. It is precisely the hypothesis that $A$ is $t$-independent that is necessary and sufficient for Section 2.3 to go through. Incidentally, the main reason we allow for complex-valued coefficients in this chapter is that operator theory is, generally, better facilitated over complex Banach spaces. Naturally, it comes as no surprise that functional calculus can be of assistance when solving partial differential equations; the Fourier transform being after all the “mother of all functional calculi” (see [30, Section 2.8]). At times, the contents of this chapter may feel rather algebraic in character. However, appearances may
be deceiving and this becomes apparent with the introduction of the concept of quadratic estimates in Section 2.2. Both the concept itself and the fact that the operators \( DB \) and \( BD \) satisfy quadratic estimates are deeply rooted in harmonic analysis. Quadratic estimates are inextricably intertwined with the boundedness of the functional calculus and the possibility to define the (bounded) operator \( \text{sgn}(BD) \), where \( \text{sgn} \) is equal to 1 on the right half-space and \(-1\) on the left half-space. The appropriate references are given as the chapter unfolds. We remark that the main players in this field have been A. McIntosh and his associates, see [1], [7], [45]. We also remark that these techniques are intimately related to the Kato square root problem, solved in [9]; see also Example 2.3.10.

Section 3.1 deals with the existence and the basic properties of the fundamental solution \( \Gamma \) to (1.7). Theorem 3.1.1 is the central point; the main reference here is [31]. For the construction of the fundamental solution it is sufficient to assume that \( n \geq 2 \) and that \( A \in L^\infty(\mathbb{R}^n; M_{1+\theta}(\mathbb{R})) \); i.e. that we are working in a three- or higher-dimensional space and that the coefficients are real. Insofar as the construction of the fundamental solution is concerned, whether or not the coefficient matrix is \( t \)-independent is, in contrast to the previous chapter, unimportant. Section 3.2 presents certain \( L^2 \)-estimates satisfied by the gradient of the fundamental solution on the boundary \( \partial \mathbb{R}^{1+n} = \mathbb{R}^n \). We remark that Proposition 3.2.2 contains some novelties in its proof as compared to [2]. This section makes use of the \( t \)-independence of \( A \). Sobolev trace theory (see, for example, [12, Chapter 9], [28, Section 1.5]) does not provide enough information about the \( L^2 \)-behaviour of the trace of \( \nabla \Gamma \) as a function on \( \mathbb{R}^n \). As far as the double layer potential operator is concerned, this fundamental solution is used in the same way as the standard fundamental solution to the Laplace equation: the conormal derivative of \( \Gamma \) comprises the kernel of the double layer potential integral operator. Observe that from this chapter onwards we restrict ourselves to real-valued coefficients.

In Chapter 4 we combine the results from Chapters 2 and 3 to obtain Green’s formula, from which the \( L^2 \)-boundedness of the double layer potential operator is obtained. Throughout this chapter \( n \geq 2 \) and \( A \) is both \( t \)-independent and real. The condition on the dimension and the realness of the coefficients permits us to use the fundamental solution which was constructed in Chapter 3, while the \( t \)-independence of \( A \) guarantees that the solutions to (1.7) constructed in Section 2.3 can be used. Green’s formula provides a representation formula for the solutions of (1.7) in the upper half-space and is proved in Section 4.1; In fact, Proposition 4.1.1 is the core of this section. The \( L^2 \)-boundedness of the double layer potential operator is obtained in Corollary 4.2.2 via Theorem 4.2.1. The double layer potential operator has now been identified with the normal-to-normal component (with respect to the splitting of vectors and matrices in normal and parallel components made in Section 2.3) of the operator \( \text{sgn}(BD) \). As already mentioned, this is the main result of this thesis and is found in the final section, namely Section 4.2.

Note that throughout this thesis, the “variable constant convention” is used; the symbol \( C \) stands for a generic constant which may well differ from occurrence to occurrence, even within the same formula.
2

The Functional Calculus

2.1 The functional calculus for bisectorial operators

In this section we introduce the class of bisectorial operators acting in a Hilbert space. Moreover, we introduce an appropriate notion of functional calculus for such operators and for a specific class of “nice” functions.

We start off by defining the following subsets of the complex plane.

Definition 2.1.1. Let $0 \leq \theta < \pi$. The closed $\theta$ sector is the set

$$S_{\theta^+} := \{ \zeta \in \mathbb{C} : z = 0 \text{ or } |\arg \zeta| \leq \theta \}.$$ 

For $0 < \nu < \pi$, the open $\nu$ sector is the set

$$S_{\nu^+} := \{ \zeta \in \mathbb{C} : \zeta \neq 0, |\arg \zeta| < \nu \}.$$ 

Let $0 \leq \omega < \pi/2$. The left closed sector is the set $S_{\omega^+} := -S_{\omega^+}$, while for $0 < \mu < \pi/2$ the left open sector is the set $S_{\mu^+} := -S_{\mu^+}$. The closed bisector is defined as $S_{\omega} := S_{\omega^+} \cup S_{\omega^-}$, whereas the open bisector is defined as $S_{\mu} := S_{\mu^+} \cup S_{\mu^-}$.

We are now in position to single out a particular class of closed operators in a Hilbert space $\mathcal{H}$, for which we would like to have a functional calculus. Two properties come into play. First, the location of the spectrum of the operators in the complex plane. Second, the bounds that their resolvent operators satisfy outside the spectrum.

Let $\mathcal{C}(\mathcal{H})$ denote the class of closed operators in $\mathcal{H}$. $R_T(\zeta)$ stands for the resolvent operator $(\zeta I - T)^{-1}$; we write $\sigma(T)$ for the spectrum of $T$ and we assume throughout that the resolvent set $\rho(T)$ is non-empty. Recall that if $T \notin \mathcal{B}(H)$, then $\infty \in \sigma(T) \subset \mathbb{C} \cup \{\infty\}$ by default. For the appropriate background, consult, for example, [1, Section C], [30, Appendices A,C], [51].

Definition 2.1.2. Let $0 \leq \omega < \pi/2$. An operator $T : \mathcal{D}(T) \to \mathcal{H}$ is called an $\omega$-bisectorial operator (or an operator of type $S_{\omega}$) whenever the following three properties hold
(i) $T \in \mathcal{C}(\mathcal{H})$,

(ii) $\sigma(T) \subset S_\omega \cup \{\infty\}$, and

(iii) for all $\mu \in (\omega, \pi/2)$, there exists a positive constant $C = C(\mu)$ such that, for all non-zero $\zeta \in \mathbb{C} \setminus S_\mu$

$$\|RT(\zeta)\|_{\mathcal{H} \rightarrow \mathcal{H}} \leq \frac{C \mu}{|\zeta|},$$

The angle $\omega$ appearing in the definition above is called the angle of bisectoriality of $T$. An $\omega$-sectorial operator is defined in exactly the same manner, apart from the obvious modifications (change $S_\omega$ to $S_\omega^+$ and $S_\mu^+$ to $S_\mu^+$; $\omega$ is now allowed to exceed $\pi/2$). We remark that condition (iii) is a consequence of

(iii') there exists a positive constant $C$ such that, for all $\zeta \in \mathbb{C} \setminus S_\omega$

$$\|RT(\zeta)\|_{\mathcal{H} \rightarrow \mathcal{H}} \leq \frac{C}{\text{dist}(\zeta, S_\omega)}.$$

Sectorial operators were introduced by T. Kato in [36], where they were called of type $(\omega, M)$, where $M := \sup \{||\zeta|| \|RT(\zeta)\|_X : \zeta \notin S_\omega^+ \} < \infty$. The same author, in his classic book [37], uses the term “sectorial” to describe a different class of operators. We follow McIntosh’s set-up in [45]. For more history, see [30, Section 2.8].

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{spectrum}
\caption{Location of the spectrum of a bisectorial operator in the complex plane. Whenever the operator $T^{-1}$ is not bounded, $\sigma(T)$ reaches the origin.}
\end{figure}

Taking a second look at Definition 2.1.2, we observe that there is nothing to hold us back from introducing the notion of a (bi)sectorial operator acting in a Banach space $X$, instead of in a Hilbert space $\mathcal{H}$. This is of course feasible, as can be readily seen from a number of entries in the bibliography, for example [1, 16, 22, 30]), yet for the needs of the present thesis, the Hilbert space setting suffices, since, as stated in the Introduction, we will only be dealing with the $L^2$-boundedness of the double layer potential operator.

Below we present some examples of (bi)sectorial operators.
2.1 The functional calculus for bisectorial operators

--- Example 2.1.3 ---

Let $T \in \mathcal{C}(\mathcal{H})$ be a self-adjoint operator. Then $T$ is of type $S_0$, since $\sigma(T) \subset \mathbb{R}$ and $\|R_T(\zeta)\|_{\mathcal{H} \rightarrow \mathcal{H}} \leq 1/|\text{Im}\zeta|$, see, for example, [30, Proposition C.4.2].

In particular, $-\Delta$ in $L^2(\mathbb{R}^n)$, is of type $S_{0+}$, as a (positive) self-adjoint operator, see [1, Section I] and [30, Section 8.3].

--- Example 2.1.4 ---

Let $T$ be an $\omega$-bisectorial operator. Then its adjoint operator $T^*$ is also $\omega$-bisectorial, because $\sigma(T^*) = \overline{\sigma(T)}$ and $R_T(\zeta)^* = R_T(\overline{\zeta})$, where the bars stand for complex conjugation.

Along the same lines, since $\sigma(BTB^{-1}) = \sigma(T)$ and $R_{BTB^{-1}}(\zeta) = B^{-1}R_T(\zeta)B$, for a closed operator $T$ and an invertible operator $B \in \mathcal{B}(\mathcal{H})$, we see that $BTB^{-1}$ is bisectorial, whenever $T$ is.

By noticing that $R_{T^2}(\zeta) = -R_T(\sqrt{\zeta})R_T(-\sqrt{\zeta})$ and that $\zeta \notin S_{2\alpha}$ if and only if $\pm \sqrt{\zeta} \notin S_{\alpha}$, we find that if $T$ is $\omega$-bisectorial, then $T^2$ is $2\omega$-sectorial.

Finally, if $T \in \mathcal{C}(\mathcal{H})$ bisectorial and injective, then $T^{-1} : \mathcal{R}(T) \rightarrow \Omega(T)$ is bisectorial as well. For the pertaining details, we refer to [30, Propositions 2.1.1, 7.0.1], appropriately modified to cover the bisectorial case. See also [46, Propositions 5.6.4, 5.6.5, Corollary 5.6.6].

--- Example 2.1.5 ---

For a less trivial example, let $\mathcal{H} = L^2(X, \mu)$, where $(X, \mu)$ is a $\sigma$-finite measure space, and let $T = M_\mu$, where $\mu : X \rightarrow \mathbb{C}$ is a measurable function such that

$$\text{essran}(\mu) := \{ \zeta \in \mathbb{C} : \mu(\{ x \in X : |f(x) - \zeta| < \delta \}) > 0, \text{ for all } \delta > 0 \} \subset S_{\theta},$$

for some angle $0 \leq \theta < \frac{\pi}{4}$, and

$$M_\mu : L^2(X, \mu) \rightarrow L^2(X, \mu) ; f \mapsto af.$$

Then $M_\mu$ is a densely defined, closed (bounded if and only if $a \in L^\infty(X, \mu)$, self-adjoint if and only if $a$ is real valued) $\omega$-bisectorial operator with

$$\|R_{M_\mu}(\zeta)\|_{L^2(X, \mu) \rightarrow L^2(X, \mu)} = \frac{1}{\text{dist}(\zeta, \text{essran}(\mu))}.$$

In particular, the operator

$$\text{diag}_{\zeta_j} : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N}) : (x_j)_{j=1}^\infty \mapsto (\zeta_j x_j)_{j=1}^\infty,$$

where $(\zeta_j)_{j=1}^\infty \subset S_{\theta}$, is an $\omega$-bisectorial operator. For further details and more examples, for the case of sectorial operators at least, see [30, Section 2.1.1 and Chapter 8].

---

We also present the following (counter)example; see [1, Section D, Example 4].
Example 2.1.6

Let $H_k = \mathbb{C}^2$, for $k = 1, 2, \ldots$, and consider the direct sum $H := \bigoplus_{k \in \mathbb{N}} H_k$, where $u = (u_1, u_2, \ldots, u_k, \ldots) \in H$ if and only if $u_k \in H_k$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} |u_k|^2 < \infty$. $H$ is a Hilbert space equipped with the inner product $\langle u, v \rangle_H := \sum_{k=1}^{\infty} \langle u_k, v_k \rangle_{H_k}$. For $k \in \mathbb{N}$, let $T_k := \begin{bmatrix} 2^{-k} - 1 & 0 \\ 0 & 2^{-k} \end{bmatrix}$ and define the operator $T : H \rightarrow H : u \mapsto \bigoplus_{k \in \mathbb{N}} T_k u_k$.

Then, for all $\omega \in [0, \pi)$, $\sigma(T) \subset S_{\omega}^+$ and for $\zeta < 0$ a direct computation shows that

$$R_T(\zeta) = \bigoplus_{k \in \mathbb{N}} \begin{bmatrix} (2^{-k} - \zeta)^{-1} & -(2^{-k} - \zeta)^{-2} \\ 0 & (2^{-k} - \zeta)^{-1} \end{bmatrix},$$

hence

$$\|R_T(\zeta)\|_{H \rightarrow H}^2 \leq \sup_{k \in \mathbb{N}} \left| 2^{-k} - \zeta \right|^{-2} \geq \frac{1}{|\zeta|^2}.$$ 

Thus, for $-1 < \zeta < 0$, the third requirement of Definition 2.1.2 is not satisfied, so $T$ is not $\omega$-sectorial for any $\omega \in [0, \pi)$.

For a self-adjoint operator $T$, we have that $\mathcal{D}(T) = H$ by definition, while it is well known that

$$H = \mathcal{N}(T) \bigoplus \mathcal{R}(T),$$

see, for example, [12, Corollary 2.18]. We use the symbol “$\bigoplus$” to emphasize that the subspaces are orthogonal to each other. Similarly, for the multiplication operator $M_a$ from Example 2.1.5, we have that $\mathcal{D}(M_a) = L^2(X, \mu)$. These properties, the orthogonality of the splittings excluded, are in fact shared by all bisectorial operators. The key hypothesis is that such operators satisfy resolvent bounds as in Definition 2.1.2(iii). For a proof of the following theorem, see [30, Proposition 7.0.1].

Theorem 2.1.7. Let $T$ be an $\omega$-bisectorial operator in a Hilbert space $H$. Then $H$ can be written in the following way

$$H = \mathcal{N}(T) \bigoplus \mathcal{R}(T),$$

where the splitting is purely topological, in the sense of Banach spaces, i.e. no orthogonality is implied by the symbol “$\bigoplus$”. Moreover, $\mathcal{D}(T) = H$.

It follows, simply by taking $\mathcal{N}(T) = \{0\}$ in the direct sum (2.1), that an injective bisectorial operator necessarily has dense domain and dense range, see [16, Theorem 2.3]. In fact, Theorem 2.1.7 remains true even in the more general framework of Banach spaces, provided the Banach space is reflexive, see [16, Theorem 3.8], [30, Theorem 2.1.1].
2.1 The functional calculus for bisectorial operators

Now, let $T$ be an $\omega$-bisectorial operator, which is not necessarily injective. Using the decomposition (2.1), we define the restriction of $T$ in $\mathbb{R}(T)$, namely

$$T|_{\mathbb{R}(T)} : \mathbb{R}(T) \rightarrow \mathbb{R}(T).$$

It turns out that $T|_{\mathbb{R}(T)}$ is an injective $\omega$-bisectorial operator in $\mathbb{R}(T)$, with dense domain and dense range, see [16, Theorem 3.8]. Thus, there is no real loss in generality in considering only injective bisectorial operators and this is something we exploit in some of the proofs that appear in Section 2.2. Nevertheless, we shall always keep track of what happens when the operator is not injective, since this will be of interest in coming sections.

We need to lay down some more groundwork before we give the next example of an $\omega$-bisectorial operator.

**Definition 2.1.8.** Let $0 \leq \omega \leq \pi/2$. An operator $T : D(T) \rightarrow \mathcal{H}$ is called an $\omega$-accretive operator whenever the following three properties hold

(i) $T \in \mathcal{C}(\mathcal{H})$,

(ii) $\sigma(T) \subset S_{\omega^+} \cup \{\infty\}$, and

(iii) $\langle Tu, u \rangle_{\mathcal{H}} \in S_{\omega^+}$, for all $u \in D(T)$.

The angle $\omega$ appearing in the definition is called the angle of accretivity. Condition (iii), which can also be expressed by means of the numerical range (see, for example, [37, Section 5.3.2]) by saying that

$$W(T) := \{ \langle Tu, u \rangle_{\mathcal{H}} : u \in D(T), \|u\|_{\mathcal{H}} = 1 \} \subset S_{\omega^+},$$

means that $|\text{Im} \langle Tu, u \rangle_{\mathcal{H}}| \leq \tan \omega \text{Re} \langle Tu, u \rangle_{\mathcal{H}}$, for all $u \in D(T)$. If $\zeta \notin S_{\omega^+}$, it is deduced that

$$\left|\frac{\langle Tu, u \rangle_{\mathcal{H}}}{\|u\|_{\mathcal{H}}^2} - \zeta\right| \geq \text{dist}(\zeta, S_{\omega^+}),$$

which in turn implies that $\|R_T(\zeta)\|_{\mathcal{H}^* \rightarrow \mathcal{H}} \leq 1/\text{dist}(\zeta, S_{\omega^+})$. Thus, an $\omega$-accretive operator is always $\omega$-sectorial.

The proposition which follows is not only interesting because it describes a whole class of $\omega$-bisectorial operators but also because it is closely related to the results in Section 2.3; see [1, Theorem H], or even its predecessor in [45, Section 9].

**Proposition 2.1.9.** Let $B \in \mathcal{B}(\mathcal{H})$ be an invertible $\omega$-accretive operator on a Hilbert space $\mathcal{H}$ and $D \in \mathcal{C}(\mathcal{H})$ be an injective, self-adjoint operator. Then $T := BD$ is an injective $\omega$-bisectorial operator.

Before proving this proposition, we present a very important example of an $\omega$-bisectorial operator, as promised. In some sense, this can be interpreted as a simpler, two-dimensional analogue of the situation we encounter in Equation (2.19). This will be illustrated more clearly in Section 2.3.
Example 2.1.10

Let \( g : \mathbb{R} \to \mathbb{R} \) be a Lipschitz function, such that \( \|g'\|_{\infty} \leq L < \infty \). Consider the Lipschitz curve in the complex plane given by \( \gamma := \{ z = x + ig(x) \in \mathbb{C} : x \in \mathbb{R} \} \) and the space (of equivalence classes) of functions \( L^2(\gamma) := \{ u : \gamma \to \mathbb{C} : u \) measurable, \( \|u\|_{L^2(\gamma)} < \infty \}, \)

where

\[
\|u\|_{L^2(\gamma)} := \left( \int_{\gamma} |u(z)|^2 \, |dz| \right)^{\frac{1}{2}}.
\]

Define the derivative of a Lipschitz function \( u \) on \( \gamma \) by

\[
d \frac{d}{dz} u(z) := \lim_{h \to 0} \frac{u(z + h) - u(z)}{h},
\]

for almost every \( z \) on \( \gamma \). The chain rule yields

\[
d \frac{d}{dz} u(x + ig(x)) = \frac{1}{1 + g'(x)} d \frac{d}{dz} u(x + ig(x)).
\]

We then use duality, with respect to the inner product \( \langle u, v \rangle_{L^2(\gamma)} := \int_{\gamma} u(z) \overline{v(z)} \, |dz| \), in order to define the operator \( D_{\gamma} \) as the closed operator with the largest domain in \( L^2(\gamma) \) that satisfies

\[
\langle D_{\gamma} u, v \rangle_{L^2(\gamma)} = \left\langle u, -i \frac{d}{dz} \right|_{\gamma} v \right\rangle_{L^2(\gamma)},
\]

for all compactly supported Lipschitz functions \( v \in L^2(\gamma) \). It turns out that

\[
D(D_{\gamma}) = W^{1,2}(\gamma) := \{ u \in L^2(\gamma) : D_{\gamma} u \in L^2(\gamma) \} = \{ u \in L^2(\gamma) : u \circ \gamma \in W^{1,2}(\mathbb{R}) \}.
\]

By considering the isomorphism between \( L^2(\gamma) \) and \( L^2(\mathbb{R}) \) given by

\[
V : L^2(\gamma) \longrightarrow L^2(\mathbb{R}) : u \longmapsto V(u) := u \circ \gamma,
\]

we see that \( V \circ D_{\gamma} = BD \circ V \), where \( D := -i \frac{d}{dz} \) and \( B := M_{(1 + ig')^{-1}} \), with respective domains \( D(D) = W^{1,2}(\mathbb{R}) \) and \( D(M_{(1 + ig')^{-1}}) = L^2(\mathbb{R}) \). In other words, the diagram below is commutative

\[
\begin{array}{ccc}
L^2(\gamma) & \xrightarrow{V} & L^2(\mathbb{R}) \\
D_{\gamma} & \downarrow & \downarrow BD \\
L^2(\gamma) & \xrightarrow{V} & L^2(\mathbb{R}).
\end{array}
\]

It is not hard to see that \( B \), the operator of multiplication by the bounded function \( (1 + ig')^{-1} \), is a bounded, invertible, \( \omega \)-accretive operator, with \( \omega = \arctan L \). What is more, \( D \) is an injective, self-adjoint operator. Applying Proposition (2.1.9), we have that \( BD \) is
2.1 The functional calculus for bisectorial operators

an injective \( \omega \)-bisectorial operator in \( L^2(\mathbb{R}) \). Thus, \( V^{-1}BD \) is an injective \( \omega \)-bisectorial operator in \( L^2(\gamma) \) (recall Example 2.1.4). Rademacher’s theorem (see, for example, [41, Theorem 11.49]), which ensures the almost everywhere differentiability of Lipschitz functions has been used freely throughout. For a full exposition of this example, see [47] and [1, Sections O and P].

Proof of Proposition 2.1.9: Let \( \zeta \in \mathbb{C} \setminus \mathcal{S}_a \). Then, for all \( 0 \neq u \in \mathcal{D}(D) \)

\[
\left| \langle (B^{-1}BD - \zeta)u, u \rangle_{\mathcal{H}} \right| = \left| \langle Du, u \rangle_{\mathcal{H}} - \zeta \langle B^{-1}u, u \rangle_{\mathcal{H}} \right| 
\]

so \( \left| \langle B^{-1}u, u \rangle_{\mathcal{H}} \right| \left| \frac{\langle Du, u \rangle_{\mathcal{H}}}{\langle B^{-1}u, u \rangle_{\mathcal{H}}} - \zeta \right| \leq \|B^{-1}\|_{\mathcal{H} \to \mathcal{H}} \|BD - \zeta\|_{\mathcal{H}} \|u\|_{\mathcal{H}}, \)

by the Cauchy-Schwartz inequality. Since \( B \) is \( \omega \)-accretive, it follows that \( \langle B^{-1}u, u \rangle_{\mathcal{H}} \in S_{a^+} \). Since \( D \) is self-adjoint, \( \langle Du, u \rangle_{\mathcal{H}} \in \mathbb{R} \). These yield that \( \langle Du, u \rangle_{\mathcal{H}} / \langle B^{-1}u, u \rangle_{\mathcal{H}} \in \mathcal{S}_a \).

Accordingly

\[
\left| \frac{\langle Du, u \rangle_{\mathcal{H}}}{\langle B^{-1}u, u \rangle_{\mathcal{H}}} - \zeta \right| \geq \text{dist}(\zeta, \mathcal{S}_a),
\]

hence, there exists a positive constant \( C \) such that

\[
C \|u\|_{\mathcal{H}} \text{dist}(\zeta, \mathcal{S}_a) \leq \|(BD - \zeta)u\|_{\mathcal{H}}, \text{ for all } u \in \mathcal{D}(D).
\] (2.2)

Now, (2.2) implies that \( \zeta - BD \) is injective and has closed range. In order to show that its range is also dense, we consider its adjoint operator and note that

\[
(\zeta - BD)^* = (\zeta - DB)^* = B^{-1}(\zeta - B'D)B^*,
\]

since \( D \) is self-adjoint, \( (BD)^* = D'B^* \) and \( B \) is invertible. Since \( \zeta - B'D \) is of exactly the same form as \( \zeta - BD \), the arguments that lead to (2.2) can be repeated to show that \( \zeta - B'D \) is also injective. But then \( (\zeta - BD)^* = (\zeta - B'D)^* \) is injective as well, thus \( R(\zeta - BD) = \mathcal{H} \) (see, for example, [12, Theorem 2.19]), i.e. the operator \( \zeta - BD \) is surjective. Thus, since the operator \( BD \) is closed, the Closed Graph Theorem (see, for example, [12, Theorem 2.9]) yields that \( \zeta - BD \) is invertible in the sense of unbounded operators. Finally

\[
\|R_{BD}(\zeta)\|_{\mathcal{H} \to \mathcal{H}} \leq \frac{1}{C\text{dist}(\zeta, \mathcal{S}_a)},
\]

follows from (2.2).

It is evident that the operator \( DB = B^{-1}(BD)B \) is also bisectorial (see Example 2.1.4). Moreover, had \( D \) been a positive operator, i.e. \( \langle Du, u \rangle_{\mathcal{H}} \geq 0 \) for all \( u \in \mathcal{H} \), then \( BD \) and \( DB \) would have been \( \omega \)-sectorial operators.

Having developed some of the theory of bisectorial operators, we now turn our attention to the classes of functions for which we would like to define a functional calculus of such operators.
Definition 2.1.11. Let $\mu \in (0, \pi/2)$. We define the following classes of holomorphic functions on an open bisector $S^\mu$

(i) The set of all holomorphic functions on $S^\mu$
$$H(S^\mu) := \{ f : S^\mu \rightarrow \mathbb{C} : f \text{ is holomorphic} \}.$$ 

(ii) The Banach algebra of all bounded holomorphic functions on $S^\mu$
$$H^\infty(S^\mu) := \{ f \in H(S^\mu) : \|f\|_{L^\infty(S^\mu)} < \infty \},$$
where $\|f\|_{L^\infty(S^\mu)} := \sup\{|f(\zeta)| : \zeta \in S^\mu\}$.

(iii) The set $\Psi(S^\mu)$ of regularly decaying functions on $S^\mu$
$$\Psi(S^\mu) := \{ f \in H(S^\mu) : \text{there exists } C > 0 \text{ and } \alpha > 0 \text{ such that } |f(\zeta)| \leq C \frac{|\zeta|^\alpha}{1 + |\zeta|^{2\alpha}} \text{ for all } \zeta \in S^\mu \}.$$ 

It is easily seen that $f \in \Psi(S^\mu)$ if and only if there exist positive constants $C$ and $\alpha$ such that
$$|f(\zeta)| \leq C \min \{|\zeta|^\alpha, |\zeta|^{-\alpha}\},$$
for all $\zeta \in S^\mu$. For alternative descriptions of the class $\Psi$, see [30, Lemma 2.2.2]. Clearly, the space $\Psi(S^\mu)$ is not dense in $H^\infty(S^\mu)$ with respect to the $\|\cdot\|_{L^\infty(S^\mu)}$-norm topology.

Note that neither the term “regularly decaying”, nor the notation $\Psi(S^\mu)$ are standard in the literature, see [30, Comment 2.2].

Now, consider an $\omega$-bisectorial operator $T$ and an angle $\mu \in (\omega, \pi/2)$. Let $f \in \Psi(S^\mu)$. Through Dunford-Riesz (sometimes called Holomorphic) calculus (see, for example, [1, Sections A, B and C], [22, Chapter 7] and [30, Section 1.1]) we define the operator
$$f(T) := \frac{1}{2\pi i} \int f(\zeta) R_T(\zeta) \, d\zeta,$$  
(2.3)
where $\gamma$ is the unbounded contour $\{te^{i\theta} : t > 0\} \cup \{-te^{i\theta} : t > 0\}$, $\omega < \theta < \mu$, parameterized counterclockwise around $S_\omega$.

Since $\zeta \mapsto f(\zeta)R_T(\zeta)$ is holomorphic for $\zeta \not\in S_\omega$, it follows from Cauchy’s theorem (for Banach valued functions and under a suitable truncation of the paths, see [46, Theorem 4.1.9]) that the integral in (2.3) is independent of the angle $\theta \in (\omega, \mu)$; thus, $f(T)$ is well-defined. Moreover, the bounds on $f$ and the resolvent guarantee that it converges uniformly, as seen by

$$\|f(T)\|_{\mathcal{B}(\mathcal{H})} \leq C \int_{\gamma} \|f(\zeta)\| \|R_T(\zeta)\|_{\mathcal{B}(\mathcal{H})} |d\zeta| \leq C \int_{\gamma} \frac{|\zeta|^\alpha}{1 + |\zeta|^{2\alpha}} |d\zeta|$$
$$\leq C \int_0^\infty \frac{t^\alpha}{1 + t^{2\alpha}} \frac{dt}{t} \leq C \frac{1}{\alpha} < \infty,$$  
(2.4)
2.1 The functional calculus for bisectorial operators

since \( \int_0^\infty t^\alpha / (1 + i t^{2\alpha}) \, dt = \pi / 2\alpha \). It follows that \( f(T) \in \mathcal{B}(\mathcal{H}) \). We emphasize that \( f(T) \) has so far only been defined for regularly decaying functions.

We mention the following simple proposition, which will be used in conjunction with Theorem 2.1.7 later on; see [30, Theorem 2.3.3].

**Proposition 2.1.12.** Let \( T \) be an \( \omega \)-bisectorial operator, \( \mu > \omega \) and \( f \in \Psi(S^\mu_\omega) \). Then \( \mathcal{N}(T) \subset \mathcal{N}(f(T)) \).

**Proof:** Notice that for \( u \in \mathcal{N}(T) \), \( R_T(\zeta)u = \zeta^{-1}u \), for \( \zeta \notin \sigma(T) \). Thus

\[
f(T)u = \left( \frac{1}{2\pi i} \int_\gamma f(\zeta)R_T(\zeta) \, d\zeta \right) u = \frac{1}{2\pi i} \int_\gamma f(\zeta)R_T(\zeta) u \, d\zeta
\]

where the last integral is zero by Cauchy’s theorem. For a proof that follows a different route, see [46, Corollary 4.1.11].

The next proposition presents probably the most important property of this construction of the operators \( f(T) \) via (2.3). From this, it follows that \( f(T)g(T) = g(T)f(T) \).

**Proposition 2.1.13.** Let \( f, g \in \Psi(S^\mu_\omega) \), then

\[
f(T)g(T) = (fg)(T).
\]

**Proof:** Let \( f(T) = (1/2\pi i) \int_\gamma f(\zeta)R_T(\zeta) \, d\zeta \) and \( g(T) = (1/2\pi i) \int_\delta g(z)R_T(z) \, dz \) where the contours have been chosen so that \( \delta \) encircles \( \gamma \) (i.e. \( \omega < \theta_T < \theta_\delta < \pi/2 \)). A direct computation shows that

\[
f(T)g(T) = \left( \frac{1}{2\pi i} \int_\gamma f(\zeta)R_T(\zeta) \, d\zeta \right) \left( \frac{1}{2\pi i} \int_\delta g(z)R_T(z) \, dz \right)
\]

\[
= \frac{1}{(2\pi i)^2} \int_\gamma \int_\delta f(\zeta)g(z)R_T(\zeta)R_T(z) \, d\zeta \, dz
\]

\[
= \frac{1}{(2\pi i)^2} \int_\gamma \int_\delta f(\zeta)g(z) \frac{1}{z - \zeta} (R_T(\zeta) - R_T(z)) \, d\zeta \, dz
\]

\[
= \frac{1}{(2\pi i)^2} \int_\gamma \int_\delta f(\zeta)R_T(\zeta) \left( \int_\delta \frac{g(z)}{z - \zeta} \, dz \right) \, d\zeta
\]

\[
- \frac{1}{(2\pi i)^2} \int_\gamma \int_\delta f(\zeta) \left( \int_\delta \frac{g(z)R_T(z)}{z - \zeta} \, dz \right) \, d\zeta
\]

\[
= \frac{1}{2\pi i} \int_\gamma f(\zeta)R_T(\zeta)g(\zeta) \, d\zeta - 0 = (fg)(T),
\]
where we have used the resolvent equation \( R_T(\zeta) - R_T(z) = (z - \zeta)R_T(\zeta)R_T(z) \), Cauchy’s integral formula for holomorphic functions and Cauchy’s theorem. The order of integration can be interchanged due to the absolute convergence of the integrals.

More is true in fact, as it turns out that the mapping

\[
\Psi(S_\mu^0) \ni f \mapsto f(T) \in B(\mathcal{H}),
\]

(2.5)
is an algebra homomorphism and does indeed satisfy other formal requirements one might expect from a functional calculus, for example that \( \sigma(f(T)) = f(\sigma(T)) \), or that \( f(T^+) = (\overline{T}(\zeta))^\ast \) (where \( \overline{T}(\zeta) = \overline{f(\zeta)} \); notice that \( \overline{T} \in \Psi(S_\mu^0) \), whenever \( f \in \Psi(S_\mu^0) \)).

Special mention goes out to the fact that the definition of \( f(T) \) is consistent with the familiar one given for rational functions, which are holomorphic at infinity and have no poles in \( \sigma(T) \setminus \{0\} \). See, for example, [16, Section 2], [30, Lemma 2.3.1, Proposition 7.0.1], [45, Section 4]. In fact, one could start by constructing a functional calculus first for the class of polynomials, then for the class of rational functions and culminating in formula (2.3) for regularly decaying \( f \), checking at each step that the definitions of \( f(T) \) agree whenever the function belongs to two different classes at once. This is done in [1], [45]; see also [22, Chapter 7]. We refrain from giving further details as a full exposition of the theory of holomorphic functional calculi lies beyond the scope of this thesis.

We remark that \( \zeta \mapsto \zeta e^{-|\zeta|} \) and \( \zeta \mapsto \zeta/(1 + \zeta^2) \) comprise examples of holomorphic functions belonging to the class \( \Psi(S_\mu^0) \). Here

\[
\chi_+(\zeta) := \begin{cases} 
1, & \text{if } \Re \zeta > 0, \\
0, & \text{if } \Re \zeta \leq 0,
\end{cases}
\]

and \( \chi_-(\zeta) = 1 - \chi_+(\zeta) \). In other words, \( \chi_{\pm} = \chi_{S_\mu^0} \) are the characteristic functions of the right and left open \( \mu \)-sectors. Furthermore, \( \sgn(\zeta) = \chi_+(\zeta) - \chi_-(\zeta) \), i.e.

\[
\sgn(\zeta) := \begin{cases} 
1, & \text{if } \Re \zeta > 0, \\
0, & \text{if } \Re \zeta = 0, \\
-1, & \text{if } \Re \zeta < 0,
\end{cases}
\]

and \( |\zeta| := \zeta \sgn(\zeta) \). Notice that \( |\zeta| \) does not denote absolute value for non-real \( \zeta \), in this context. Hence, using (2.3) we see that the operators \( Te^{-|T|} \) and \( T/(1 + T^2) \) are all bounded operators in \( \mathcal{H} \). Notice however that none of the functions \( \chi_+, \chi_-, \sgn \) or \( \zeta \mapsto e^{-|\zeta|} \) belong to the class \( \Psi(S_\mu^0) \). This happens because they do not meet the right decay criteria at zero and/or infinity. It is this, rather than failure of holomorphy, that prevents them from belonging to \( \Psi(S_\mu^0) \). Actually, all the aforementioned functions are holomorphic on \( S_\mu^\ast \). Obviously, the function which is identically 1 on \( S_\mu \) does not belong to \( \Psi(S_\mu^0) \), so (2.5) is definitely not a unital algebra homomorphism.

Nevertheless, in order to use the tool of functional calculus to solve equation (2.19), defined later in Section 2.3, one needs to define spectral projections \( \chi_{\pm}(T) \) as well as the operators \( \sgn(T), e^{-|T|} \). Thus, it is necessary to extend the functional calculus constructed so far, to general bounded holomorphic functions on a bisector. This is done in the next section.
In this section we introduce the key concept of quadratic estimates, that first appeared in [45], see [30, Comment 5.6]. This will allow us to increase the domain of definition of the functional calculus defined in the previous section, to all \( f \in \mathcal{H}^{\infty}(S) \). This is achieved through Theorem 2.2.3 and Proposition 2.2.8, that are the central points of this section.

We state the definition right away.

**Definition 2.2.1.** An injective \( \omega \)-bisectorial operator \( T \) satisfies quadratic estimates (or square function estimates) with respect to \( \psi \in \Psi(S) \), if there exist positive constants \( m = m(\psi) \) and \( M = M(\psi) \) such that

\[
m \| u \|_{\mathcal{H}}^2 \leq \int_0^\infty \| \psi(tT)u \|_{\mathcal{H}}^2 \frac{dt}{t} \leq M \| u \|_{\mathcal{H}}^2,
\]

for all \( u \in \mathcal{H} \).

Note that the integral in (2.6) makes sense as an improper Riemann integral, since the mapping \( t \mapsto \psi(tT)u \) is continuous, see [30, Lemma 5.2.1, Theorem 5.2.2].

In the literature sometimes one encounters a slightly different definition: an operator \( T \) satisfies a quadratic estimate if the second inequality in (2.6) holds, while \( T \) satisfies a reverse quadratic estimate if the first inequality in (2.6) is true, see [1, Section E], [47, Section 5].

It is known that \( T \) satisfies a quadratic estimate if and only if its adjoint \( T^* \) satisfies a reverse quadratic estimate, see [1, Theorem E, Corollary E], [47, Theorems 5.2, 5.3].

Furthermore, if \( T \) satisfies quadratic estimates with respect to some particular \( \psi \in \Psi(S) \), then \( T \) satisfies quadratic estimates with respect to every non-zero \( \psi \) from the same class. This follows from Theorem 2.2.3, Proposition 2.2.8 and Corollary 2.2.9; for details, see [1, Proposition E]. Accordingly, there is no ambiguity in suppressing the reference to a specific function \( \psi \) when saying that an operator satisfies quadratic estimates.

We now turn to some examples.

---

**Example 2.2.2**

(i) Consider a self-adjoint operator \( T \), like in Example 2.1.3. Then, \( T \) satisfies quadratic estimates, with respect to the function

\[
\psi : S \rightarrow \mathbb{C} ; \ \zeta \mapsto \frac{\zeta}{1 + \zeta^2},
\]

and therefore, by the aforementioned comment, with respect to any other function from the class \( \Psi(S) \), for some (therefore, by Theorem 2.2.10, for any) \( \mu > 0 \). Indeed, using Lemma 2.2.5 and performing computations similar to those in [1,
Section G] (see also [47, Section 5]), one obtains that
\[ \int_0^\infty \| \psi(tT)u \|_{\mathcal{H}}^2 \frac{dt}{t} = \int_0^\infty \langle \psi(tT) \psi(tT)u, u \rangle_{\mathcal{H}} \frac{dt}{t} = \int_0^\infty \langle \psi(tT) \psi^*(tT)u, u \rangle_{\mathcal{H}} \frac{dt}{t} \]

where
\[ M = \frac{1}{m} = \max \left\{ \int_0^\infty |\psi(t)|^2 \frac{dt}{t}, \int_0^\infty |\psi(-t)|^2 \frac{dt}{t} \right\} = \frac{1}{2}. \]

(ii) The multiplication operator \( M_a \) from Example 2.1.5 satisfies quadratic estimates, with respect to some (therefore any) \( \psi \in \Psi(S'_\mu) \), for some (therefore, again by Theorem 2.2.10, for any) \( \mu > \omega \). See [47, Section 5].

(iii) The operator \( \frac{d}{dz} \) of differentiation on a Lipschitz curve \( \gamma \) satisfies quadratic estimates, as shown in [47, Section 7]. This is a non-trivial result and in fact equivalent to the \( L^2 \)-boundedness of the principal-value Cauchy integral operator
\[ C_\gamma : L^2(\gamma) \rightarrow L^2(\gamma) \ : \ u \mapsto C_\gamma u, \]
where
\[ C_\gamma u(z) := \text{p.v.} \int_{\gamma} \frac{1}{z-\zeta} u(\zeta) \, d\zeta, \quad z \in \gamma. \]

See [47] and also Theorems 2.2.3 and 2.2.10 in the sequel. The boundedness of the Cauchy integral on Lipschitz curves was first proved by Calderón for small Lipschitz constant in [13], and by Coifman, McIntosh and Meyer for arbitrary Lipschitz constant in [14]. Investigations on the boundedness of the Cauchy integral were the origins of the \( T(b) \)-theorem, which is a criterion for the \( L^2 \)-boundedness of non-convolution integral operators; see [20].

It is not true that all bisectorial operators satisfy quadratic estimates. Counterexamples, with (bi)sectorial operators that do not satisfy quadratic estimates, can be found in [4, Section 4.5.2] and [48].

Now, fix \( \psi \in \Psi(S'_\mu) \) such that \( \psi|_{S_{2\mu}} \neq 0 \) and let \( T \) be an injective bisectorial operator. For \( u \in \mathcal{H} \), define the quadratic norm associated to this operator by
\[ \|u\|_T := \left( \int_0^\infty \| \psi(tT)u \|_{\mathcal{H}}^2 \frac{dt}{t} \right)^{\frac{1}{2}}. \]
Let

\[ H^0_T := \{ u \in \mathcal{H} : \| u \|_T < \infty \} \subset \mathcal{H}. \] (2.8)

It is seen that \( \| \cdot \|_T \) is a norm on \( H^0_T \) and that this norm is induced by the inner product

\[ (u,v)_T := \int_0^\infty \langle \psi(tT)u, \psi(tT)v \rangle_{\mathcal{H}} \, \frac{dt}{t}, \] (2.9)

where \( u, v \in \mathcal{H} \), which turns \( H^0_T \) into a pre-Hilbert space. The only non-obvious property of \( \| \cdot \|_T \) is the positive-definiteness, for which Lemma 2.2.5 is needed. The completion of the space \( H^0_T \) with respect to the norm \( \| \cdot \|_T \) is denoted by \( H_T \). Therefore, we see that to say that an operator \( T \) satisfies quadratic estimates is equivalent to saying that \( H_T = \mathcal{H} \) and that the quadratic norm \( \| \cdot \|_T \) is equivalent to the original norm \( \| \cdot \|_{\mathcal{H}} \).

Note that for non-injective \( T \), \( \| \cdot \|_T \) is only a seminorm, as there may well exist non-zero \( u \in \mathcal{H} \) such that \( u \in \text{N}(T) \), hence, by Proposition 2.1.12, \( \psi(tT)u = 0 \) for all \( t > 0 \) and, consequently, \( \| u \|_T = 0 \).

The following key theorem brings forth the importance and the strength of quadratic estimates; see [5, Proposition 6.3], [1, Lemma E, Proposition E]. The significance of estimate (2.10) lies in the particular nature of the upper bound for \( \| f(T) \|_{\mathcal{H} \to \mathcal{H}} \), namely a constant times the sup-norm of \( f \). For regularly decaying functions, that \( \| f(T) \|_{\mathcal{H} \to \mathcal{H}} < \infty \) was already seen in (2.4).

**Theorem 2.2.3.** Assume that an injective \( \omega \)-bisectorial operator \( T \) satisfies quadratic estimates. Then, there exists a finite constant \( C \) such that

\[ \| f(T) \|_{\mathcal{H} \to \mathcal{H}} \leq C \| f \|_{L^\infty(S_\mu)}, \] (2.10)

for all \( f \in \Psi(S^0_\mu) \).

In order to prove this theorem we make use of the following fact of holomorphic functional calculus, which is a particular case of [30, Theorem 5.2.6]. It can be viewed as a property analogous to the standard resolution of the identity in the sense of [51, Definition 12.7]; for details, which belong to the realm of functional calculi for, say, normal operators we refer to [30, Appendix D] and [51, Chapters 12, 13].

We first recall an elementary fact of Functional Analysis, which however inconspicuous it may seem, it simplifies things considerably. Its proof involves a standard “\( \varepsilon/3 \)”-argument and is omitted, see [46, Proposition 2.6.3].

**Lemma 2.2.4.** Let \( X \) be a Banach space and let \((B_n)_{n=1}^\infty \in X \) be a sequence of operators satisfying \( \sup_n \| B_n \|_X \leq C < \infty \), for some non-negative constant \( C \). Suppose that \( Y \) is a dense subset of \( X \) and that \( B_nY \) converges for all \( y \in Y \). Then \( B_nx \) converges for all \( x \in X \).

**Lemma 2.2.5.** Let \( T \) be an injective \( \omega \)-bisectorial operator and let \( \psi(\zeta) = \zeta/(1 + \zeta^2) \), for \( \zeta \in S^0_\mu \), where \( \mu \in (\omega, \pi/2) \). Then

\[ \int_0^\infty \psi^2(tT)u \, \frac{dt}{t} = \frac{1}{2} u, \]

for all \( u \in \mathcal{H} \).
Proof: Let $0 < a < b < \infty$. Then, by the definition of the operators $\psi(tT)$ and Fubini’s theorem we have that

$$2 \int_a^b \psi^2(tT)u \frac{dt}{t} = \frac{1}{2\pi i} \int \left( -\frac{1}{1 + b^2 \xi^2} + \frac{1}{1 + a^2 \xi^2} \right) R_T(\xi)u \, d\xi$$

$$= \left( -\frac{I}{I + b^2 T^2} + \frac{I}{I + a^2 T^2} \right) u \longrightarrow u,$$

as $a \to 0$ and $b \to \infty$, for all $u \in \mathcal{H}$. Observe that, since $T$ is sectorial, the operators $-(I + b^2 T^2)^{-1}$ and $(I + a^2 T^2)^{-1}$ are uniformly bounded. This follows from the identity

$$-(I + b^2 T^2)^{-1} = \frac{1}{b^2} \left( \frac{1}{b^2} - T \right)^{-1} = \frac{1}{2ib} \left( \frac{1}{ib} - T \right)^{-1} - \left( \frac{1}{ib} - T \right)^{-1},$$

and similarly for $(I + a^2 T^2)^{-1}$. Thus, to verify the strong limit it suffices, in light of the aforementioned Lemma 2.2.4, to consider $u \in D(T) \cap R(T)$, the latter set being dense in $\mathcal{H}$ by Proposition 2.2.13. Let $u = Tv$ for some $v \in D(T)$, then

$$-(I + b^2 T^2)^{-1}u = -\frac{Tv}{I + b^2 T^2} = -\frac{1}{b} \frac{bTv}{I + b^2 T^2} \longrightarrow 0, \quad b \longrightarrow \infty,$$

since $bT(I + b^2 T^2)^{-1}$ is also uniformly bounded, as seen by the identity

$$bT(I + b^2 T^2)^{-1} = -\frac{1}{2b} \left( \frac{1}{ib} - T \right)^{-1} + \left( \frac{1}{ib} - T \right)^{-1}. $$

Similarly, one shows that $(I + a^2 T^2)^{-1}$ converges strongly to the identity operator $I$. \hfill \Box

In other words, after rescaling a function $\psi$ such that $\psi|_{\mathbb{R}_{\infty}^+} \neq 0$, by dividing it by a non-zero number $C$ if necessary, the operator

$$\mathcal{H} \ni u \longmapsto \int_0^\infty \psi^2(tT)u \frac{dt}{t} \in \mathcal{H},$$

acts like the identity on $\mathcal{H}$. Of course, for a non-injective $T$ it acts like the identity only on $R(T)$, being zero on $N(T)$, by Proposition 2.1.12. It would have thus been a bounded (not necessarily orthogonal) projection on $\mathcal{H}$. This justifies the use of the term “resolution of the identity”. Note that from this we recover the fact that $\mathcal{H} = N(T) \oplus \overline{R(T)}$, when $T$ is a bisectorial operator; see Theorem 2.1.7 and [45, Section 7].

Quadratic estimates can also be interpreted in the following way. Consider an operator $T \in C(\mathcal{H})$ for which there exists a family of bounded spectral projections $\{\chi_\lambda(T)\}_{\lambda=1}^\infty$, corresponding to the spectral decomposition of $\mathcal{H}$ associated with $T$. This is possible (allowing for a continuous parameter $\lambda \in \mathbb{R}$) whenever $T$ is a (bounded or unbounded) self-adjoint operator. The important details of the “spectral measure approach” of the spectral theorem can be found in [37, Section 6.5] and [51, Chapter 13]; see also [46,
Section 2.4. For the “multiplier approach” see [30, Appendix D]. In such a situation, 
\[ \sum_{k=1}^{N} \chi_k(T)u = u \] 
and
\[ \frac{1}{C} \|u\|_{\mathcal{H}}^2 \leq \sum_{k=1}^{N} \|\chi_k(T)u\|_{\mathcal{H}}^2 \leq C \|u\|_{\mathcal{H}}^2, \] 
(2.11)
for some positive constant C and for all \( u \in \mathcal{H} \). Thereupon, (2.6) can be viewed as a continuous analogue of (2.11) for the functions \( \psi_t(\cdot) = \psi(t\cdot) \) and the parameter \( t \) varying in the measure space \( ((0,\infty);t^{-1}dt) \). In fact, for a bisectorial operator, as Theorem 2.2.10 and Propositions 2.2.12 and 2.2.13 reveal, it is possible to define spectral projections (only) for the left and right sectors of the spectrum and split the space into spectral subspaces (corresponding to the parts of the spectrum lying in the left and right half-planes), if the operator satisfies quadratic estimates.

**Proof of Theorem 2.2.3:** Since \( T \) satisfies quadratic estimates, by the left-hand-side of (2.6) we have that
\[ \|f(T)u\|_{\mathcal{H}}^2 \leq C \int_0^\infty \|\psi(tT)f(T)u\|_{\mathcal{H}}^2 \frac{dt}{t}. \]
Using Lemma 2.2.5, plugging in \( u = C \int_0^\infty \psi^2(sT)us^{-1}ds \) and using the homomorphism property of the functional calculus, one sees that
\[ \|f(T)u\|_{\mathcal{H}}^2 \leq C \int_0^\infty \int_0^\infty \left\| (\psi(tT)f(T)\psi(sT))\frac{ds}{s} \right\|_{\mathcal{H}}^2 \frac{dt}{t}. \]
An application of the Cauchy-Schwartz inequality gives
\[ \|f(T)u\|_{\mathcal{H}}^2 \leq C \int_0^\infty \left( \int_0^\infty \|\psi(tT)f(T)\psi(sT)\frac{ds}{s} \right)^2 \frac{dt}{t}, \]
where the subscripts of the norms have been suppressed to ease notation. Call the quantity appearing in the right-hand-side \( I \). We need to estimate \( I \). To this end, note that the
homomorphism property of the functional calculus and the fact that $f, \psi \in \Psi(S^1)$ yield

$$
\|\psi(tT)f(T)\psi(sT)\|_{\mathcal{F} \to \mathcal{F}} \leq \left\| \frac{1}{2\pi i} \int f(\xi)\psi(t\xi)\psi(s\xi)R_T(\xi) \, d\xi \right\|_{\mathcal{F} \to \mathcal{F}} 
$$

$$
\leq C \int \|\psi(t\xi)\psi(s\xi)\| \frac{|d\xi|}{|\xi|} \leq C \|f\|_{L^\infty(S^1)} k\left(\frac{t}{s}\right), \quad \text{(2.12)}
$$

where $k(t/s) := \min\{(t/s)^\alpha, (t/s)^{−\alpha}\}(1 + |\log(t/s)|)$. To arrive at the expression for $k$ we use the estimates on the function $\psi$ and the definition of the contour integral to get

$$
\int |\psi(t\xi)\psi(s\xi)| \frac{|d\xi|}{|\xi|} \leq C \int_0^\infty \frac{(tx)^\alpha}{1 + (tx)^{2\alpha}} \frac{s}{1 + (sx)^{2\alpha}} \frac{dx}{x},
$$

and then express the last integral as a sum of three integrals $f_{1/0} + f_{1/s} + f_{1/s}$, for $t < s$ and similarly for the case when $s < t$. Notice that $\int_0^\infty k(t/s)^{−1} \, dt$ and $\int_0^\infty k(t/s) s^{−1} \, ds$ are both finite. This is used to bound $I$ from above as follows

$$
I \leq C \int_0^\infty \left( \int_0^\infty \|f\|_{L^\infty(S^1)} k\left(\frac{t}{s}\right) \frac{dx}{s} \right) \left( \int_0^\infty \|f\|_{L^\infty(S^1)} k\left(\frac{t}{s}\right) \|\psi(sT)u\|_{\mathcal{F} \to \mathcal{F}} \frac{dx}{s} \right) \frac{dt}{t} 
$$

$$
\leq C \|f\|_{L^2(S^1)}^2 \left( \sup_{t > 0} k\left(\frac{t}{s}\right) \frac{dx}{s} \right) \left( \sup_{s > 0} \int k\left(\frac{t}{s}\right) \frac{dt}{t} \right) \int_0^\infty \|\psi(sT)u\|_{\mathcal{F} \to \mathcal{F}}^2 \frac{dx}{s} 
$$

$$
\leq C \|f\|_{L^2(S^1)}^2 \|u\|_{\mathcal{F} \to \mathcal{F}},
$$

where in the last step the right hand side of (2.6) was used. \hfill \Box

We wish to take a second look into the proof presented above. Let

$$
K : (0,\infty) \times (0,\infty) \to \mathbb{R} ; (t, s) \longmapsto \|\psi(tT)f(T)\psi(sT)\|_{\mathcal{F} \to \mathcal{F}}.
$$

It follows from the estimates in (2.12) and the subsequent comments about $k$, that $K$ is locally integrable on the product measure space $(\mathbb{R}_+, s^{-1} \, ds) \times (\mathbb{R}_+, t^{-1} \, dt)$ and that

$$
\sup_{t > 0} \int_0^\infty |K(t,s)| \frac{dt}{t} < \infty, \quad \sup_{s > 0} \int_0^\infty |K(t,s)| \frac{dt}{t} < \infty.
$$

Then, interpolation between $L^1(\mathbb{R}_+; t^{-1} \, dt)$ and $L^\infty(\mathbb{R}_+; s^{-1} \, ds)$, also known as Schur’s lemma, shows that the integral operator with kernel $K$

$$
\|\psi(\cdot T)u\|_{\mathcal{F} \to \mathcal{F}} \longmapsto \int_0^\infty K(\cdot, s) \|\psi(sT)u\|_{\mathcal{F} \to \mathcal{F}} \frac{dx}{s},
$$

is $L^1(\mathbb{R}_+; t^{-1} \, dt) \to L^2(\mathbb{R}_+; t^{-1} \, dt)$ bounded, see [24, Theorem 6.18] or [27, Appendix I]. From this, the estimate for the quantity $I$ defined in the proof of Theorem 2.2.3 follows promptly.
The property appearing in the conclusion of Theorem 2.2.3 is very important; so important in fact that it merits its own definition.

**Definition 2.2.6.** Let \( T \) be an \( \omega \)-bisectorial operator. If (2.10) holds, we say that the operator \( T \) has a bounded \( \Psi(S^0_\mu) \) functional calculus.

Thereupon, Theorem 2.2.3 says that all injective \( \omega \)-bisectorial operators that satisfy quadratic estimates have a bounded \( \Psi(S^0_\mu) \) functional calculus. Of course, since \( f(T)u = 0 \) for all \( u \in \mathbb{N}(T) \) when \( f \in \Psi(S^0_\mu) \), by Proposition 2.1.12, then a non-injective \( T \) satisfying quadratic estimates on \( \mathbb{N}(T) \), also has a bounded \( \Psi \) functional calculus.

**Example 2.2.7**

All operators from Example 2.2.2 satisfy quadratic estimates, so they have a bounded \( \Psi(S^0_\mu) \) functional calculus.

(i) For a self-adjoint operator \( T \) at least, this should come as no surprise, since we know that in this case the holomorphic functional calculus extends via spectral integration to a Borel functional calculus, such that the estimate \( \|f(T)\|_{\mathcal{L}(\mathbb{H})} \leq \|f\|_{L^\infty(\sigma(T))} \) holds, for all Borel measurable functions \( f : \sigma(T) \to \mathbb{C} \), see [30, Appendix D].

(ii) This was to be expected for multiplication operators \( M_a \) as well, since the holomorphic functional calculus obtained through Dunford integration by (2.3) is consistent with the one defined via \( f \mapsto M_{f \circ a} \); in other words \( f(M_a) = M_{f \circ a} \), see [30, Section 1.4 and Example 2.3.15].

(iii) One can show that the operator \( BD \) has a bounded \( \Psi(S^0_\mu) \) functional calculus in \( L^2(\gamma) \) if and only if \( D_\gamma \) has a bounded \( \Psi(S^0_\mu) \) functional calculus in \( L^2(\gamma) \). Indeed, this follows from the fact that the operators \( D_\gamma \) and \( BD \) are similar (\( D_\gamma = V^{-1}BDV \), where \( V : L^2(\gamma) \to L^2(\mathbb{R}) \) is the isomorphism defined in Example 2.1.10), see [1, Section O].

Keep in mind that our goal is to extend the functional calculus of Section 2.1 to functions from the class \( \mathbb{H}^\infty \), i.e. to have a definition for \( f(T) \) where \( f \) is now drawn from the larger class \( \mathbb{H}^\infty(S^0_\mu) \) instead of its subset \( \Psi(S^0_\mu) \). To do this, we use the existence of such a good bound for \( \|f(T)\|_{\mathcal{L}(\mathbb{H})} \) together with forthcoming Proposition 2.2.8, usually referred to as The Convergence Lemma, see [5, Proposition 6.4], [45, Section 5].

**Proposition 2.2.8.** Assume that \( T \) is an injective, \( \omega \)-bisectorial operator satisfying quadratic estimates. Let \( f \in \mathbb{H}^\infty(S^0_\mu) \) and let \( (f_n)_{n=1}^\infty \in \Psi(S^0_\mu) \) be a sequence of functions such that \( \sup_n \|f_n\|_{L^\infty(S^0_\mu)} < \infty \) and \( f_n \to f \) pointwise. Then the operators \( f_n(T) \) converge strongly to a bounded operator \( f(T) \); that is, for all \( u \in \mathcal{H} \)

\[
\lim_{n \to \infty} f_n(T)u = f(T)u.
\]

We postpone the proof for a while, in order to show how this crucial proposition is used to define operators \( f(T) \) for functions \( f \in \mathbb{H}^\infty(S^0_\mu) \) and for bisectorial operators \( T \) that satisfy quadratic estimates.
Assume the validity of Proposition 2.2.8 and consider the sequence of functions from [46, Lemma 5.7.10]

\[
\psi_n(\zeta) := \frac{\cosh n\zeta}{\cosh n\zeta - 1}, \quad \zeta \in S^\mu, \quad n = 1, 2, 3, \ldots.
\]  

(2.13)

Let \( n \in \mathbb{N} \). Then, there exists a suitable constant \( C \), such that \( |\psi_n(\zeta)| \leq C \min\{|\zeta|, |\zeta|^{-1}\} \), for all \( \zeta \in S^\mu \), so \( \psi_n \in \Psi(S^\mu) \). Moreover, the sequence \( \psi_n \) is uniformly bounded, since \( \|\psi_n\|_{L^\infty(S^\mu)} \leq (\cos^2 \mu)^{-1} \). Furthermore, for every \( \zeta \in S^\mu \), \( \psi_n(\zeta) \to 1 \), as \( n \to \infty \). Let \( f \in \mathcal{H}^\infty(S^\mu) \) and consider \( f_n := \psi_n f \). It is immediate that \( f_n \) belongs to \( \Psi(S^\mu) \) for all \( n \), that \( f_n \to f \) pointwise (so \( f_n \to f \) uniformly on the compact subsets of \( S^\mu \)), see [30, Proposition 5.1.1] and that

\[
\|f_n\|_{L^\infty(S^\mu)} \leq \frac{1}{\cos^2 \mu} \|f\|_{L^\infty(S^\mu)}. 
\]  

(2.14)

Taking into account Proposition 2.2.8, we define

\[
f(T) : \mathcal{H} \longrightarrow \mathcal{H}; \quad u \longrightarrow f(T)u := \lim_n f_n(T)u.
\]

Since, by hypothesis, \( T \) satisfies quadratic estimates, Theorem 2.2.3 holds, so for every \( n \in \mathbb{N} \) we know that

\[
\|f_n(T)\|_{\mathcal{H} \longrightarrow \mathcal{H}} \leq C \|f_n\|_{L^\infty(S^\mu)} \quad \text{by (2.10)}
\]

\[
\leq C \|f\|_{L^\infty(S^\mu)} \quad \text{by (2.14)},
\]

hence \( \|f(T)\|_{\mathcal{H} \longrightarrow \mathcal{H}} \leq C \|f\|_{L^\infty(S^\mu)} \) and \( f(T) \in \mathcal{B}(\mathcal{H}) \).

We have actually proved the following corollary (see [16, Corollary 2.2]), where a slightly different sequence is used, namely \((1 + \zeta/n)^{-1} - (1 + n\zeta)^{-1}\).

**Corollary 2.2.9.** Let \( T \in \mathcal{C}(\mathcal{H}) \) be an injective bisectorial operator. Then \( T \) has a bounded \( \Psi(S^\mu) \) functional calculus if and only if it has a bounded \( \mathcal{H}^\infty(S^\mu) \) functional calculus, i.e. there exists a positive constant \( C \), such that

\[
\|f(T)\|_{\mathcal{H} \longrightarrow \mathcal{H}} \leq C \|f\|_{L^\infty(S^\mu)},
\]

for all \( f \in \Psi(S^\mu) \), if and only if, there exists a positive constant \( C \), such that

\[
\|f(T)\|_{\mathcal{H} \longrightarrow \mathcal{H}} \leq C \|f\|_{L^\infty(S^\mu)},
\]

for all \( f \in \mathcal{H}^\infty(S^\mu) \).

We remark that the preceding process goes through for a non-injective bisectorial operator \( T \) as well; simply repeat the same arguments for \( u \in \mathcal{P}(\mathcal{T}) \), whereas for \( u \in \mathcal{N}(\mathcal{T}) \), set \( f(T)u = f(0)u \), where now the function \( f \) is drawn from the class \( \mathcal{H}^\infty(S^\mu \cup \{0\}) := \{f : S^\mu \cup \{0\} \rightarrow \mathbb{C} : f|_{S^\mu} \in \mathcal{H}^\infty(S^\mu)\} \). Notice that functions from this class are not necessarily continuous, let alone holomorphic, at zero, so \( f(0) \) is just a finite complex number. Thus, for any bisectorial operator \( T \), the operator \( f(T) \) is defined for \( f \in \mathcal{H}^\infty(S^\mu \cup \{0\}) \) by

\[
f(T)u = f(0)u + \lim_{n \to \infty} \psi_n(T)u,
\]  

(2.15)
for all $u \in \mathcal{H}$, where $\psi_n$ are suitable uniformly bounded functions in $\Psi(S^\mu_\omega)$ that tend to $f$ uniformly on compact subsets of $S^\mu_\omega$. This is consistent with the equivalent definitions of the functional calculus we mentioned earlier, see [30, Theorem 2.3.3]. For an injective bisectorial operator $T$, the map $H^\omega(S^\mu_\omega) \ni f \mapsto f(T) \in \mathcal{B}(\mathcal{H})$ is a continuous (unital) Banach algebra homomorphism.

It turns out that the converse of Theorem 2.2.3 also holds. In fact, regarding the boundedness of the holomorphic functional calculus, we summarize the situation in the following theorem, essentially drawn from [45, Section 8]. See [1, Theorem F] and [30, Theorem 7.3.1].

**Theorem 2.2.10.** Let $T$ be an injective $\omega$-bisectorial operator in a Hilbert space $\mathcal{H}$. Then, the following statements are equivalent:

(i) $T$ has a bounded $H^\omega(S^\mu_\omega)$ functional calculus, for some $\mu \in (\omega, \pi/2)$.

(ii) $T$ has a bounded $H^\omega(S^\mu_\omega)$ functional calculus, for all $\mu \in (\omega, \pi/2)$.

(iii) $T$ satisfies quadratic estimates.

Therefore, there is no ambiguity in saying that an operator $T$ has bounded $H^\omega(S^\mu_\omega)$ functional calculus without specifying the precise angle $\mu > \omega$. Due to Corollary 2.2.9, each of the statements (i)-(iii) is also equivalent to:

(iv) $T$ has a bounded $\Psi(S^\mu_\omega)$ functional calculus, for some/all $\mu \in (\omega, \pi/2)$.

It is clear that the same operators from [4] and [48] that do not satisfy quadratic estimates, do not have a bounded functional calculus.

We also note that there is a close relationship between quadratic estimates, bounded imaginary powers of operators (i.e. expressions of the form $T^\mu$) and interpolation spaces. For these matters, that lie beyond the scope of this thesis, we refer to [1, Section F], [30, Chapter 3] and the appropriate references therein.

We now turn to the proof of the Convergence Lemma. Recall that for an $\omega$-bisectorial operator $T$, the operator $T((1 + T^2)^{-1})$ is well-defined. If $T$ is assumed to be injective, then the same goes for $T((1 + T^2)^{-1})$. Moreover,

$$R(T(I + T^2)^{-1}) = TR((I + T^2)^{-1}) = TD(I + T^2) = T(1 + T^2)^{-1} \cap R(T),$$

and when $T$ is injective $R(T(I + T^2)^{-1}) = \mathcal{H}$, as follows from [16, Theorem 3.8].

**Proof of Proposition 2.2.8:** Since $\mathcal{H}$ is complete, it suffices to show that $(f_n(T)u)_{n=1}^\infty$ is a Cauchy sequence, for all $u \in \mathcal{H}$. Because $\sup_n \|f_n\|_{L^\infty(S^\mu_\omega)} < \infty$ and $T$ satisfies quadratic estimates, Theorem 2.2.3 implies that the operators $f_n(T)$ are uniformly bounded, i.e. $\sup_n \|f_n(T)\|_\mathcal{H} \leq C$. Thus, in light of Lemma 2.2.4, it suffices to show that $(f_n(T)u)_{n=1}^\infty$ is Cauchy, for all $u = \psi(T)v$, $v \in \mathcal{H}$, where $\psi(\zeta) = \zeta(1 + \zeta^2)^{-1} \in \Psi(S^\mu_\omega)$, since $\overline{R(\psi(T))} = \mathcal{H}$.
To that end
\[
\|f_n(T)u - f_m(T)u\|_{\mathcal{H}} = \|f_n(T)(\psi(T)v) - f_m(T)(\psi(T)v)\|_{\mathcal{H}}
\]
\[
\leq C \int_{\gamma} |f_n(\xi) - f_m(\xi)| \frac{\psi(\xi)}{\xi} |\psi(\xi)\xi^{-1}| |d\xi|
\]
\[
\leq C \int_{\gamma} |f_n(\xi) - f_m(\xi)| \frac{\psi(\xi)}{\xi} |v||\psi(\xi)| |d\xi|
\]

Now, since \(f_n\) converges pointwise, the sequence \((f_n(\xi))_n\) is Cauchy, for every \(\xi \in \mathbb{S}_\mu^0\). Thus, \(|f_n(\xi) - f_m(\xi)| \to 0\), as \(n, m \to \infty\), for all \(\xi \in \mathbb{S}_\mu^0\). Furthermore \(|f_n(\xi) - f_m(\xi)| \leq 2\sup_n \|f_n\|_{L^\infty(\mathbb{S}_\mu^0)} < \infty\) and \(\int_{\gamma} |\psi(\xi)| |d\xi| < \infty\). Therefore, by Lebesgue's dominated convergence theorem

\[
\|f_n(T)u - f_m(T)u\|_{\mathcal{H}} \longrightarrow 0, \quad n, m \to \infty.
\]

Notice that the functions \(\psi_n\), where \(\psi_n\) is given by (2.13), converge pointwise to the functions \(\psi\) and that \(\psi_n\) converges pointwise to the signum function. The following figures illustrate this phenomenon.

**Figure 2.2(A):** The sequence of real-valued regularly decaying functions converging pointwise to the signum function.

**Figure 2.2(B):** The sequence of real-valued regularly decaying functions converging pointwise to the signum function.

If the operator \(T\) admits a bounded \(\Psi(\mathbb{S}_\mu^0)\) functional calculus, the operators \(\chi_{\pm}(T)\), \(\text{sgn}(T)\) are defined via Proposition 2.2.8. Subsequently, we are lead to the following definition.

**Definition 2.2.11.** Let \(T\) be an injective, \(\omega\) bisectorial operator that has bounded \(H^\infty(\mathbb{S}_\mu^0)\) functional calculus. Then the operators

\[
E_T^\pm := \chi_{\pm}(T) \quad \text{and} \quad E_T := \text{sgn}(T),
\]  

(2.16)
are well defined and bounded operators on \( \mathcal{H} \). The operators \( E_T^\pm \) are called the generalised Hardy projections and \( E_T \) is called the generalised Cauchy operator; see [7, Section 3].

The next proposition is an immediate consequence of the homomorphism property of the functional calculus, see [1, Section H].

**Proposition 2.2.12.** Let \( T \) be an injective, \( \omega \)-bisectorial operator that has bounded \( \mathcal{H}^\infty(S_\mu^\omega) \) functional calculus. Then the generalised Hardy projections \( E_T^\pm \in \mathcal{B}(\mathcal{H}) \) satisfy the following identities

\[
\begin{align*}
(i) & \quad (E_T^\pm)^2 = E_T^\pm, \\
(ii) & \quad E_T^+ E_T^- = 0, \\
(iii) & \quad E_T^+ + E_T^- = I_{\mathcal{H}}, \\
(iv) & \quad E_T^+ - E_T^- = \text{sgn}(T).
\end{align*}
\]

We remark that the use of the term “projection” for the operators \( E_T^\pm \) is justified by the first identity of the aforementioned proposition. Identities (ii) and (iii) tell us that the projections are complementary.

For a non-injective operator, the operator \( E_T^0 = \chi_0(T) \) can also be introduced, where \( \chi_0(0) = 1 \) and \( \chi_0(\zeta) = 0 \), for all \( \zeta \neq 0 \). It is seen that \( E_T^0 = P_{\mathcal{N}(T)} \), where \( P_{\mathcal{N}(T)} \) is the (not necessarily orthogonal) projection on the nullspace of \( T \) along \( \mathcal{R}(T) \), see [46, Proposition 5.7.15].

The generalised Hardy projections give the following splitting of the Hilbert space \( \mathcal{H} \), see [1, Section H]. For generalities on projections and complementary subspaces see, for example, [12, Section 2.4], [22, Section 6.3].

**Proposition 2.2.13.** Let \( T \) be an injective, \( \omega \)-bisectorial operator that has bounded \( \mathcal{H}^\infty(S_\mu^\omega) \) functional calculus. Then

\[
\mathcal{H} = \mathcal{R}(E_T^+) \oplus \mathcal{R}(E_T^-),
\]

where the splitting is topological, in the sense of Banach spaces.

**Proof:** It is well known that two bounded complementary projections on a Banach space provide such a splitting (and vice versa), see [37, Section 3.3.4].

The notation \( \mathcal{H}^\pm := \mathcal{R}(E_T^\pm) = E_T^\pm \mathcal{H} \) is used for the spectral subspaces of \( \mathcal{H} \). Note that for a non-injective \( T \), the Hilbert space \( \mathcal{H} \) splits as

\[
\mathcal{H} = \mathcal{N}(T) \oplus \mathcal{H}^+ \oplus \mathcal{H}^-.
\]

We emphasize that the decomposition in (2.17) is not orthogonal in general. However, for a self-adjoint operator \( T \), it can be seen that \( \|E_T^\pm\|_{\mathcal{H} \to \mathcal{H}} = 1 \), so the splitting turns out to be orthogonal in this case.

Now, let \( T \) be an \( \omega \)-bisectorial operator that has bounded \( \mathcal{H}^\infty(S_\mu^\omega) \) functional calculus. Then the operator \( |T| \), where \( |T|u = \pm Tu \) if \( u \in \mathcal{D}(T) \cap \mathcal{H}^\pm \) and \( |T|u = 0 \) if \( u \in \mathcal{N}(T) \) (i.e. \( |T| = T\text{sgn}(T) \)), is an \( \omega \)-sectorial operator in \( \mathcal{H} \). This is because \( \pm T|_{\mathcal{H}^\pm} \) is \( \omega \)-sectorial.
in \( \mathcal{H}^+ \). Indeed, let \( u \in \mathcal{D}(T) \cap \mathcal{H}^+ \), then \( T|_{\mathcal{H}^+} u = TE^+_T u = E^+_T Tu \in \mathcal{H}^+ \) by Proposition 2.1.13, so \( T|_{\mathcal{H}^+} : \mathcal{H}^+ \to \mathcal{H}^+ \) is well-defined. Moreover, \( T|_{\mathcal{H}^+} \in C(\mathcal{H}^+) \) since \( T \in C(\mathcal{H}) \) and \( \text{sgn}(T) \in \mathcal{B}(\mathcal{H}) \) (see, for example, [46, Proposition 3.1.12(i)]). Furthermore, for \( \zeta \notin \mathbb{S}_{\omega^+} \cup \{ \infty \} \), consider the holomorphic function

\[
f(z) := \begin{cases} \frac{1}{\zeta - z}, & \text{if } z \in \mathbb{S}_{\mu^+}, \\ 0, & \text{if } z \in \mathbb{S}_{\mu^-}, \end{cases}
\]

where \( \mu \in (\omega, \pi/2) \) is such that \( \| f \|_{L^\infty(B)} < \infty \). It follows that \( (\zeta - z)f(z) = \chi_{\mathbb{S}_{\mu^+}}(z) \), hence on the closed subspace \( \mathcal{H}^+ \)

\[
(\zeta - T|_{\mathcal{H}^+})f(T) = E^+_T = I_{\mathcal{H}^+}.
\]

Therefore, \( \sigma(T|_{\mathcal{H}^+}) \subset \mathbb{S}_{\omega^+} \cup \{ \infty \} \). Corollary 2.2.9 with \( f(z) = (\zeta - z)^{-1}\chi_{\mathbb{S}_{\omega^+}}(z) \) for \( z \in \mathbb{S}_{\mu^+} \) guarantees that there exists a positive constant \( C \) such that

\[
\left\| R_{T|_{\mathcal{H}^+}}(\zeta) \right\|_{\mathcal{H}^+ \to \mathcal{H}^+} = \| f(T) \|_{\mathcal{H}^+ \to \mathcal{H}^+} \leq C \| f \|_{L^\infty(B)} \leq \frac{C}{\text{dist}(\zeta, \mathbb{S}_{\omega^+})},
\]

for all \( \zeta \in \mathbb{C} \setminus \mathbb{S}_{\omega^+} \). This concludes the proof that \( T|_{\mathcal{H}^+} \) is \( \omega \)-sectorial. The operator \( T|_{\mathcal{H}^+} \) can be treated similarly.

Clearly, Proposition 2.2.8 can be adapted to cover the sectorial case as well. Thus, if we start with an \( \omega \)-sectorial operator, since \( \zeta \mapsto e^{-it\zeta}, \ t > 0, \ \text{belongs to} \ \mathcal{H}^+(\mathbb{S}_{\mu^+}) \), the operator \( e^{-it\zeta} \) is well defined for \( t \in (0, \infty) \); moreover, \( e^{-it\zeta} \in \mathcal{B}(\mathcal{H}) \). In fact, \( e^{-it\zeta} \) defines a bounded holomorphic semigroup, in the sense of [37, Section 9.1.6]; see also [30, Section 3.4]. In particular, this semigroup has the properties listed in the proposition below. Proofs can be found in [30, Section 3.4] and [37, Chapter 9.1]; see also [4, Chapters 1, 2]. For a generalization of the concept to more general linear spaces, see [55, Chapter 9]; for elaborations on evolution equations, see [12, Chapter 7].

**Proposition 2.2.14.** Let \( T \) be an \( \omega \)-sectorial operator satisfying quadratic estimates. Then

(i) The family of operators \( (e^{-it\zeta})_{0 \leq t \leq \infty} \) is uniformly bounded in \( t \), i.e. there exists \( C > 0 \) such that \( \| e^{-it\zeta} \|_{\mathcal{H}^+ \to \mathcal{H}^+} \leq C \), for all \( t \in [0, \infty] \).

(ii) The family of operators \( (e^{-it\zeta})_{0 \leq t \leq \infty} \) satisfies the semigroup law, i.e. \( e^{-it\zeta} e^{-is\zeta} = e^{-(i(t+s)\zeta)} \), for all \( t, s \in [0, \infty] \).

(iii) The semigroup \( (e^{-it\zeta})_{0 \leq t \leq \infty} \) is continuous, i.e. the mapping \( (0, \infty) \to \mathcal{B}(\mathcal{H}) \); \( t \mapsto e^{-it\zeta} \), is continuous.

(iv) The semigroup \( (e^{-it\zeta})_{0 \leq t \leq \infty} \) is strongly differentiable in \( t \) and \( \frac{d}{dt} e^{-it\zeta} = -Te^{-it\zeta} \in \mathcal{B}(\mathcal{H}) \).

(v) If \( u \in \mathcal{N}(T) \), then \( e^{-it\zeta} u = u \), for all \( t \geq 0 \).

(vi) For all \( u \in \mathcal{R}(T) \), \( \lim_{t \to \infty} e^{-it\zeta} u = 0 \).
(vii) For all \( u \in \mathcal{H} \), \( \lim_{t \to 0} e^{-tT}u = u \); i.e. the semigroup is strongly continuous at zero, with limit \( I_\mathcal{H} \).

(viii) The function \( u : (0, \infty) \to \mathcal{B}(\mathcal{H}) ; t \mapsto e^{-tT}u_0 \) is the unique solution to the evolution problem

\[
\begin{aligned}
\frac{d}{dt} u(t) + Tu(t) &= 0, \\
\lim_{t \to 0} u(t) &= u_0, \\
\lim_{t \to \infty} u(t) &= 0,
\end{aligned}
\]

for any fixed initial value \( u_0 \in \mathcal{H} \).

The aforementioned proposition will be fully exploited in the next section. We emphasize that for an \( n \)-bisectorial operator \( T \), we must necessarily consider \( e^{-t|T|} \), since \( e^{-tT} \) is not well-defined for any \( t \neq 0 \).

### 2.3 The equation \( \text{div} A \nabla u = 0 \)

Consider the divergence form, second order partial differential equation

\[
\text{div}_{x} A(x) \nabla_{x} u(t, x) = 0,
\]

defined in the upper half space \( \mathbb{R}^{1+n}_+ := \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t > 0\} \), for \( n \geq 1 \). The coefficient matrix \( A \) is assumed to be a \( t \)-independent matrix of complex-valued \( L^\infty \) coefficients, i.e. \( A = (A_{ij}(x))_{i,j=0}^n \in L^\infty (\mathbb{R}^n, M_{1+n}(\mathbb{C})) \), that additionally is strictly pointwise accretive, i.e. that there exists a positive constant \( C \), such that

\[
C |\xi|^2 \leq \text{Re} \langle A(x)\xi, \xi \rangle = \sum_{i,j=0}^n A_{ij}(x) \xi_i \bar{\xi}_j, \tag{2.20}
\]

for all \( \xi \in \mathbb{C}^{1+n} \) and for almost every \( x \in \mathbb{R}^n \). In other words, the divergence form equation is assumed uniformly elliptic. Here and in the sequel, whenever \( \xi \) is a vector in the finite dimensional space \( \mathbb{C}^{1+n} , |\xi|^2 = \sum_{j=0}^n |\xi_j|^2 \) will stand for its standard norm; \( \langle \xi, \zeta \rangle = \xi \cdot \zeta = \sum_{j=0}^n \xi_j \bar{\zeta}_j \) will denote the standard inner product of \( \xi, \zeta \in \mathbb{C}^{1+n} \). Clearly, the adjoint matrix \( A^* \) is also strictly pointwise accretive whenever \( A \) is and vice versa.

A function \( u \) is a solution of equation (2.19) if it is a weak solution, i.e. if \( u \in W^{1,2}_{\text{loc}} (\mathbb{R}^{1+n}_+) \) and for all complex-valued \( \phi \in C_0^\infty (\mathbb{R}^{1+n}_+) \)

\[
\iint_{\mathbb{R}^{1+n}} \langle A(x)\nabla_{x} u(t, x), \nabla_{x} \phi(t, x) \rangle \, dt \, dx = 0. \tag{2.21}
\]

Recall that

\[
W^{1,2}_{\text{loc}} (\mathbb{R}^{1+n}_+) = \{ f \in L^2_{\text{loc}} (\mathbb{R}^{1+n}_+) : \phi f \in W^{1,2} (\mathbb{R}^{1+n} ), \text{ for all } \phi \in C_0^\infty (\mathbb{R}^{1+n}_+) \}.
\]

Such standard solutions are obtained in the usual, abstract functional analytic way, by applying the Lax-Milgram lemma in suitable function spaces, see [7, Section 5] and, for example, [12, Section 9.5].
A point worth making is that, due to the $t$-independence of the coefficients, if $u$ solves equation (2.19), then $\partial_t u$ is also a solution. Indeed, it is easy to see that

$$\int_{\mathbb{R}^{1+n}} \langle A(x) \nabla_{t,x} (\partial_t u(t,x)), \nabla_{t,x} \phi(t,x) \rangle \, dx = \int_{\mathbb{R}^{1+n}} \langle \nabla_{t,x} (\partial_t u(t,x)), A(x)^* \nabla_{t,x} \phi(t,x) \rangle \, dx$$

$$= -\int_{\mathbb{R}^{1+n}} \langle A(x) \nabla_{t,x} u(t,x), \nabla_{t,x} (\partial_t \phi(t,x)) \rangle \, dx$$

$$= 0,$$

for all complex-valued test functions $\phi \in C_0^\infty(\mathbb{R}^{1+n})$, which is the same as saying that

$$0 = \partial_t \left( \text{div}_{t,x} A(x) \nabla_{t,x} u(t,x) \right) = \text{div}_{t,x} A(x) \nabla_{t,x} (\partial_t u(t,x),$$

in the sense of distributions.

The first main result of this section is the reformulation of (2.19) as an equivalent vector-valued ordinary differential equation (see Proposition 2.3.3) and the solution of this equation by means of the functional calculus developed in Sections 2.1 and 2.2 (see Theorem 2.3.12). This first-order approach was first used in [6], where (2.19) was solved with respect to the gradient $\nabla_{t,x} u$. In [7] the conormal gradient $\nabla_A u$ (see Definition 2.3.2 below) was taken as the unknown in (2.19) and it is there where the simple structure of the – intimately related to $A$ – operator $DB$ (where $B$ and $D$ are as in (2.30) further down) was exploited; see also [5, Section 3].

The second main result is the construction of solutions to the Dirichlet problem for (2.19) in the upper half-space via functional calculus. For this, similar functional calculus-theoretic arguments as before, are applied to a modified first-order vector-valued ordinary differential equation. Moreover, certain $L^2$-estimates are obtained for this particular class of solutions. All this takes place in Subsection 2.3.1.

As seen by the proposition below, the strict pointwise accretivity of the coefficient matrix appearing in equation (2.19) is a condition of considerable strength.

**Proposition 2.3.1.** Consider the Hilbert space

$$\mathcal{H} := L^2(\mathbb{R}^n; \mathbb{C}^{1+n}),$$

and let $A \in L^\infty(\mathbb{R}^n; M_{1+n}(\mathbb{C}))$ be strictly pointwise accretive in the sense of (2.20). Define the operator $M_A$ acting on vector fields by multiplication

$$M_A : \mathcal{H} \rightarrow \mathcal{H} : f \mapsto Af.$$

Then, $M_A$ is strictly accretive on $\mathcal{H}$, in the sense that there exists a positive constant $C$, such that

$$C \|f\|^2_{\mathcal{H}} \leq \text{Re} \langle Af, f \rangle_{\mathcal{H}} = \text{Re} \int_{\mathbb{R}^n} \langle A(x)f(x), f(x) \rangle \, dx,$$

for all $f \in \mathcal{H}$. Moreover, $M_A$ is bounded and has a bounded inverse.
2.3 The equation \( \text{div} \nu u = 0 \)

**Proof:** That \( M_A \in \mathcal{B}(\mathcal{H}) \) follows from the fact that \( A \in L^\infty(\mathbb{R}^n; M_{1+n}(\mathbb{C})) \). Since \( f(x) \in C^{1+n} \) for almost every \( x \in \mathbb{R}^n \), integrating (2.20) with respect to \( x \) yields (2.24). From the strict pointwise accretivity of \( A \), it follows that \( A^{-1}(x) : C^{1+n} \rightarrow C^{1+n} \) exists for almost every \( x \in \mathbb{R}^n \) and that there exists a positive constant \( C \) such that \( |A^{-1}(x)| \leq C \), for almost every \( x \in \mathbb{R}^n \). Thus, the operator

\[
M_{A^{-1}} : \mathcal{H} \rightarrow \mathcal{H} : f \mapsto A^{-1} f,
\]

is well defined and bounded (even strictly accretive; see, for example, [46, Proposition 5.3.7]). It is easily verified that \( (M_A)^{-1} = M_{A^{-1}} \). \( \Box \)

We need to fix some more notation and conventions. The standard basis of the ordinary Euclidean space \( \mathbb{R}^{1+n} \) is \( \{e_0, e_1, \ldots, e_n\} \), where \( e_0 \) points upwards into \( \mathbb{R}^{1+n} \). For the vertical coordinate we use \( t \) instead of \( x_0 \). Thus, the vertical derivative is mostly written \( \partial_t \) instead of \( \partial_{x_0} \).

A vector \( v \in C^{1+n} \) has normal and tangential parts, written as \( v_\parallel \equiv [v_\parallel, 0, \ldots, 0]^T \) and \( v_\perp \equiv [0, v_1, v_2, \ldots, v_n]^T \) respectively. The vector itself is written as a column, so

\[
v = [v_\parallel, v_\perp]^T = \begin{bmatrix} v_\parallel \\ v_\perp \end{bmatrix}.
\]

Notice that \( \langle v, e_0 \rangle = v \cdot e_0 = v_\perp \). For functions \( f : \mathbb{R}^{1+n} \rightarrow \mathbb{C} \) (the full) gradient is

\[
\nabla_{t,x} f(t,x) := [\partial_t f(t,x), \partial_{x_1} f(t,x), \ldots, \partial_{x_n} f(t,x)]^T \in \mathbb{C}^{1+n}, \quad (t,x) \in \mathbb{R}^{1+n},
\]

with tangential counterpart

\[
\nabla_{\parallel} f(t,x) := [\partial_{x_1} f(t,x), \ldots, \partial_{x_n} f(t,x)]^T = (\nabla_{t,x} f(t,x))_{\parallel} \in \mathbb{C}^{n}, \quad (t,x) \in \mathbb{R}^{1+n}.
\]

For vector fields

\[
f : \mathbb{R}^{1+n} \rightarrow \mathbb{C}^{1+n} : (t,x) \mapsto f(t,x) := \begin{bmatrix} f_\parallel(t,x) \\ f_\perp(t,x) \end{bmatrix} = \begin{bmatrix} f_\parallel(t,x), f_1(t,x), \ldots, f_n(t,x) \end{bmatrix}^T,
\]

the (full) divergence is defined by

\[
\text{div}_{t,x} f(t,x) := \sum_{j=0}^n \partial_{x_j} f_j(t,x) \in \mathbb{C}, \quad (t,x) \in \mathbb{R}^{1+n},
\]

with tangential counterpart \( \text{div}_{\parallel} f(t,x) := \sum_{j=1}^n \partial_{x_j} f_j(t,x) \). When it is necessary to specify the variable with respect to which differentiations take place we will use the more cumbersome symbols \( \nabla^\perp \) and \( \text{div}^\perp \).

The statement “\( \text{curl}_t f = 0 \)” means that \( \partial_{x_j} f_j = \partial_{x_i} f_i \) for all \( i, j = 0, 1, \ldots, n \). As expected, “\( \text{curl}_x f = 0 \)” means that \( \partial_{x_j} f_j = \partial_{x_i} f_i \) for all \( i, j = 1, \ldots, n \) (the vertical component and the vertical derivative are left alone this time). Needless to say that \( \text{curl}_t^\parallel \) will be used when it is necessary to specify the variable with respect to which differentiation takes place, as earlier.

**Definition 2.3.2.** The **conormal gradient** (with respect to the coefficient matrix \( A \)) of a function \( u : \mathbb{R}^{1+n} \rightarrow \mathbb{C} \) is given by

\[
\nabla_A u := [\partial_{x_1} u, \nabla_{\parallel} u]^T,
\]

where \( \partial_{x_1} u := \langle A \nabla_{t,x} u \rangle_{\parallel} = \langle A \nabla_{t,x} u, e_0 \rangle \) is the conormal derivative of \( u \) for the inward-pointing unit normal.
The coefficient matrix \( A \) can be written as

\[
A(x) = \begin{bmatrix}
A_{\perp \perp}(x) & A_{\perp}(x) \\
A_{\parallel \perp}(x) & A_{\parallel}(x)
\end{bmatrix}, \quad x \in \mathbb{R}^n,
\]

where \( A_{\perp \perp}(x), A_{\perp \parallel}(x), A_{\parallel \parallel}(x) \) are, respectively, \( 1 \times n, n \times 1, n \times n \) matrices with complex entries and \( A_{\perp \perp}(x) \), being a \( 1 \times 1 \) matrix, is just a complex number. This decomposition is consistent with the splitting of vectors to normal and tangential parts. Furthermore, define the following auxiliary matrices:

\[
\overline{A} := \begin{bmatrix}
A_{\perp \perp} & A_{\perp \parallel} \\
0 & I
\end{bmatrix} \quad \text{and} \quad A := \begin{bmatrix}
1 & 0 \\
A_{\perp \parallel} & A_{\parallel \parallel}
\end{bmatrix}.
\]

Notice that \( \nabla A u = \overline{A} \nabla_{t,u} \).

Due to the strict pointwise accretivity of \( A \), it is easy to see that the diagonal blocks, namely \( A_{\perp \perp} \) and \( A_{\parallel \parallel} \), are also strictly pointwise accretive. Thus, similarly to Proposition 2.3.1, both of the multiplication operators

\[
M_{\overline{A}} : \mathcal{H} \to \mathcal{H} \quad \text{and} \quad M_{\overline{A}} : \mathcal{H} \to \mathcal{H},
\]

where \( \overline{A} \) and \( A \) are given by (2.26) and \( \mathcal{H} = L^2(\mathbb{R}^n; \mathbb{C}^{1+n}) \) as in Proposition 2.3.1, are bounded, strictly accretive and invertible.

After these preliminaries, we turn to the core of this section which is the reduction of the equation to an equivalent vector-valued ordinary differential equation and its solution via functional calculus. In what follows, all derivatives are taken in distributional sense.

By setting \( g(t,x) = \nabla_{t,x} u(t,x) \), we see that (2.19) is equivalent to the first order system

\[
\begin{cases}
\text{div} \, t, x A(x) g(t,x) = 0, \\
\text{curl} \, t, x g(t,x) = 0,
\end{cases}
\]

where the second equation expresses the fact that \( g \) is an irrotational vector field, e.g. a gradient. By splitting normal and tangential derivatives, we see that (2.27) is equivalent to the system

\[
\begin{cases}
\frac{\partial}{\partial t} (Ag)_{\perp} + \text{div}_{\parallel} (Ag)_{\parallel} = 0, \\
\frac{\partial}{\partial t} g_{\parallel} - \nabla_{\parallel} g_{\perp} = 0, \\
\text{curl}_{\parallel} g_{\parallel} = 0,
\end{cases}
\]

which we rewrite as

\[
\begin{cases}
\frac{\partial}{\partial t} \begin{bmatrix}(Ag)_{\perp} \\
g_{\parallel}
\end{bmatrix} + \begin{bmatrix} 0 & \text{div}_{\parallel} \\
-\nabla_{\parallel} & 0
\end{bmatrix} \begin{bmatrix} g_{\perp} \\
(Ag)_{\parallel}
\end{bmatrix} = 0, \\
\text{curl}_{\parallel} g_{\parallel} = 0.
\end{cases}
\]

By unwrapping the definition of \( g \) and making use of the matrices \( \overline{A} \) and \( A \), we see that this is the same as

\[
\begin{cases}
\frac{\partial}{\partial t} (\nabla A u) + \begin{bmatrix} 0 & \text{div}_{\parallel} \\
-\nabla_{\parallel} & 0
\end{bmatrix} \begin{bmatrix} \overline{A}^{-1} \nabla A u \\
\overline{A}^{-1} \nabla A u_{\parallel}
\end{bmatrix} = 0, \\
\text{curl}_{\parallel} (\overline{A}^{-1} \nabla A u_{\parallel}) = 0.
\end{cases}
\]

(2.29)
Finally, after setting
\[ B := \Delta A^{-1} \quad \text{and} \quad D := \begin{bmatrix} 0 & \text{div}\| \nabla \| \\ -\nabla \| & 0 \end{bmatrix}, \quad (2.30) \]
and taking note of the fact that \( \text{curl} (\Delta A^{-1} \nabla_A u)\| = 0 \) if and only if \( \text{curl}_{t,x} (\nabla_{t,x} u)\| = 0 \), we see that solving (2.19) for the conormal gradient \( f = \nabla_A u \) is equivalent to solving
\[ \partial_t f + DB f = 0, \quad (2.31) \]
for vector fields \( f : \mathbb{R}^1 + n \to \mathbb{C}^{1+n} \), such that \( f(t, \cdot) \in \mathcal{H}_{\text{curl}} \), for every fixed \( t \in \mathbb{R}_+ \), where
\[ \mathcal{H}_{\text{curl}} := \{ h = [h_l, h_r]^t \in L^2(\mathbb{R}^n; \mathbb{C}^{1+n}) : \text{curl}_l h_l = 0 \}. \quad (2.32) \]
Note that only derivatives along the boundary \( \partial \mathbb{R}^1 + n = \mathbb{R}^n \) appear in \( D \). It is clear that if \( g = [g_\perp, g]\| \) is a curl-free vector field in \( L^2(\mathbb{R}^1 + n; \mathbb{C}^{1+n}) \), there exists a function \( u \in W_{\text{loc}}^{1,2}(\mathbb{R}^1 + n) \), such that \( g = \nabla_{t,x} u \) and satisfying (2.21), i.e. \( u \) is a solution to (2.19). Thus, starting from a vector field \( f \) satisfying (2.31) and such that \( f(t, \cdot) \in \mathcal{H}_{\text{curl}} \), for all \( t > 0 \) and considering \( g = [BF]_\perp, f]\| \) which is seen to satisfy the system (2.27), one ends up with a solution to (2.19). Ergo, the original elliptic equation (2.19) and equation (2.31) are indeed equivalent.

Equation (2.31) is sometimes referred to as a generalized Cauchy-Riemann equation, since in light of Proposition 2.2.14(vii), when \( f = u + iv \in L^2(\mathbb{R}^2) \), \( D = -i \frac{\partial}{\partial x} \) and \( B = 1 \), a simple computation shows that the solutions correspond to holomorphic functions, see [1, Section J], [46, Example 5.7.22].

It is important to notice that the operator \( DB \) appearing in (2.31) depends only on the variable \( x \) and not on the variable \( t \). Thus one could try to obtain a solution to the equation by resorting to Proposition 2.2.14 for \( T = DB \). However, Proposition 2.2.14 cannot be directly applied, as \( DB \) is not a sectorial operator, but a bisectorial one; see Proposition 2.3.7 below. In order to overcome this technical problem, one needs to consider appropriate spectral subspaces on which the holomorphic semigroup is well-defined, using the generalized Hardy projections. For this to work, it is necessary for the operator \( DB \) to have a bounded functional calculus, which is indeed the case, since \( DB \) satisfies quadratic estimates; see Theorem 2.3.9 below.

The following proposition sheds more light on the equivalence between the system (2.27) and equation (2.31); see [7, Proposition 3.2], [5, Proposition 4.1].

**Proposition 2.3.3.** There exists a one-to-one correspondence between solutions \( g \in L^2_{\text{loc}}(\mathbb{R}_+; \mathcal{H}) \) to the system (2.27) and solutions \( f \in L^2_{\text{loc}}(\mathbb{R}_+; \mathcal{H}_{\text{curl}}) \) to equation (2.31), given by
\[ g \mapsto f := [(Ag)\perp, g]\|, \]
with inverse
\[ f \mapsto g := [(Bf)\perp, f]\|, \]
where \( B \) and \( D \) are as in (2.30) and the derivatives are taken in distributional sense. Moreover, the mapping \( \Phi : A \to B \) is a self-inverse bijective transformation on the set of bounded strictly pointwise accretive matrices in \( L^\infty(\mathbb{R}^n; M_{(1+n)}(\mathbb{C})) \).
that end, since $B$ is strictly pointwise accretive. We shall show that $\Phi$ is bijective and self-inverse. Now, let $A$ be strictly pointwise accretive. To that end, since $A_{\perp\perp}$ is also strictly pointwise accretive, $A^{-1}(x)$ exists for almost every $x \in \mathbb{R}^n$. Thus, for almost every $x \in \mathbb{R}^n$, for every $\xi \in C^{1+n}$ there exists a $\zeta \in C^{1+n}$ such that $\xi = A(x)\zeta$. Hence

$$\Re \langle B(x)\xi, \xi \rangle = \Re \langle A(x)A^{-1}(x)A(x)\zeta, A(x)\zeta \rangle = \Re \langle A(x)\xi, A(x)\zeta \rangle$$

$$= \Re \left\langle \begin{bmatrix} 1 & 0 \\ A_{\perp\perp}(x) & A_{\parallel\parallel}(x) \end{bmatrix} \begin{bmatrix} \xi_{\perp} \\ \zeta_{\perp} \end{bmatrix}, \begin{bmatrix} A_{\perp\perp}(x) & A_{\parallel\parallel}(x) \end{bmatrix} \begin{bmatrix} \xi_{\perp} \\ \zeta_{\perp} \end{bmatrix} \right\rangle$$

$$= \Re \langle A(x)\xi, \zeta \rangle \geq C|\xi|^2 = C\|A^{-1}(x)\xi\|^2 \geq C|\xi|^2,$$

for all $\xi \in C^{1+n}$ and for almost every $x \in \mathbb{R}^n$. The converse, that $A$ is strictly pointwise accretive if $B$ is strictly pointwise accretive, follows immediately from the fact that $\Phi^{-1} = \Phi$ or from reversing the previous computations. 

Proof: Noting the $L^\infty$-boundedness of the coefficient matrix $A$, ergo of $B$ as well, it is easy to see that $f$ is locally square integrable if and only if $g$ is locally square integrable. Therefore, the one-to-one correspondence $f \leftrightarrow g$ follows from the preceding discussion. It is routine, using (2.28), to check that $\Phi$ is bijective and self-inverse. Now, let $A$ be strictly pointwise accretive. We shall show that $B$ is also strictly pointwise accretive. To that end, since $A_{\perp\perp}$ is also strictly pointwise accretive, $A^{-1}(x)$ exists for almost every $x \in \mathbb{R}^n$. Thus, for almost every $x \in \mathbb{R}^n$, for every $\xi \in C^{1+n}$ there exists a $\zeta \in C^{1+n}$ such that $\xi = A(x)\zeta$. Hence

$$\Re \langle B(x)\xi, \xi \rangle = \Re \langle A(x)A^{-1}(x)A(x)\zeta, A(x)\zeta \rangle = \Re \langle A(x)\xi, A(x)\zeta \rangle$$

$$= \Re \langle A(x)\xi, A(x)\zeta \rangle \geq C|\xi|^2 = C\|A^{-1}(x)\xi\|^2 \geq C|\xi|^2,$$

for all $\xi \in C^{1+n}$ and for almost every $x \in \mathbb{R}^n$. The converse, that $A$ is strictly pointwise accretive if $B$ is strictly pointwise accretive, follows immediately from the fact that $\Phi^{-1} = \Phi$ or from reversing the previous computations.

Lemma 2.3.4. Let $T$ be a bounded and strictly accretive operator. Then there exists an angle $\omega \in [0, \pi/2)$ such that $T$ is bounded and $\omega$-accretive; see Definition 2.1.8.

Proof: By the accretivity hypothesis, there exists a positive constant $C$ such that for all $u \in \mathcal{H}$

$$C \|u\|^2_{\mathcal{H}} \leq \Re \langle Tu, u \rangle_{\mathcal{H}} \leq \|T\|_{\mathcal{H} \to \mathcal{H}} \|u\|^2_{\mathcal{H}}.$$

If $C = \|T\|_{\mathcal{H} \to \mathcal{H}}$, then $\langle Tu, u \rangle_{\mathcal{H}} \in \mathbb{R}$ for all $u \in \mathcal{H}$ and the operator $T$ is bounded, self-adjoint (see, for example, [30, Lemma C.4.1]) and positive, hence also $\omega$-accretive, with $\omega = 0$. If $C < \|T\|_{\mathcal{H} \to \mathcal{H}}$, then choosing $\omega \in (0, \pi/2)$ such that $C \tan \omega = \|T\|_{\mathcal{H} \to \mathcal{H}}$, guarantees that $\langle Tu, u \rangle_{\mathcal{H}} \in S_{\omega+}$, for all $u \in \mathcal{H}$.

Proposition 2.3.5. Let $B$ be as in (2.30) and $\mathcal{H}, \mathcal{H}_{\text{cut}}$ as in (2.23), (2.32) respectively. Then, $D(B) = \mathcal{H}$, and $B$ is invertible. Moreover, $B$ is pointwise strictly accretive on $C^{1+n}$, so strictly accretive on $\mathcal{H}$ and $\mathcal{H}_{\text{cut}}$ as well. In addition, $B$ is $\omega$-accretive, where

$$\omega := \sup_{f \in \mathcal{H}} \left| \arg \langle Bf, f \rangle_{\mathcal{H}} \right| < \frac{\pi}{2}.$$

Similar considerations apply to the adjoint operator $B^*$. 

As far as the properties of the constant-coefficient, first order differential operator $D$ are concerned, we have the following proposition.
2.3 The equation \( \text{div} \nabla u = 0 \)

**Proposition 2.3.6.** Let \( D \) be as in \((2.30)\) and \( \mathcal{H}, \mathcal{H}_{\text{curl}} \) as in \((2.23), (2.32)\) respectively. Then, \( D : \mathcal{D}(D) \to \mathcal{H} \) is a densely defined, closed, self-adjoint operator in \( \mathcal{H} \). Furthermore, \( \mathcal{R}(D) = \mathcal{H}_{\text{curl}} \), \( \mathcal{N}(D) = (\mathcal{H}_{\text{curl}})^\perp \) and there exists a positive constant \( C \), such that

\[
\| \nabla f_k \|_{L^2(\mathbb{R}^n; \mathbb{C}^n)} \leq C \| D f \|_{\mathcal{H}},
\]

for all \( f = [f_0, f_1, f_2, \ldots, f_n]^T \in \mathcal{D}(D) \cap \mathcal{R}(D) \) and all \( k = 0, 1, 2, \ldots, n \).

**Proof:** Using the Fourier transform componentwise on \( \mathcal{H} \), it is seen that \( D \) corresponds to the Fourier multiplier

\[
\hat{D} : \left[ \hat{f}_\perp, \hat{f}_\parallel \right] \mapsto \left[ i\xi \cdot \hat{f}_\parallel, i\xi \hat{f}_\perp \right],
\]

and that \( \mathcal{H}_{\text{curl}} \) is the subspace \( \{ [\hat{f}_\perp, \hat{f}_\parallel]^T : \xi \wedge \hat{f}_\parallel = 0 \} \), where “\( \wedge \)” stands for the wedge product, see [44, Chapter 7] and also [27, Chapter 2]. From this it follows that \( D \) is closed, densely defined and self-adjoint in the sense of unbounded operators, with stated range and nullspace. Furthermore, the lower bound follows from Lagrange’s identity

\[
|\xi|^2 |\hat{f}_\parallel|^2 = |\xi \cdot \hat{f}_\parallel|^2 + |\xi \wedge \hat{f}_\parallel|^2 = |\xi \cdot \hat{f}_\parallel|^2,
\]

if \( \xi \wedge \hat{f}_\parallel = 0 \), i.e. if \( \xi \in \mathcal{N}(D) \). \( \square \)

We now come to the properties of the product \( DB \), see [7, Proposition 3.3], [8, Proposition 6.1].

**Proposition 2.3.7.** Let \( B \) and \( D \) be as in \((2.30)\) and \( \mathcal{H}, \mathcal{H}_{\text{curl}} \) as in \((2.23), (2.32)\) respectively. Then

(i) The operator \( DB \) is closed and densely defined in \( \mathcal{H} \). Moreover, \( \mathcal{R}(DB) = \mathcal{H}_{\text{curl}} \) and \( \mathcal{N}(DB) = B^{-1}\mathcal{N}(D) \).

(ii) The restriction of \( DB \) to \( \mathcal{H}_{\text{curl}} \), namely

\[
DB|_{\mathcal{H}_{\text{curl}}} : \mathcal{H}_{\text{curl}} \to \mathcal{H}_{\text{curl}} : f \mapsto DBf,
\]

is a densely defined, injective, \( \omega \)-bisectorial operator, with dense range, where \( \omega \) is as in Proposition 2.3.5.

(iii) The operator \( DB \) is a densely defined \( \omega \)-bisectorial operator in \( \mathcal{H} \).

The proof uses the properties of the operators \( B \) and \( D \) from Propositions 2.3.5 and 2.3.6 respectively, together with standard identities as in [12, Proposition 1.9, Remark 6, Section 2.5 and Corollary 2.18], the reflexivity of Hilbert spaces (to get \( \mathcal{N}(A) = \mathcal{R}(A^*) \)), for a closed densely defined operator \( A \), see [12, Remark 17]) and the fact that if \( X_1, X_2 \) are two closed subspaces of a Banach space \( X \) such that \( X_1 \cap X_2 = \{0\} \), then \( X_1 + X_2 \) is closed if and only if there exists a positive constant \( C \) so that \( \|x_1\|_X + \|x_2\|_X \leq C \|x_1 + x_2\|_X \), for all \( x_1 \in X_1 \) and all \( x_2 \in X_2 \); see [46, Definition 1.2.4]. See also [37, Chapter 4].
Note that this decomposition is in accordance with Theorem 2.1.7. Since
\[ N(D) = N(B^*D) = \mathcal{R}(DB)^\perp \quad \text{and} \quad B^*\mathcal{R}(D) = \mathcal{R}(B^*D) = N(DB)^\perp, \]
and due to the stability of splittings under taking orthogonal complements, to show \((2.34)\) it suffices to show that
\[ \mathcal{H} = N(D) \oplus B^*\mathcal{R}(D). \quad (2.35) \]
Clearly, \(N(D)\) and \(B^*\mathcal{R}(D)\) are closed subspaces of \(\mathcal{H}\). To prove that \(N(D) \cap B^*\mathcal{R}(D) = \{0\}\), we make use of the accretivity of \(B^*\). Indeed, let \(u = \lim_{n \to \infty} B^*Dv_n \in B^*\mathcal{R}(D)\) for some sequence \(v_n \in \mathcal{D}(D)\) and assume also that \(Du = 0\). Then there exists a positive constant \(C\) such that for all \(n \in \mathbb{N}\)
\[ 0 = \text{Re} \langle Du, v_n \rangle_{\mathcal{H}} = \text{Re} \langle B^*Dv_n, Dv_n \rangle_{\mathcal{H}} \geq C \|Dv_n\|^2_{\mathcal{H}}, \]
hence \(Dv_n = 0\) for all \(n \in \mathbb{N}\), thus \(a = 0\) as well. To show that \(N(D) + B^*\mathcal{R}(D)\) is closed, it suffices to show that
\[ \|u\|_{\mathcal{H}} \leq C \|u + B^*Dv\|_{\mathcal{H}} \quad \text{and} \quad \|B^*Dv\|_{\mathcal{H}} \leq C \|u + B^*Dv\|_{\mathcal{H}}, \]
for some \(C > 0\) and for all \(u \in N(D)\) and all \(v \in \mathcal{D}(D)\). To verify the first of the aforementioned estimates, let \(u \in N(D)\), \(v \in \mathcal{D}(D)\) and proceed, using the accretivity of \((B^*)^-1\), the boundedness of \(B^{-1}\), the self-adjointness of \(D\) and the properties of the adjoint, as follows:
\[ \|u\|_{\mathcal{H}}^2 \leq C \text{Re} \langle (B^*)^-1u, u \rangle_{\mathcal{H}} + C \|Dv, u\|_{\mathcal{H}} \leq C \|u + B^*Dv\|_{\mathcal{H}} \|B^{-1}u\|_{\mathcal{H}} \leq C \|u + B^*Dv\|_{\mathcal{H}} \|u\|_{\mathcal{H}}. \]

The same train of thought yields
\[ \|B^*Dv\|_{\mathcal{H}}^2 \leq C \text{Re} \langle B^{-1}B^*Dv, B^*Dv \rangle_{\mathcal{H}} = C \text{Re} \langle Dv + u, (B^*)^-1B^*Dv \rangle_{\mathcal{H}} \leq C \|B^*Dv + u\|_{\mathcal{H}} \|B^*Dv\|_{\mathcal{H}}, \]
where \(u \in N(D)\) and \(v \in \mathcal{D}(D)\) as before. From this, the second estimate follows readily. To show that \(N(D) + B^*\mathcal{R}(D)\) is dense in \(\mathcal{H}\), we use the fact that \(N(D) \oplus \mathcal{R}(D) = \mathcal{H}\), which holds since \(D\) is a self-adjoint operator, together with the accretivity of \(B^*\) to show...
The equation \( \text{div} A \nabla u = 0 \) contains only the zero vector. Indeed, let \( u = \lim_{n} Dv_n \in N(D) \cap \overline{R(B^*D)} \cap \overline{R(D)}. \) Then

\[
C \|u\|_2^2 \leq \Re \langle B^* u, u \rangle_{\mathcal{H}} = \Re \left( \langle B^* \lim_{n} Dv_n, u \rangle_{\mathcal{H}} \right) = 0,
\]

for some positive constant \( C. \) This implies that \( u = 0. \)

It is clear that the operator \( DB|_{\mathcal{H}_{\text{curl}}} \) is well-defined and injective. Since \( D(DB|_{\mathcal{H}_{\text{curl}}}) = \mathcal{H}_{\text{curl}} \cap D(DB) \) it is also densely defined. To show that \( DB|_{\mathcal{H}_{\text{curl}}} \) is \( \omega \)-bisectorial we argue in a similar fashion as in Proposition 2.1.9. Let \( \zeta \notin S_{\omega} \) and \( 0 \neq u \in D(DB|_{\mathcal{H}_{\text{curl}}}). \) Then, since \( \langle Bu, Bu \rangle_{\mathcal{H}} \in \mathbb{R} \) and \( B \) is \( \omega \)-accretive

\[
\left| \langle Bu, u \rangle_{\mathcal{H}} \right| \leq \| (\zeta I - DB)u \|_{\mathcal{H}} \| Bu \|_{\mathcal{H}},
\]

hence there exists a positive constant \( C \) such that

\[
C \text{dist}(\zeta, S_{\omega}) \|u\|_{\mathcal{H}} \leq \| (\zeta I - DB)u \|_{\mathcal{H}}, \quad \text{for all } u \in D(DB|_{\mathcal{H}_{\text{curl}}}).
\]

Thus, \( \zeta I - DB \) is injective and has closed range. Similarly, \( \overline{\zeta I} - B^*D \) is shown to be injective. Accordingly, \( \zeta I - DB \) is invertible and

\[
\| R_{DB}(\zeta) \|_{\mathcal{H} \to \mathcal{H}} \leq C \frac{1}{\text{dist}(\zeta, S_{\omega})},
\]

for some \( C \) and for all \( \zeta \notin S_{\omega}. \) This proves (ii).

As far as (iii) is concerned, we simply extend the action of \( DB|_{\mathcal{H}_{\text{curl}}} \) in all of \( \mathcal{H} \) by using (2.34). \( \square \)

Not surprisingly, the properties of the operator \( BD \) are very closely related to those of \( DB. \) For example, that \( BD \) is also an \( \omega \)-bisectorial operator in \( \mathcal{H} \) can be deduced immediately from the fact that the operators are similar, i.e.

\[
BD = B(DB)B^{-1},
\]

which of course is a consequence of the fact that \( B \) is strictly accretive on all \( \mathcal{H}; \) see also Example 2.1.4. Of course, it is possible to treat the operator \( BD \) first, and then deduce the properties of \( DB. \) For example, to prove that \( BD \) restricted on its range is an injective \( \omega \)-bisectorial operator, instead of (2.34) we use the splitting \( \mathcal{H} = \overline{R(BD)} \oplus N(BD). \)

More specifically, with regards to the properties of the operator \( BD, \) the following proposition holds, see [8, Proposition 3.1].

**Proposition 2.3.8.** Let \( B \) and \( D \) be as in (2.30) and \( \mathcal{H}, \mathcal{H}_{\text{curl}} \) as in (2.23), (2.32) respectively. Then

(i) The operator \( BD \) is closed and densely defined in \( \mathcal{H}. \) Moreover, \( D(BD) = D(D), \)

\[
N(BD) = N(D) \text{ and } \overline{R(BD)} = \overline{R(D)}.
\]
(ii) The restriction of $BD$ to $\mathbb{R}(BD)$, namely

$$BD|_{\mathbb{R}(BD)} : \mathbb{R}(BD) \rightarrow \mathbb{R}(BD) ; f \mapsto BDf,$$

is a densely defined, injective, $\omega$-bisectorial operator, with dense range, where $\omega$ is as in Proposition 2.3.5.

(iii) The operator $BD$ is a densely defined $\omega$-bisectorial operator in $\mathcal{H}$.

The following deep result was proved in [11], see [7, Theorem 3.4]. The proof of the theorem uses heavy machinery and techniques from harmonic analysis, such as Carleson measures and the $Tb$ theorem, and is omitted. For a more direct approach see [8].

**Theorem 2.3.9.** Let $D$ be a self-adjoint homogeneous first order differential operator with constant coefficients that satisfies (2.33) as in Proposition 2.3.6. Let $B$ be a bounded multiplication operator on $L^2(\mathbb{R}^n; C_1 + \mathbb{R})$ that is strictly accretive on $\mathbb{R}(D)$. Then, the operator $DB$ satisfies quadratic estimates. Moreover, the same is true for the operator $BD$.

In a slightly modified framework, the aforementioned theorem answers the Kato square root problem. This is the content of the next example, see [7, Section 2], [46, Examples 5.4.18 and 5.7.21].

**Example 2.3.10**

Let $A \in L^\infty(\mathbb{R}^n; M_n(\mathbb{C}))$ be a strictly pointwise accretive matrix and consider the second order uniformly elliptic partial differential operator $L = -\text{div} A\nabla$. Then, due to the accretivity hypothesis, for all $u \in D(L) \subset W^{1,2}(\mathbb{R}^n)$ we find that

$$\langle Lu, u \rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \langle A(x)u(x), u(x) \rangle \, dx = \langle A\nabla u, \nabla u \rangle_{L^2(\mathbb{R}^n; C_n)} \in S_{\omega^+},$$

for some $\omega \in [0, \pi/2)$, see also Lemma 2.3.4. Moreover, $\sigma(L) \subset S_{\omega^+}$, since for $\zeta \notin S_{\omega^+}$, $\mathcal{U} - L$ is certainly injective and has closed range (see also [46, Proposition 5.4.10, Remark 5.4.11]) and by the Lax-Milgram lemma (see, for example, [30, Theorem C.5.3]) its range is dense. Thus, $L$ is an $\omega$-accretive, therefore an $\omega$-sectorial operator. Using functional calculus, it is possible to define the square root of $L$, which is denoted $\sqrt{T} = T^{1/2}$, which turns out to be an $\omega/2$-accretive/sectorial operator, see [30, Chapter 3], [46, Section 5.5]. When $\omega = 0$, equivalently when $A = A^*$, equivalently when $L = L^*$, it is easy to see that $D(\sqrt{L}) = W^{1,2}(\mathbb{R}^n)$ and that $\|\sqrt{L}u\|_{L^2(\mathbb{R}^n)}$ is equivalent to $\|\nabla u\|_{L^2(\mathbb{R}^n)}$. The question asked by T. Kato in the 1960’s was whether this is also true for an operator which is not self-adjoint:

Suppose that $\omega \neq 0$. Is it true that

$$\begin{cases} D(\sqrt{L}) = W^{1,2}(\mathbb{R}^n), \\
\|\sqrt{L}u\|_{L^2(\mathbb{R}^n)} = \|\nabla u\|_{L^2(\mathbb{R}^n)}, \ u \in W^{1,2}(\mathbb{R}^n) ?
\end{cases}$$
2.3 The equation $\text{div}A \nabla u = 0$

For $n = 1$, an affirmative answer to this conjecture was given in [14]. In [9] it was shown to hold for arbitrary $n$.

To verify this, set

$$D := \begin{bmatrix} 0 & -\text{div} \\ \nabla & 0 \end{bmatrix}, \quad B := \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix}. $$

The operators $B$ and $D$ satisfy the hypotheses of Theorem 2.3.9, so, by Theorem 2.2.3, $BD$ has a bounded $H^\infty$ functional calculus. In particular, $\text{sgn}(BD)$ is a well-defined bounded operator and $\text{sgn}(BD)BD = \sqrt{(BD)^2}$ (see [46, Proposition 5.7.18]), where

$$(BD)^2 = \begin{bmatrix} -\text{div}A & 0 \\ 0 & -A\text{div} \end{bmatrix} = \begin{bmatrix} L & 0 \\ 0 & -A\text{div} \end{bmatrix}. $$

and

$$\sqrt{(BD)^2} = \begin{bmatrix} \sqrt{-\text{div}A} & 0 \\ 0 & \sqrt{-A\text{div}} \end{bmatrix} = \begin{bmatrix} \sqrt{L} & 0 \\ 0 & \sqrt{-A\text{div}} \end{bmatrix}. $$

Accordingly, there exist positive constants $m$ and $M$, such that

$$m \|BDu\|_{\mathcal{H}} \leq \left\|\sqrt{(BD)^2}u\right\|_{\mathcal{H}} \leq M \|BDu\|_{\mathcal{H}}, $$

for all $u \in D(BD) \subset \mathcal{H} = L^2(\mathbb{R}^n; \mathbb{C}^{1+n})$. Hence, considering vector fields of the form $u = [f, 0]^t$, where $f \in W^{1,2}(\mathbb{R}^n)$, it follows that there exist positive constants $m$ and $M$, such that

$$m \|\nabla f\|_{L^2(\mathbb{R}^n)} \leq \left\|\sqrt{L}f\right\|_{L^2(\mathbb{R}^n)} \leq M \|\nabla f\|_{L^2(\mathbb{R}^n)}, $$

for all $f \in W^{1,2}(\mathbb{R}^n)$.

---

**Example 2.3.11**

For this example we make an exception to the convention that $n \geq 1$ (as far as the range space $\mathbb{C}^{1+n}$ of the vector fields are concerned) and allow the multiplication and the differential operator to act on complex-valued functions instead of vector fields; in other words, the space $\mathcal{H}$ is now $L^2(\mathbb{R}) = L^2(\mathbb{R}, \mathbb{C})$ and $B, D$ are $1 \times 1$-matrices with complex entries. Both Theorem 2.3.9 and the formalism pertaining to the functional calculus are valid in this setting as well; see also [1, Section J].

The operators $B$ and $D$ appearing in Example 2.1.10 satisfy the hypotheses of Theorem 2.3.9. Arguing as before, it is seen that $\text{sgn}(BD)$ is a well-defined bounded operator on $L^2(\mathbb{R})$ which is similar to the Cauchy integral on a Lipschitz curve, yielding the $L^2$-boundedness of the latter. See also Example 2.2.2(iii), Example 2.2.7(iii) and [7, Section 2].
Combining Theorem 2.3.9 with Proposition 2.2.14, we arrive at the following very important conclusion concerning the solutions of Equation (2.31), see [5, Proposition 4.3], [7, Theorem 2.3] and also [8, Proposition 7.1].

**Theorem 2.3.12.** Let $A$ be a $t$-independent matrix of complex-valued $L^\infty$ coefficients, strictly pointwise accretive, as in (2.19). Let $H$ and $H_{\text{curl}}$ defined by (2.23) and (2.32) respectively and let $B, D$ be as in (2.30). Then, $f_0 \in H_{\text{curl}}$ is in one-to-one correspondence with a pair of vector fields

$$f^\pm : \mathbb{R}^{1+n}_\pm \rightarrow \mathbb{C}^{1+n} ; (t, x) \mapsto f^\pm (t, x) = f^\pm_t (x). \quad (2.37)$$

having the following properties

(i) For each fixed $t \in \mathbb{R}_\pm$, $f^\pm_t \in H_{\text{curl}}$ and the mapping

$$\mathbb{R}_\pm \rightarrow H_{\text{curl}} ; t \mapsto f^\pm (t, \cdot),$$

is continuous, i.e. $f^\pm \in C^0 (\mathbb{R}_\pm ; H_{\text{curl}})$.

(ii) The mapping

$$\mathbb{R}_\pm \rightarrow H ; t \mapsto f^\pm (t, \cdot),$$

is continuously differentiable, i.e. $f^\pm \in C^1 (\mathbb{R}_\pm ; H)$.

(iii) The vector field $f^\pm$ satisfies equation (2.31) in $\mathbb{R}_\pm^{1+n}$.

(iv) $\lim_{t \to \pm \infty} f^\pm (t, \cdot) = 0$, where the limit is taken in the $L^2$ sense, i.e. $\| f^\pm (t, \cdot) \|_{2f} \to 0$, as $t \to \pm \infty$.

(v) $\lim_{t \to 0^\pm} f^\pm (t, \cdot) = f^\pm_0 (\cdot)$, where the limit is again taken in the $L^2$ sense as above and

$$f^\pm_0 : \mathbb{R}^n \rightarrow \mathbb{C}^{1+n} ; x \mapsto f^\pm_0 (x),$$

are square integrable and satisfy $f_0 = f^+_0 + f^-_0$.

Moreover, for a situation as in (vi), the norms $\| \cdot \|_{2f}$ and $\| \cdot \|_{2f^+} + \| \cdot \|_{2f^-}$ are equivalent, in the sense that there exist positive constants $m$ and $M$, such that

$$m \| f_0 \|_{2f} \leq \| f^+_0 \|_{2f^+} + \| f^-_0 \|_{2f^-} \leq M \| f_0 \|_{2f}, \quad (2.38)$$

for all $f_0 \in H_{\text{curl}}$. What is more, there exist positive constants $C$, such that

$$\| f^+_0 \|_{2f} \leq C \sup_{t \geq 0} \| f^+ (t, \cdot) \|_{2f} \leq C \left( \int_0^\infty \| \partial_t f^+ (t, \cdot) \|_{2f}^2 t \, dt \right)^{\frac{1}{2}} \leq C \| f^+_0 \|_{2f}, \quad \text{and}$$

$$\| f^-_0 \|_{2f} \leq C \sup_{t < 0} \| f^- (t, \cdot) \|_{2f} \leq C \left( \int_{-\infty}^0 \| \partial_t f^- (t, \cdot) \|_{2f}^2 |t| \, dt \right)^{\frac{1}{2}} \leq C \| f^+_0 \|_{2f}. \quad (2.39)$$
The main ingredient of the upcoming proof is quadratic estimates. These are used in an essential way, through the boundedness of the generalized Hardy projections of the operator $DB$. They also help to establish the norm equivalences. In addition, properties of the holomorphic semigroup from Proposition 2.2.14 are used freely. Recall that

$$ |DB| = DB \text{sgn}(DB) = DB(E^+_DB - E^-_{DB}), $$

so that

$$ |DB| f_0 = \begin{cases} DB f_0, & \text{if } f_0 \in E^+_DB \mathcal{H}_{\text{curl}}, \\ -DB f_0, & \text{if } f_0 \in E^-_{DB} \mathcal{H}_{\text{curl}}, \end{cases} $$

is a well-defined sectorial operator in $\mathcal{H}_{\text{curl}}$.

**Proof:** Since, by Theorem 2.3.9 the operator $DB$ satisfies quadratic estimates, by Proposition 2.2.13, (2.18) and because $E^\pm_{DB}$ is zero on $N(DB)$ (see 2.2.11 and subsequent comments), we get the following splitting

$$ \mathcal{H} = N(DB) \oplus \mathcal{H}_{\text{curl}} = N(DB) \oplus E^+_DB \mathcal{H} \oplus E^-_{DB} \mathcal{H} $$

$$ = N(DB) \oplus E^+_DB \mathcal{H}_{\text{curl}} \oplus E^-_{DB} \mathcal{H}_{\text{curl}}. $$

Thus, starting from a vector field $f_0 \in \mathcal{H}_{\text{curl}}$ on the boundary, we arrive at the following well-defined vector fields

$$ f^\pm(t,x) = e^{-\mp|DB|} E^\pm_{DB} f_0(x), \ (t,x) \in \mathbb{R}^{1+n}_+. $$

Properties (i)-(v) follow from the corresponding abstract properties of the holomorphic semigroup and the very definition of the generalized Hardy projections: (i) and (ii) follow from items (iii) and (iv) of Proposition 2.2.14 and the fact that $DB$ has a bounded functional calculus; (iv) and (v) follow from items (vi) and (vii) of Proposition 2.2.14, after setting $f^\pm_0 := E^\pm_{DB} f_0$. Differentiating $f^\pm$ at $(t,x) \in \mathbb{R}^{1+n}_+$ gives

$$ \partial_t f^\pm(t,x) = \mp |DB| e^{-\mp|DB|} E^\pm_{DB} f_0(x) = -DB f^\pm(t,x), $$

thus

$$ \partial_t f^\pm(t,x) + DB f^\pm(t,x) = 0, \ (t,x) \in \mathbb{R}^{1+n}_+. $$

Conversely, consider a vector field $f : \mathbb{R}^{1+n}_+ \to \mathcal{C}^{1+n}$ satisfying equation (2.31) in $\mathbb{R}^{1+n}_+$ and having the properties (i)-(v). Then, $f(t, \cdot) \in \mathcal{H}_{\text{curl}}$, for each fixed $t > 0$, and by applying the generalized Hardy projections we can write $f(t, \cdot) = f^+(t, \cdot) + f^-(t, \cdot)$, where $f^\pm(t, \cdot) = E^\pm_{DB} f(t, \cdot)$. This leads to the two equations

$$ \partial_t f^\pm(t,x) \pm |DB| f^\pm(t,x) = 0, \ (t,x) \in \mathbb{R}^{1+n}_+. $$

Multiplying with suitable integrating factors, we get the differential equations

$$ \partial_s \left( e^{-(t-s)|DB|} f^+(s,x) \right) = 0, \ s \in (0, t), $$

$$ \partial_s \left( e^{(t-s)|DB|} f^-(s,x) \right) = 0, \ s \in (0, t). $$
and
\[ \partial_t \left( e^{(s-t)DB} f^-(s,x) \right) = 0, \quad s \in (t, \infty), \]
which we integrate, using the limit in (iv) for the second one and the limit in (v) for the first one. It follows that \( f^-(t, \cdot) = 0 \) and \( f^+(t, \cdot) = e^{-tDB} f_0(\cdot) \), where \( f_0 \in \mathcal{H}_{\text{cont}} \), for all \( t > 0 \). Repeating the same procedure for a vector field that solves (2.31) in the lower half plane, we get the analogous results, with \( f_0 \in \mathcal{H}_{\text{cont}} \). This concludes the proof of the one-to-one correspondence.

Proceeding to the norm equivalences, it is easy to see that the boundedness of the generalized Hardy projections immediately yields (2.38). By the uniform boundedness – in \( t \) – of the holomorphic semigroup we obtain the equivalence of the equivalence of \( \|f_0^+\|_{\mathcal{A}} \) and \( \sup_{t>0} \|f^\pm(t, \cdot)\|_{\mathcal{A}} \).

Using the fact the the operator \( DB \) satisfies quadratic estimates (with respect to the function \( \psi(\zeta) = \zeta e^{-|\zeta|^2} \)) it immediately follows from Definition 2.2.1 that there exist positive constants \( m \) and \( M \), such that
\[ m \|f_0^+\|_{\mathcal{A}} \leq \sup_{t>0} \|\partial_t f^\pm(t, \cdot)\|_{\mathcal{A}} \leq M \|f_0^+\|_{\mathcal{A}}, \]
which establishes the remaining norm equivalence. \( \square \)

2.3.1 Construction of solutions to equation (2.19) using BD

As mentioned in the introductory remarks of Section 2.3, in this subsection we construct solutions for the Dirichlet problem for (2.19) in the upper half-space, with square integrable boundary data, through functional calculus. Moreover, certain \( L^2 \)-estimates are proved for this particular class of solutions.

Consider the first-order vector-valued ordinary differential equation in \( \mathbb{R}^{1+n}_+ \)
\[ \partial_t \nu(t,x) + BD \nu(t,x) = 0, \quad (2.40) \]
where \( \nu = [v_\perp, v_\parallel]^T : \mathbb{R}^{1+n}_+ \to \mathbb{C}^{1+n} \) is a vector field on \( \mathbb{R}^n \) such that \( \nu(t, \cdot) \in \mathcal{D}(D) \subset \mathcal{H} \), for all \( t \in \mathbb{R}_+ \). This equation is of interest because of the following reason. Applying the operator \( D \) to (2.40) yields
\[ \partial_t (D \nu(t,x)) + DB (D \nu(t,x)) = 0, \quad (2.41) \]
since \( D \) does not involve any differentiation in the \( t \)-variable. We have already seen that solving the generalised Cauchy-Riemann equation \( \partial_t f + DB f = 0 \) in \( \mathbb{R}^{1+n}_+ \), with \( f = \nabla_A u \), is equivalent to solving (2.19) in \( \mathbb{R}^{1+n}_+ \). Thus, if we could express \( f = \nabla_A u \) as \( f = D \nu \), i.e. write
\[ \begin{bmatrix} A_{\perp} \partial_t u + A_{\parallel} \nabla \parallel u \\ \nabla \parallel u \end{bmatrix} = \begin{bmatrix} f_{\perp} \\ f_{\parallel} \end{bmatrix} = \begin{bmatrix} \text{div} v_\parallel \\ -\nabla_\parallel v_\perp \end{bmatrix}, \quad (2.42) \]
then \( u = -v_\perp \) (plus a constant) would be a solution to (2.19) in the upper half-space. Notice that only the normal part of the vector field \( \nu \) is used in the solution.
Having explained the significance of equation (2.40), we would like to have a theorem analogous to Theorem 2.3.12, giving a one-to-one correspondence between trace vector fields $v_0(\cdot)$ belonging to a particular subspace of $\mathcal{H}$ and solutions $v(\cdot, \cdot)$ to (2.40) in $\mathbb{R}_+^{1+n}$ satisfying the “quadratic estimate”

$$\int_{\mathbb{R}_+^{1+n}} |\partial_t v(t,x)|^2 t \, dt \, dx < \infty. \quad (2.43)$$

Since $B$ is bounded and invertible, this is the same as requiring $\int_{\mathbb{R}_+^{1+n}} |Dv(t,x)|^2 t \, dt \, dx < \infty$, for a solution $v$ of (2.40) on $\mathbb{R}_+^{1+n}$. Notice that for $u = -v_+$, (2.43) implies that

$$\int_{\mathbb{R}_+^{1+n}} |\nabla_t u(t,x)|^2 t \, dt \, dx < \infty. \quad (2.44)$$

Estimate (2.44) is sometimes referred to as a “square function estimate”, see [5, Section 1]. Notice that if $\nabla_t u$ satisfies (2.44), then $\nabla_t u \in L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^n; C^{1+n}))$. We remark that it is precisely because the operator $BD$ satisfies quadratic estimates as in Section 2.2 that it is possible to prove such estimates as (2.43) and (2.44). This was the case in Theorem 2.3.12 as well.

The operator $BD$ is a bisectorial operator in $\mathcal{H}$ (by Proposition 2.3.8) that satisfies quadratic estimates (by Theorem 2.3.9). Therefore, it is possible, using the bounded spectral projections $E_{BD}^+$ and Proposition 2.3.8, to split the Hilbert space $\mathcal{H}$ as the topological direct sum

$$\mathcal{H} = \overline{R(BD)} \oplus N(BD) = B\overline{R(D)} \oplus N(D) = B\mathcal{H}_{\text{curl}} \oplus (\mathcal{H}_{\text{curl}})\perp$$

$$= E_{BD}^+ \mathcal{H}_{\text{curl}} \oplus E_{BD}^- \mathcal{H}_{\text{curl}} \oplus N(D),$$

where $N(D) = E_{BD}^0 \mathcal{H}$; see Proposition 2.2.13. The restriction of $BD$ on the spectral subspace $E_{BD}^+ \mathcal{H}_{\text{curl}} \subset \mathcal{H}$ is well-defined and sectorial. Thus, the bounded holomorphic semigroup $(e^{-tBD})_{t>0} \in B(E_{BD}^+ \mathcal{H}_{\text{curl}})$ is also well-defined; see Proposition 2.2.14. Accordingly, we arrive at the following proposition whose proof is similar to that of Theorem 2.3.12 and is omitted; again, the crucial ingredient is quadratic estimates.

**Proposition 2.3.13.** Let $A$ be a $t$-independent matrix of complex-valued $L^\infty$ coefficients, strictly pointwise accretive, as in (2.20). Let $\mathcal{H}$ be as in (2.23), $\mathcal{H}_{\text{curl}}$ as in (2.32) and let $B$, $D$ be as in (2.30). Then all solutions of (2.40) in $\mathbb{R}_+^{1+n}$ satisfying the estimate (2.43), are of the form $v(t,x) = e^{-tBD} E_{BD}^+ v_0(x) + c$, for a unique $v_0 \in E_{BD}^+ \mathcal{H}_{\text{curl}}$ and some constant $c \in C^{1+n}$. In addition, we have the $L^2$-limits $\lim_{t \to \infty} v(t,\cdot) = c$ and $\lim_{t \to 0} v(t,\cdot) = v_0(\cdot) + c$. Moreover, there exist positive constants $C$ such that

$$\|v_0 - c\|_{\mathcal{H}} \leq C \sup_{t \geq 0} \|v(t,\cdot) - c\|_{\mathcal{H}} \leq C \left( \int_0^\infty \|\partial_t v(t,\cdot)\|_{\mathcal{H}}^2 t \, dt \right)^{\frac{1}{2}} \leq C \|v_0 - c\|_{\mathcal{H}}.$$

Next we investigate the relationship between solutions of (2.31) (such solutions satisfy $\int_0^\infty \|f(t,\cdot)\|_{\mathcal{H}}^2 t \, dt < \infty$) and solutions of (2.40). We aim to prove the following theorem
Theorem 2.3.14. Let $A$ be a $t$-independent matrix of complex-valued $L^m$ coefficients, strictly pointwise accretive, as in (2.20). Let $H$ be as in (2.23), $\mathcal{H}_{\text{curl}}$ as in (2.32) and let $B, D$ be as in (2.30). Then $u \in W^{2, 1}_0(\mathbb{R}^{1+n}_+)$ is a solution to (2.19) satisfying the square function estimate (2.44), if and only if it is of the form

$$u(t, x) = - (e^{-tBD} E_{\mathbb{R}^2} v_0(x))_+ + c, \quad (t, x) \in \mathbb{R}^{1+n}_+,$$

for some constant $c \in \mathbb{C}$ and some vector field $v_0 \in E_{\mathbb{R}^2}^{1+n} \mathcal{H}_{\text{curl}}$.

Proof: The “only if” part follows from Proposition 2.3.16. The “if” part is a consequence of Proposition 2.3.15. □

Consider a vector field $v$ satisfying the quadratic estimate and (2.43), vanishing at infinity and such that $(\partial_t + BD)v = 0$ in the upper half-space. By Proposition 2.3.13 there exists a unique $v_0 \in E_{\mathbb{R}^2}^{1+n} \mathcal{H}_{\text{curl}}$, i.e. $v_0 = E_{\mathbb{R}^2}^{1+n} w_0$ for some $w_0 \in \mathcal{H}_{\text{curl}}$, such that

$$v(t, x) = e^{-tBD} E_{\mathbb{R}^2}^{1+n} v_0(x) = e^{-tBD} E_{\mathbb{R}^2}^{1+n} B w_0(x), \quad (t, x) \in \mathbb{R}^{1+n}_+.$$

Then, as already mentioned, the vector field $Dv$ satisfies equation $(\partial_t + DB)(Dv) = 0$ in the upper half-space. By Theorem 2.3.12, there exists a unique $f_0 \in E_{\mathbb{R}^2}^{1+n} H$ such that

$$Dv(t, x) = e^{-tDB} E_{\mathbb{R}^2}^{1+n} f_0(x), \quad (t, x) \in \mathbb{R}^{1+n}_+, \quad (2.45)$$

Note that (2.45) is the same as (2.42) for $f$ equal to the conormal gradient of a solution to (2.19). Through standard identities of the functional calculus (see, for example, [46, Corollary 5.6.6] and also Example 2.1.4 and Proposition 2.1.13), such as

$$e^{-tBD} = e^{-B(tDB)B^{-1}} = Be^{-tDB} B^{-1}, \quad DB e^{-tDB} = e^{-tDB} DB \quad \text{and} \quad E_{\mathbb{R}^2}^{1+n} B = BE_{\mathbb{R}^2}^{1+n},$$

and by considering the strong limit $t \to 0$ in (2.45), it follows that $Dv_0 = f_0$. In other words $f_0 \in \mathcal{R}(D) \cap E_{\mathbb{R}^2}^{1+n} \mathcal{H}_{\text{curl}}$, the latter being a dense subspace of $E_{\mathbb{R}^2}^{1+n} \mathcal{H}_{\text{curl}}$. Thus, loosely speaking, starting from a vector field $v$ as in Proposition 2.3.13 we can construct a unique vector field $f$ as in Theorem 2.3.12.

Conversely, let $f$ be a vector field satisfying $\int_0^\infty \| f(t, \cdot )\|^2_H t \, dt < \infty$ and (2.31) in the upper half-plane. By Theorem 2.3.12, it is of the form $f(t, x) = e^{-tDB} E_{\mathbb{R}^2}^{1+n} f_0(x), (t, x) \in \mathbb{R}^{1+n}_+$, for a unique vector field $f_0 \in E_{\mathbb{R}^2}^{1+n} \mathcal{H}_{\text{curl}}$. Recall that the operator $DB|_{\mathcal{H}_{\text{curl}}} : \mathcal{H}_{\text{curl}} \to \mathcal{H}_{\text{curl}}$ is densely defined, closed, injective and has dense range, by Proposition 2.3.7. Its restriction on $E_{\mathbb{R}^2}^{1+n} \mathcal{H}_{\text{curl}}$, namely, abusing notation, $DB : E_{\mathbb{R}^2}^{1+n} \mathcal{H}_{\text{curl}} \to E_{\mathbb{R}^2}^{1+n} \mathcal{H}_{\text{curl}}$, is a well-defined operator with the same properties. It follows that the operator

$$D : E_{\mathbb{R}^2}^{1+n} \mathcal{H}_{\text{curl}} \to E_{\mathbb{R}^2}^{1+n} \mathcal{H}_{\text{curl}},$$

is well-defined, closed, injective, with dense domain and dense range. Consequently, there exists a sequence $f_{0,n} = Dv_{0,n} \in E_{\mathbb{R}^2}^{1+n} \mathcal{H}_{\text{curl}}$, such that $v_{0,n} \in E_{\mathbb{R}^2}^{1+n} \mathcal{H}_{\text{curl}}$, converging to $f_0$, as $n \to \infty$. Since both the operators $BD$ and $DB$ satisfy quadratic estimates by
Theorem 2.3.9, the estimates from Proposition 2.3.13 and from Theorem 2.3.12, together with the boundedness of $B$ and standard identities of the functional calculus, yield

$$
\|v_{0,n} - v_{0,m}\|_{\mathcal{H}} = \|E_{BD}^n(v_{0,n} - v_{0,m})\|_{\mathcal{H}} \leq C \left( \int_0^\infty \|BD e^{-tBD} E_{BD}^n(v_{0,n} - v_{0,m})\|_{\mathcal{H}}^2 t \, dt \right)^{1/2} 
$$

$$
\leq C \left( \int_0^\infty \|De^{-tBD} E_{BD}^n(v_{0,n} - v_{0,m})\|_{\mathcal{H}}^2 t \, dt \right)^{1/2} \leq C \left( \int_0^\infty \|e^{-tBD} E_{BD} D(v_{0,n} - v_{0,m})\|_{\mathcal{H}}^2 t \, dt \right)^{1/2} 
$$

$$
\leq C \|f_{0,n} - f_{0,m}\|_{\mathcal{H}},
$$

for $n, m \in \mathbb{N}$. Since the sequence $f_{0,n}$ converges it follows that $v_{0,n}$ is Cauchy. Since $E_{BD}^n B\mathcal{H}_{\text{curl}}$ is closed, there exists $v_0 \in E_{BD}^n B\mathcal{H}_{\text{curl}}$ such that $v_{0,n} \to v_0$ as $n \to \infty$. Due to the fact that the operator $D$ is closed, $Dv_0 = f_0$. This boundary vector field $v_0$ corresponds to vector fields $v(t, x) = e^{-tBD} E_{BD}^+ v_0(x) + c$, $(t, x) \in \mathbb{R}^{1+\eta}$, $c \in \mathbb{C}^{1+\eta}$. That $v_0$ does not depend on the sequence $f_{0,n}$ follows from the fact that $D$ is injective in $E_{BD}^n B\mathcal{H}_{\text{curl}}$. Thus, loosely speaking, starting from a vector field $f$ as in Theorem 2.3.12, it is possible to construct a unique vector field $v$ as in Proposition 2.3.13. This $v$ is sometimes referred to as “the vector-valued potential” of $f$, see [5, Section 3].

The following proposition makes the preceding discussion precise, see [5, Theorem 9.2].

**Proposition 2.3.15.** Let $A$ be a $t$-independent matrix of complex-valued $L^\infty$ coefficients, strictly pointwise accretive, as in (2.20). Let $\mathcal{H}$ be as in (2.23), $\mathcal{H}_{\text{curl}}$ as in (2.32) and let $A$, $B$ be as in (2.30). Let $f : \mathbb{R}^{1+\eta} \to \mathbb{C}^{1+\eta}$ be a vector field for which the quadratic estimate $\int_0^\infty \|f(t, \cdot)\|_{\mathcal{H}}^2 t \, dt < \infty$ holds. Then $f$ satisfies (2.31), i.e. $\partial_t f + DBf = 0$, in the upper half-plane if and only if $f = De^{-tBD} E_{BD}^+ v_0$ for some $v_0 \in E_{BD}^+ B\mathcal{H}_{\text{curl}}$. In this case, the vector field $v = e^{-tBD} E_{BD}^+ v_0$ satisfies (2.40), i.e. $\partial_t v + BDv = 0$, in the upper half-plane and has $L^2$-limits $\lim_{t \to \infty} v(t, \cdot) = v_0(\cdot)$ and $\lim_{t \to \infty} \partial_t v(t, \cdot) = 0$. Moreover, there exist positive constants $C$ such that

$$
\|v_0\|_{\mathcal{H}}\leq C \sup_{t > 0} \|v(t, \cdot)\|_{\mathcal{H}} \leq C \left( \int_0^\infty \|f(t, \cdot)\|_{\mathcal{H}}^2 t \, dt \right)^{1/2} \leq C \left( \int_0^\infty \|\partial_t v(t, \cdot)\|_{\mathcal{H}}^2 t \, dt \right)^{1/2} \leq C \|v_0\|_{\mathcal{H}}.
$$

**Proof:** The one-to-one correspondence $f \leftrightarrow v_0$ has already been established. The $L^2$-limits of $v_0$ and the equivalence of norms follow from Proposition 2.3.13. They also follow directly from the fact that $BD$ satisfies quadratic estimates using properties of the holomorphic semigroup from Proposition 2.2.14 and standard properties of the functional calculus.

We now interpret the aforementioned proposition in terms of solutions $u$ to (2.19), i.e. $\text{div} Au = 0$, in the upper half-space, satisfying the square function estimate (2.44). See [5, Theorem 9.3], [7, Corollary 2.4].

**Proposition 2.3.16.** Let $A$ be a $t$-independent matrix of complex-valued $L^\infty$ coefficients, strictly pointwise accretive, as in (2.20). Let $\mathcal{H}$ be as in (2.23), $\mathcal{H}_{\text{curl}}$ as in (2.32) and
let $B$, $D$ be as in (2.30). Let $u \in W^{1,2}_{0}(\mathbb{R}^{1+n})$ satisfy (2.19) and (2.44). Then there exists a complex number $c$ and a vector field $v \in C_{0}^{1}(\mathbb{R}^{1+n}; H)$ such that $u = c - v_{\perp}$. Defining $u_{0} := c - v_{0,\perp} : \mathbb{R}^{n} \to C$ we have the $L^{2}$-limits $\lim_{t \to 0} u(t, \cdot) = u_{0}$ and $\lim_{t \to \infty} u(t, \cdot) = c$. Moreover, there exists positive constants $C$ such that

$$
\|u_{0} - c\|_{g_{t}} \leq C \sup_{t > 0} \|u(t, \cdot) - c\|_{g_{t}} \leq C \left( \int_{0}^{\infty} \|\nabla_{t, u}(t, \cdot)\|_{g_{t}}^{2} \, dt \right)^{\frac{1}{2}} \leq C \|v_{0}\|_{g_{t}}.
$$

Observe that the proposition above does not assume any $(L^{2})$-limits for $u$ as $t \to 0$ or $t \to \infty$; instead, it is shown that the $L^{2}$-trace of $u$ on $\mathbb{R}^{n}$ is $-v_{0,\perp}$.

**Proof:** Let $u \in W^{1,2}_{0}(\mathbb{R}^{1+n})$ satisfy (2.19) and (2.44). Then its conormal gradient $f = \nabla_{A}u = [(A\nabla_{t, u})_{\perp}, \nabla_{t, u}]^{\perp} \in L^{2}_{0}(\mathbb{R}^{1+n}; H_{\text{curl}})$ is well-defined and satisfies $\int_{0}^{\infty} \|f(t, \cdot)\|_{g_{t}}^{2} \, dt < \infty$ as in Proposition 2.3.3; this $f$ satisfies (2.31). By Proposition 2.3.15, there exists a boundary vector field $v_{0} \in E_{B}^{+}B_{H}^{+}$ such that $f = Dv$, where $v(t, x) = e^{-tBD}E_{B}^{+}v_{0}(x)$, $(t, x) \in \mathbb{R}^{1+n}$ solves (2.40). To show $u = c - v_{\perp}$ it suffices to show that $\nabla_{t, u} = -\nabla_{t, v_{\perp}}$ as weak derivatives (i.e. in $L^{2}$-distribution sense). Since $\nabla_{A}u = f = Dv$, that $\nabla_{t}u = f|_{t=0} = -\nabla_{t}v_{\perp}$ follows immediately. Since $v$ satisfies (2.40), it is clear that $\partial_{t}v_{\perp} + (BDv)_{\perp} = 0$, so, since $B = A\nabla$ and $\nabla_{A}u = \nabla v_{\perp}$, it follows that $\partial_{t}v_{\perp} = -(BDv)_{\perp} = (Bf)_{\perp} = (\nabla_{t, u})_{\perp} = \partial_{t}u$. The estimates follow from those of Proposition 2.3.13, or those of Proposition 2.3.15, and from the fact that $A$ is $L^{\infty}$-bounded. \hfill \Box

Utilizing the regularly decaying function $\psi(\xi) = \zeta e^{-\xi}$, for $\zeta \in S_{\mu, +}$, so that

$$
\psi(t\xi) = t\zeta e^{-t\xi} = e^{-t\xi}t\xi = -t\partial_{t}e^{-t\xi}, \quad \zeta \in S_{\mu, +}, \quad t > 0,
$$

and using the fact that the operator $BD$ has a bounded functional calculus and that $B^{-1}$ is a bounded operator, shows that for $f$ and $v_{0}$ as in Proposition 2.3.15, there exists a positive constant $C$

$$
\|f(t, \cdot)\|_{g_{t}} \leq C \frac{1}{t} \|v_{0}\|_{g_{t}}, \quad (2.46)
$$

for all $t > 0$. The constant $C$ depends on the $L^{\infty}(S_{\mu}^{0})$-norm of the function $\psi$ and on the operator $B$. Combining the estimates for $f$ from Theorem 2.3.12 with those in (2.46), we conclude that

$$
\|f(t, \cdot)\|_{g_{t}} \leq C \|\nabla_{t, u}(t, \cdot)\|_{g_{t}} \leq C \|f(t, \cdot)\|_{g_{t}} \leq C \min \left\{ \frac{1}{t} \|v_{0}\|_{g_{t}}, \|Dv_{0}\|_{g_{t}} \right\}, \quad (2.47)
$$

for appropriate positive constants $C$ and for all $t > 0$. This certainly refines the initial estimate for $f$ of Theorem 2.3.12(iv), since for large $t$, $\|f(t, \cdot)\|_{g_{t}}$ decays like $1/t$. Note that when $v_{0} \in D(D)$, we have

$$
\|f(t, \cdot)\|_{g_{t}} \leq C \min \left\{ \frac{1}{t}, 1 \right\}, \quad (2.48)
$$

for all $t > 0$. This estimate will be used later on, in Section 4.1.
The Fundamental Solution

3.1 Construction and interior estimates

As means of motivation, let us briefly revisit the classical case when $A$ is the $(1+n) \times (1+n)$-identity matrix and equation (2.19) is simply the Laplace equation in $\mathbb{R}^{1+n}$

$$\triangle_{t,x} u(t,x) = \partial_{t}^{2} u(t,x) + \partial_{x}^{2} u(t,x) + \ldots + \partial_{x}^{2} u(t,x) = 0,$$

as in Section 1.1.

For $n \geq 2$, the fundamental solution is

$$\Gamma(t,x;s,y) = -\frac{1}{(n-1)\sigma_{n}} |(t,x) - (s,y)|^{1-n}, \quad (3.1)$$

where $(t,x), (s,y) \in \mathbb{R}^{1+n}$ and $(t,x) \neq (s,y)$; $\sigma_{n}$ denotes the area of the unit sphere in $\mathbb{R}^{1+n}$. It can be verified that $\triangle_{t,x} \Gamma(t,x;s,y) = \delta_{(s,y)}(t,x)$, in the sense of distributions, where $\delta_{(s,y)}(\cdot, \cdot)$ is the Dirac delta distribution in $\mathbb{R}^{1+n}$ with pole at $(s,y)$. In other words

$$\int_{\mathbb{R}^{1+n}} \nabla_{t,x} \Gamma(t,x;s,y) \cdot \nabla_{t,x} \phi(t,x) \, dt \, dx = \int_{\mathbb{R}^{1+n}} \delta_{(s,y)}(t,x) \phi(t,x) \, dt \, dx = \phi(s,y), \quad (3.2)$$

for all test functions $\phi \in C_{c}^{\infty}(\mathbb{R}^{1+n})$. It is obvious that $\Gamma$ and, by a simple differentiation, its gradient $\nabla \Gamma$ satisfy the estimates

$$|\Gamma(t,x)| \leq C |(t,x)|^{1-n} \quad \text{and} \quad |\nabla_{t,x} \Gamma(t,x)| \leq C |(t,x)|^{-n}, \quad (3.3)$$

for some constant $C > 0$ and for all $(t,x) \neq (0,0)$, see, for example, [33, Theorem 3.3.2] and also the references given in Section 1.1.
In this section we shall construct a certain function, having properties and satisfying estimates like those in (3.2) and (3.3), for the equation (2.19) under the additional assumptions that the matrix $A$ is real-valued and that $n \geq 2$. More precisely, we consider the second order uniformly elliptic equation

$$L_{t,x}u(t,x) := - \text{div}_{t,x}A(x)\nabla_{t,x}u(t,x) = 0,$$

where $A \in L^\infty(\mathbb{R}^n; M_{1+n}(\mathbb{R}))$ is $t$-independent strictly accretive, i.e. there exists a constant $C > 0$ such that

$$C|\xi|^2 \leq A(x)\xi \cdot \xi,$$

for all $\xi \in \mathbb{R}^{1+n}$ and for almost all $x \in \mathbb{R}^n$. Here and in the sequel, “·” stands for the standard inner product in $\mathbb{R}^{1+n}$; $n \geq 2$. It immediately follows that the transpose matrix $A^T$ also satisfies (3.5), so the transpose operator $L^T := - \text{div}A^T \nabla$ is also uniformly elliptic.

We emphasize that the coefficient matrix $A$ is not assumed symmetric.

Intuitively, in this general setting, where no smoothness of the coefficients is assumed, one cannot hope to replicate the particularly nice pointwise estimates (3.3) for the corresponding fundamental solution and its gradient, in their entirety. In fact, as will be shown later in this chapter, the pointwise estimate for the gradient of the fundamental solution is replaced by a local $L^p$ estimate. In a sense, this is the “next best thing”: these local $L^p$ estimates are the same estimates that one would have gotten after integrating (averaging) the pointwise estimates over a ball; see Theorem 3.1.1, Proposition 3.1.12 and Proposition 3.1.13.

The ensuing theorem will be proved in this section, see [29, Theorem 1.1], [31, Theorem 3.1].

**Theorem 3.1.1.** Let $L = - \text{div}A \nabla$ be a uniformly elliptic second order partial differential operator with real $L^\infty$ coefficients, as in (3.4). Then, there exists a unique function $\Gamma : \mathbb{R}^{1+n} \times \mathbb{R}^{1+n} \rightarrow \mathbb{R}$, such that for each $(s,y) \in \mathbb{R}^{1+n}$, $\Gamma(\cdot, \cdot; s,y)$ has the following properties:

(i) For every $\phi \in C_c^\infty(\mathbb{R}^{1+n})$

$$\int_{\mathbb{R}^{1+n}} A(x)\nabla_{t,x} \Gamma(t,x; s,y) \cdot \nabla_{t,x} \phi(t,x) \, dx = \phi(s,y). \tag{3.6}$$

(ii) For every $f \in C_c^\infty(\mathbb{R}^{1+n})$, the function

$$u(t,x) := \int_{\mathbb{R}^{1+n}} \Gamma(t,x; s,y) f(s,y) \, ds \, dy,$$

is a weak solution to $Lu = f$, i.e.\n
$$\int_{\mathbb{R}^{1+n}} A(x)\nabla_{t,x} u(t,x) \cdot \nabla_{t,x} \phi(t,x) \, dx = \int_{\mathbb{R}^{1+n}} f(t,x) \phi(t,x) \, dx, \tag{3.7}$$

for all $\phi \in C_c^\infty(\mathbb{R}^{1+n})$. 

Corollary 3.1.2. Let $\Gamma : \mathbb{R}^{1+n} \times \mathbb{R}^{1+n} \to \mathbb{R}$ be as in Theorem 3.1.1. Then

(i) For every $p \in [1, (n+1)/(n-1))$, there exists a positive constant $C = C(p)$, such that

$$\|\Gamma(\cdot, \cdot; s, y)\|_{L^p(B(s,y; R))} \leq CR^{1-n+1/(np)},$$

for all $R > 0$.

(ii) For every $p \in [1, (1+n)/n)$, there exists a positive constant $C = C(p)$, such that

$$\|\nabla \Gamma(\cdot, \cdot; s, y)\|_{L^p(B(s,y; R) ; \mathbb{R}^{1+n})} \leq CR^{-n+1/(np)},$$

for all $R > 0$.

(iii) There exists a positive constant $C$, such that

$$\iint_{A(s,y;R,2R)} |\Gamma(t,x;s,y)|^2 \, dx \leq CR^{3-n},$$

for all $R > 0$.

The use of the term “fundamental solution” for the function $\Gamma$ is justified by (3.6) together with (3.7). Note that for symmetric coefficients, i.e., for matrices $A$ such that $A = A^T$, Theorem 3.1.1 can also be found in [39, Theorem 1.2.8]. Moreover, in this case, it also follows that $\Gamma(t,x; s,y) = \Gamma(s,y; t,x)(= \Gamma^T(s,y; t,x))$, see [29, Theorem 1.3]. This is in accordance with what happens in the familiar case of the Laplace operator. For further interesting properties of the fundamental solution, such as the H"{o}lder continuity of $\Gamma(\cdot, \cdot; s,y)$ in $\mathbb{R}^{1+n} \setminus \{(s,y)\}$ (see [31, Sections 3.6, 3.7]), we refer to [29] and [31] yet again. We stress that the $t$-independence of the coefficients is not used anywhere in the proofs of this section. So, all conclusions regarding the existence and properties
of the fundamental solutions are valid for any \( A \in L^\infty(\mathbb{R}^{1+n}, M_{1+\gamma}(\mathbb{R}) \). That said, for \( t \)-independent coefficients, considering the translation \( t \mapsto t + s \), the identity

\[
\Gamma(t, x, s, y) = \Gamma(t - s, x; 0, y),
\]

is also valid, see [2, Section 1].

Fundamental solutions for uniformly elliptic equations have been studied, among numerous other places, in [29] and in [42]. For systems of elliptic equations, existence and properties of fundamental solutions were studied in [31]. Here we follow the functional theoretic approach of [29] and [31] (mainly the latter’s as we work in the unbounded domain \( \mathbb{R}^{1+n} \)) and avoid potential theory altogether.

We recall some results and definitions concerning the local behaviour of the weak solutions, see [2, Section 1], [31, Lemmata 2.3, 2.4], [25, Section III.1], and [49]. We emphasize that the hypothesis that the matrix \( A \) is real is crucial for the following theorem to hold.

**Theorem 3.1.3.** Consider a second order uniformly elliptic operator with real \( L^\infty \) coefficients as in (3.4) and let \( \Omega \) be a domain in \( \mathbb{R}^{1+n} \). Let \( u \in W^{1,2}_{\text{loc}}(\mathbb{R}^{1+n}) \) be a weak solution of \( Lu = 0 \) in \( \Omega \). Then \( u \) is locally H"older continuous in \( \Omega \), i.e. there exist positive constants \( C \) and \( \alpha \), such that for every ball \( B(t_0, x_0; R) \), whose concentric double ball \( B(t_0, x_0; 2R) \) is contained in \( \Omega \), we have that

\[
|u(t, x) - u(s, y)| \leq C \left( \frac{|(t, x) - (s, y)|}{R} \right)^\alpha \left( \frac{1}{|B(t_0, x_0; 2R)|} \int_{B(t_0, x_0; 2R)} |u(t, x)|^2 \, dx \right)^{\frac{1}{2}},
\]

for all \((t, x), (s, y) \in B(t_0, x_0; R)\). Furthermore, \( u \) satisfies Moser’s local boundedness estimate meaning that for every \( p > 0 \) there exists a positive constant \( C = C(p) \), such that

\[
||u(\cdot, \cdot)||_{L^p(B(t_0, x_0; R))} \leq C \left( \frac{1}{|B(t_0, x_0; 2R)|} \int_{B(t_0, x_0; 2R)} |u(t, x)|^p \, dx \right)^{\frac{1}{p}},
\]

for all \((t_0, x_0) \in \Omega \) and for all \( R > 0 \) such that \( B(t_0, x_0; 2R) \subset \Omega \).

In our setting, since we are dealing with a single equation whose coefficient matrix \( A \) is assumed real, the DeGiorgi-Nash theorem, see [25, Section II.2] and also [21, 50], guarantees that Theorem 3.1.3 is true, with \( C \) and \( \alpha \) depending only on dimension and the ellipticity parameters; see [26, Sections 8.6, 8.9] and [39, Lemma 1.1.8, Corollary 1.1.10], for the symmetric case at least. Loosely speaking, a weak solution has some regularity, provided the coefficients are real. This is also true for \( t \)-dependent coefficients. However, this is not generally true for systems of elliptic equations, see [25, Section II.3, Chapter III]. It is precisely here that additional hypotheses on the regularity of weak solutions of \( L \) are necessary in [31] (in particular, the Morrey-type estimate in Definition 2.1 in [31]).
3.1 Construction and interior estimates

3.1.1 The averaged fundamental solution

We now introduce the function space with which we shall work in order to construct the
fundamental solution, see [31, Section 2.2]. The space \( \mathcal{Y} \) consists of all weakly differen-
tiable functions in \( L^{(2n+2)/(n-1)}(\mathbb{R}^{1+n}) \) whose first weak derivatives are square integrable,
that is
\[
\mathcal{Y} := \{ u : \mathbb{R}^{1+n} \rightarrow \mathbb{R} : u \in L^{2n+2} (\mathbb{R}^{1+n}), \nabla u \in L^2(\mathbb{R}^{1+n};\mathbb{R}^{1+n}) \}.
\]
(3.15)
By the Sobolev embedding theorem (see, for example, [41, Theorem 11.2]), we know that
\[
\| u \|_{L^{2n+2} (\mathbb{R}^{1+n})} \leq C \| \nabla u \|_{L^2(\mathbb{R}^{1+n};\mathbb{R}^{1+n})},
\]
(3.16)
for all \( u \in \mathcal{Y} \), where the positive constant \( C \) is allowed to depend on dimension. It follows
that the bilinear form
\[
\langle u, v \rangle_{\mathcal{Y}} := \iint_{\mathbb{R}^{1+n}} \nabla u(t,x) \cdot \nabla v(t,x) \, dt \, dx,
\]
(3.17)
where \( u, v \in \mathcal{Y} \), defines an inner product on \( \mathcal{Y} \) and, moreover, that \( \mathcal{Y} \) equipped with this
inner product becomes a Hilbert space, with norm
\[
\| u \|_{\mathcal{Y}} := \| \nabla u \|_{L^2(\mathbb{R}^{1+n};\mathbb{R}^{1+n})},
\]
(3.18)
where \( u \in \mathcal{Y} \). In addition, the following inclusions hold
\[
W^{1,2}(\mathbb{R}^{1+n}) \subset \mathcal{Y} \subset W^{1,2}_{\text{loc}}(\mathbb{R}^{1+n}).
\]
Further properties of the function space \( \mathcal{Y} \) can be found in [43, Section 1.3.4].

Moving on, we define the (not necessarily symmetric, since the matrix \( A \) is not necessarily symmetric) bilinear form
\[
B(u,v) := \iint_{\mathbb{R}^{1+n}} A(x) \nabla u(t,x) \cdot \nabla v(t,x) \, dt \, dx,
\]
(3.19)
where \( u, v \in \mathcal{Y} \). Since, by hypothesis, \( \| A \|_{L^\infty(\mathbb{R}^n;M_{1+n}(\mathbb{R}))} \) is finite, it readily follows that \( B \) is bounded, i.e. there exists a positive constant \( C \), such that
\[
|B(u,v)| \leq C \| u \|_{\mathcal{Y}} \| v \|_{\mathcal{Y}},
\]
(3.20)
for all \( u, v \in \mathcal{Y} \). Moreover, since \( A \) is strictly pointwise accretive, it follows that \( B \) is coercive (accretive) on \( \mathcal{Y} \), in other words that there exists a positive constant \( C \) such that
\[
C \langle u, u \rangle_{\mathcal{Y}} \leq B(u,u),
\]
(3.21)
for all \( u \in \mathcal{Y} \). For more information on the interplay between sesqui/bi-linear forms and elliptic operators see [30, Section 7.3.2] and [37, Chapter 6].
We will now use the Lax-Milgram lemma (see, for example, [30, Theorem C.5.3]) in the Hilbert space $\mathcal{Y}$ in order to find an “approximate solution” to the equation $-\text{div} A \nabla = \delta$, where $\delta$ is the Dirac delta distribution with pole at $(s, y) \in \mathbb{R}^{1+n}$.

Fix arbitrary $(s, y) \in \mathbb{R}^{1+n}$ and let $\rho > 0$. Consider, as in [31, Section 3.1], [29, Theorem 1.1], the linear functional

$$
\mathcal{Y} \ni u \mapsto \frac{1}{|B(s, y; \rho)|} \int_{B(s, y; \rho)} u(t, x) \, dt \, dx \in \mathbb{R},
$$

where $|B(s, y; \rho)|$ stands for the Lebesgue volume of the ball in $\mathbb{R}^{1+n}$. This functional is chosen as an approximation to the Dirac delta function with pole at $(s, y)$ in $\mathbb{R}^{1+n}$; instead of mapping $u$ to its value $u(s, y)$ at the pole, it maps it to its average over a ball of radius $\rho$ centered at the pole. Recall that for $u$ sufficiently well-behaved, locally integrable for example, Lebesgue’s differentiation theorem (see, for example, [24, Theorem 3.21]) yields

$$
\lim_{\rho \to 0} \frac{1}{|B(s, y; \rho)|} \int_{B(s, y; \rho)} u(t, x) \, dt \, dx = u(s, y), \quad (s, y) \in \mathbb{R}^{1+n},
$$

which reassures us that we are on the right track.

Hölder’s inequality shows that

$$
\left| \frac{1}{|B(s, y; \rho)|} \int_{B(s, y; \rho)} u(t, x) \, dt \, dx \right| \leq C \rho^{\frac{1-n}{2}} \|u\|_{L^{\frac{2n+2}{n-2}}(\mathbb{R}^{1+n})} \leq C \rho^{\frac{1-n}{2}} \|u\|_{\mathcal{Y}},
$$

for all $u \in \mathcal{Y}$, so the functional in (3.22) is bounded. Thus, by the Lax-Milgram lemma, there exists a unique $\Gamma^\rho = \Gamma^\rho(\cdot, \cdot, s, y) \in \mathcal{Y}$ such that

$$
B(\Gamma^\rho, u) = \frac{1}{|B(s, y; \rho)|} \int_{B(s, y; \rho)} u(t, x) \, dt \, dx,
$$

for all $u \in \mathcal{Y}$. Inserting $\Gamma^\rho$ for $u$ in (3.24) and using (3.21), we see that

$$
\|\Gamma^\rho\|_{\mathcal{Y}} = \|\nabla \Gamma^\rho\|_{L^2(\mathbb{R}^{1+n})} \leq C \rho^{\frac{1-n}{2}}.
$$

Adapting the terminology from [31, Section 3.1] we call $\Gamma^\rho$ the averaged fundamental solution with radius $\rho$ of $L$.

Next, we turn our attention to weak solutions of the non-homogeneous transpose equation $L^T u = f$. For a given $f \in L^\infty(\mathbb{R}^{1+n})$ with compact support, consider the linear functional

$$
\mathcal{Y} \ni w \mapsto \int_{\mathbb{R}^{1+n}} f(t, x) w(t, x) \, dt \, dx \in \mathbb{R}.
$$
3.1 Construction and interior estimates

We reason in a similar way as before. Hölder’s inequality shows that

$$\left| \int \int_{\mathbb{R}^{1+n}} f(t,x)w(t,x) \, dt \, dx \right| \leq \|f\|_{L^\infty(\mathbb{R}^{1+n})} \|w\|_{L^{2n+2}(\mathbb{R}^{1+n})} \leq C \|f\|_{L^\infty(\mathbb{R}^{1+n})} \|w\|_{Y},$$

for all $w \in Y$, so the averaged fundamental solution and its gradient, see [31, Section 3.2], are of the coefficient matrix.

In (3.27) we find that

$$\left| \int \int_{\mathbb{R}^{1+n}} A^T(x)\nabla u(t,x) \cdot \nabla w(t,x) \, dt \, dx \right| = \left| \int \int_{\mathbb{R}^{1+n}} \nabla u(t,x) \cdot A(x) \nabla w(t,x) \, dt \, dx \right|$$

for all $w \in Y$. Inserting $u$ for $w$ in (3.27) and using the coercivity of $B$, we see that

$$\|u\|_Y = \|\nabla u\|_{L^2(\mathbb{R}^{1+n})} \leq C \|f\|_{L^{2n+2}(\mathbb{R}^{1+n})}. \quad (3.28)$$

In addition, inserting $\Gamma^p$ for $w$ in (3.27) we find that

$$\left| \int \int_{\mathbb{R}^{1+n}} \Gamma^p(t,x,s,y) f(t,x) \, dt \, dx \right| = \frac{1}{|B(s,y,p)|} \int_{B(s,y,p)} u(t,x) \, dx. \quad (3.29)$$

The following lemma will help us establish local estimates for the averaged fundamental solution in the sequel.

**Lemma 3.1.4.** Fix an arbitrary point $(t_0,x_0) \in \mathbb{R}^{1+n}$ and an arbitrary radius $R > 0$. Let $p > (n+1)/2$, $f \in L^p(B(t_0,x_0;R))$ and suppose that $u \in Y$ is a weak solution to $L^t u = f$. Then there exists a positive constant $C$, such that

$$\|u\|_{L^p(B(t_0,x_0;R/2))} \leq CR^{2-\frac{n+1}{p}} \|f\|_{L^p(B(t_0,x_0;R))}. \quad (3.30)$$

In the same vein as in Theorem 3.1.3, we say that $u$ satisfies an inhomogeneous local boundedness estimate. For a proof of the aforementioned lemma, which uses the regularity of the coefficient matrix $A$ through the local Hölder continuity of $u$ and an iteration argument from [25, Chapter III, Lemma 2.1], we refer to [31, Section 3.2].

The following proposition is the first in line of several providing us with estimates for the averaged fundamental solution and its gradient, see [31, Section 3.2].

**Proposition 3.1.5.** Let $f \in L^{\infty}(\mathbb{R}^{1+n})$ with supp$f \subset B(s,y;R)$ and let $\Gamma^p$ be as in (3.24). Then there exists a positive constant $C$, such that

$$\|\Gamma^p(\cdot,\cdot; s,y)\|_{L^q(B(s,y;R))} \leq CR^{1-n+\frac{1+q}{p}}, \quad (3.31)$$

for all $1 \leq q < (n+1)/(n-1)$ and for all $p < R/2$. 

Whenever for all \( R > 3p \), introducing polar coordinates, we see that

\[
\iint_{B(y;R)} |\nabla \nabla \mathbb{R}(t,x,s,y)|_{L^2}^{2n+2} \, dx dy \leq C \int_{R^1\times0} |(t,x) - (s,y)|^{-2n-2} \, dx dy \leq CR^{-n-1},
\]

for all \( R > 0 \) and for all \( p > 0 \).

**Proof:** We consider two cases, depending on the relation between the radii \( R \) and \( p \). Whenever \( R > 3p \), introducing polar coordinates, we see that

\[
\iint_{B(y;R)} |\nabla \nabla \mathbb{R}(t,x,s,y)|_{L^2}^{2n+2} \, dx dy \leq C \int_{R^1\times0} |(t,x) - (s,y)|^{-2n-2} \, dx dy \leq CR^{-n-1},
\]

for all \( R > 0 \) and for all \( p > 0 \).
3.1 Construction and interior estimates

where Proposition 3.1.6, which is valid for such a configuration of \( R \) and \( p \), was used in the first inequality. For \( R \leq 3p \) we see that

\[
\| \Gamma^p(\cdot, \cdot; s, y) \|_{L^{\frac{2n+2}{n+1}}(\mathbb{R}^{1+n} \setminus B(s, y; R))} \leq \| \Gamma^p(\cdot, \cdot; s, y) \|_{L^{n+1}(\mathbb{R}^{1+n})} \leq \| \nabla \Gamma^p(\cdot, \cdot; s, y) \|_{L^2(\mathbb{R}^{1+n}; \mathbb{R}^{1+n})} \leq C p^{\frac{1}{2n}} \leq C R^{\frac{1}{4n}},
\]

where (3.16) and (3.25) where used in the second and third inequality respectively. From this inequality (3.33) follows by taking \( (2n+2)/(n-1) \) powers.

The following Proposition is derived from the aforementioned lemma via standard \( L^p \) space theory. In particular, it shows that the averaged fundamental solution is square integrable on annuli. By choosing to work with annuli we deviate slightly from the path followed in [31], thus circumventing the usage of weak \( L^p \) spaces and distribution functions (see, for example, [24, Section 6.4]). See also [31, Section 3.5].

**Proposition 3.1.8.** Let \( \Gamma^p \) be as in (3.24). Then, for every \( p \in [1, (2n+2)/(n-1)] \), there exists a positive constant \( C = C(p) \) such that

\[
\iint_{A(s, y; R, 2R)} |\Gamma^p(t, x; s, y)|^p \, dx \, dr \leq CR^{n p + (1+n) + p} \tag{3.34}
\]

for all \( R > 0 \) and for all \( p > 0 \).

**Proof:** Since for any \( R > 0 \), \( A(s, y; R, 2R) \subset \mathbb{R}^{1+n} \setminus B(s, y; R) \), inequality (3.34) for \( p = (2n+2)/(n-1) \), follows immediately from (3.33) by the properties of the integral.

Hölder’s inequality (see also [24, Proposition 6.12]) yields

\[
\iint_{A(s, y; R, 2R)} |\Gamma^p(t, x; s, y)| \, dx \, dr \leq \left( \iint_{A(s, y; R, 2R)} |\Gamma^p(t, x; s, y)| \frac{2n+2}{n+1} \, dx \right)^{\frac{n+1}{2n+2}} \left( \iint_{A(s, y; R, 2R)} 1 \, dx \right)^{\frac{4n+1}{2n+2}}
\]

which is precisely (3.34) for \( p = 1 \).

The estimate for \( 1 < p < (2n+2)/(n-1) \) follows by interpolation (Hölder’s inequality with conjugate exponents \( 1/\lambda p \) and \( (2n+2)/p(n-1)(1-\lambda) \), where \( \lambda = (2n+2 - np + p)/(n+3) \) between \( L^1 \) and \( L^{(2n+2)/(n-1)} \), see, for example, [24, Proposition 6.10].

Now we turn to the estimates satisfied by the gradient of \( \Gamma^p \). The following proposition tells us that the gradient of the averaged fundamental solution is square integrable on annuli; see also [31, Section 3.4].

**Proposition 3.1.9.** Let \( \Gamma^p \) be as in (3.24). Then there exists a positive constant \( C \) such that

\[
\iint_{A(s, y; R, 2R)} |\nabla_{t, x} \Gamma^p(t, x; s, y)|^2 \, dx \, dr \leq CR^{1-n} \tag{3.35}
\]

for all \( R > 0 \) and for all \( p > 0 \).
Before giving the proof of this proposition, let us recall Caccioppoli’s inequality; see, for example, [39, Lemma 1.1.5] or [25, Chapter III, Proposition 2.1].

**Lemma 3.1.10 (Caccioppoli’s Inequality).** Let \( u \in W^{1,2}_{\text{loc}}(\mathbb{R}^{1+n}) \) be a weak solution of (3.4). Then, there exists a positive constant \( C \), such that

\[
\iint_{B(t_0, R)} |\nabla_{t,x} u(t,x)|^2 \, dx \leq CR^{-2} \iint_{B(t_0, 2R)} |u(t,x)|^2 \, dx,
\]

for all \((t_0, x_0) \in \mathbb{R}^{1+n} \) and all \( 0 < R < t_0/2 \).

Choosing a suitable cut-off function \( \eta \) which is identically one on \( A(x, y; R, 2R) \) such that \( \eta \Gamma \) is a solution to (3.4) in \( B(s,y;5R/2) \), Proposition 3.1.8 for \( p = 2 \) and (3.36) yield

\[
\iint_{A(s,y; R,2R)} |\nabla_{t,x} \Gamma^p(t,x,s,y)|^2 \, dx \leq C \int_{A(s,y; R/2, 2.5R)} |\eta(t,x)\Gamma^p(t,x,s,y)|^2 \, dx \leq C \frac{R^{3-n}}{R^2} = CR^{1-n}.
\]

This is the key idea to the following proof, which resembles that of Lemma 3.1.10; see [39, Lemma 1.1.5].

**Proof of Proposition 3.1.9:** Consider the case where \( p < R/6 \) first. Let \( \eta \in C^\infty(\mathbb{R}^{1+n}) \) be a cut-off function such that \( \eta \equiv 1 \) on \( A(x, y; R, 2R) \), \( \eta \equiv 0 \) on \( \mathbb{R}^{1+n} \setminus A(x, y; R/2, 5R/2) \), \( 0 \leq \eta(t,x) \leq 1 \) and \( |\nabla_{t,x} \eta(t,x)| \leq C/\|t,x\| \) for all \( (t,x) \in \mathbb{R}^{1+n} \). Setting \( u(x,y) = \eta^2(\cdot,y)\Gamma^p(\cdot,y,s,y) \) in (3.24) and applying the product rule yields

\[
0 = \iint_{\mathbb{R}^{1+n}} \eta^2(t,x)A(x)\nabla_{t,x} \Gamma^p(t,x,s,y) \cdot \nabla_{t,x} \Gamma^p(t,x,s,y) \, dx
\]

\[
+ \iint_{\mathbb{R}^{1+n}} 2\eta(t,x)\Gamma^p(t,x,s,y)A(x)\nabla_{t,x} \Gamma^p(t,x,s,y) \cdot \nabla_{t,x} \eta(t,x) \, dx
\]

\[
= \iint_{A(x,y; R/2, 2.5R/2)} \eta^2(t,x)A(x)\nabla_{t,x} \Gamma^p(t,x,s,y) \cdot \nabla_{t,x} \Gamma^p(t,x,s,y) \, dx
\]

\[
+ \iint_{A(x,y; R/2, 2.5R/2)} 2\eta(t,x)\Gamma^p(t,x,s,y)A(x)\nabla_{t,x} \Gamma^p(t,x,s,y) \cdot \nabla_{t,x} \eta(t,x) \, dx.
\]

By the strict accretivity assumption (3.5) on \( A \), the triangle inequality, the Cauchy-Schwartz inequality and the fact that \( A \) is \( L^\infty \)-bounded, this implies that

\[
\iint_{A(x,y; R/2, 2.5R/2)} |\eta(t,x)|^2 |\nabla_{t,x} \Gamma^p(t,x,s,y)|^2 \, dx \leq C \iint_{A(x,y; R/2, 2.5R/2)} (|\Gamma^p(t,x,s,y)||\nabla_{t,x} \eta(t,x)||\nabla_{t,x} \Gamma^p(t,x,s,y)|) \, dx
\]

\[
\leq C \iint_{A(x,y; R/2, 2.5R/2)} (|\Gamma^p(t,x,s,y)||\nabla_{t,x} \eta(t,x)||\nabla_{t,x} \Gamma^p(t,x,s,y)|) \, dx
\]
By the absorption inequality (namely $ab \leq \varepsilon a^2 + b^2/4\varepsilon$, where $a, b, \varepsilon > 0$) and the properties of the cut-off function $\eta$ it follows that

$$\int_{A(x,y;R,2R)} |\nabla_I \Gamma^p(t,x,s,y)|^2 \, dx \leq C \int_{A(x,y;R/2,R)} \frac{|\Gamma^p(t,x,s,y)|^2}{|(t,x)|^2} \, dx + C \int_{A(x,y;2R,5R/2)} \frac{|\Gamma^p(t,x,s,y)|^2}{|(t,x)|^2} \, dx.$$ 

Introducing polar coordinates and invoking the pointwise estimate for $\Gamma^p$ from Proposition 3.1.6 yields

$$\int_{A(x,y;R,2R)} |\nabla_I \Gamma^p(t,x,s,y)|^2 \, dx \leq C \int_{R/2}^R \frac{r^{2-2n}}{r^2} r^p \, dr + C \int_{2R}^{5R/2} \frac{r^{2-2n}}{r^2} r^p \, dr = CR^{1-n}.$$ 

Now consider the case where $p \geq R/6$. Directly from (3.25) it follows that

$$\int_{A(x,y;R,2R)} |\nabla_I \Gamma^p(t,x,s,y)|^2 \, dx \leq \int_{\mathbb{R}^{1+n}} |\nabla_I \Gamma^p(t,x,s,y)|^2 \, dx \leq C p^{1-n} \leq CR^{1-n}. \quad \square$$

The following corollary is the analogue of Proposition 3.1.8; see also [31, Section 3.5]. It is derived from Proposition 3.1.9 via standard $L^p$ space theory, see, for example, [24, Proposition 6.10].

**Proposition 3.1.11.** Let $\Gamma^p$ be as in (3.24). Then, for every $p \in [1,2]$, there exists a positive constant $C = C(p)$ such that

$$\int_{A(x,y;R,2R)} |\nabla_I \Gamma^p(t,x,s,y)|^p \, dx \leq CR^{-np+(1+n)}, \quad (3.37)$$

for all $R > 0$ and for all $p > 0$.

**Proof:** Proposition 3.1.9 covers the case $p = 2$, while the Cauchy-Schwartz inequality shows that

$$\int_{A(x,y;R,2R)} |\nabla_I \Gamma^p(t,x,s,y)| \, dx \leq \left( \int_{A(x,y;R,2R)} |\nabla_I \Gamma^p(t,x,s,y)|^2 \, dx \right)^{1/2} \left( \int_{A(x,y;R,2R)} 1 \, dx \right)^{1/2} \leq CR^{\frac{1-p}{2p}} R^{\frac{1+n}{2}} = CR,$$

which is precisely (3.37) for $p = 1$.

The estimate for $1 < p < 2$ follows by interpolation between $L^1$ and $L^2$. In detail,
Hölder’s inequality yields
\[
\int \int_{A(s,y;R,2R)} \left| \nabla_{t\tilde{\tau}} \Phi (t,x,s,y) \right|^p \, dx \, dt \leq \int \int_{A(s,y;R,2R)} \left( \left( \int \int_{A(s,y;R,2R)} \left| \nabla_{t\tilde{\tau}} \Phi (t,x,s,y) \right|^2 \, dx \, dt \right)^{2-p} \left( \int \int_{A(s,y;R,2R)} \left| \nabla_{t\tilde{\tau}} \Phi (t,x,s,y) \right|^2 \, dx \, dt \right)^{p-1} \right) \, dx \, dt
\]
\[
\leq \left( \int \int_{A(s,y;R,2R)} \left| \nabla_{t\tilde{\tau}} \Phi (t,x,s,y) \right| \, dx \, dt \right)^{2-p} \left( \int \int_{A(s,y;R,2R)} \left| \nabla_{t\tilde{\tau}} \Phi (t,x,s,y) \right|^2 \, dx \, dt \right)^{p-1} \leq CR^2 - p R^{1-n} (p-1) = CR^{-np + (1+n)}.
\]

For convenience, the various estimates the averaged fundamental solution and its gradient satisfy on annuli are summarized in the next proposition.

**Proposition 3.1.12.** Let \( \Phi \) be as in (3.24). Then, for every \( 1 \leq p \leq (2n + 2)/(n+1) \), there exists a positive constant \( C = C(p) \) such that

\[
\| \Phi (\cdot, \cdot, s, y) \|_{L^p(A(s,y;R,2R))} \leq CR^{-\frac{1+np}{p}},
\]

for all \( R > 0 \) and all \( p > 0 \). Moreover, for every \( 1 \leq p \leq 2 \), there exists a positive constant \( C = C(p) \) such that

\[
\| \nabla \Phi (\cdot, \cdot, s, y) \|_{L^p(A(s,y;R,2R);R^{1+\sigma})} \leq CR^{-\frac{1+np}{p}},
\]

for all \( R > 0 \) and all \( p > 0 \). In both cases, the constants are allowed to depend on the dimension \( 1+n \) and the ellipticity constants as well. They are however independent of the radius \( p \).

We shall now obtain similar estimates for \( \Phi \) and \( \nabla \Phi \) on balls, by decomposing them in dyadic annuli and using the proposition above. Let \( R > 0 \) and consider the ball \( B(s,y;2R) \). For \( 1 \leq p < q \leq 2 \) (the exact value of \( q \) will be determined in the sequel) it is seen that

\[
\int \int_{B(s,y;2R)} |\Phi (t,x,s,y)|^p \, dx \, dt \leq \sum_{k=0}^{\infty} \int \int_{A(s,y;2^k R,2^{k+1} R)} |\Phi (t,x,s,y)|^p \, dx \, dt \leq CR^{-np + (1+n) + p} \sum_{k=0}^{\infty} 2^{-k(1+np + (1+n) + p)}.
\]

This geometric series will converge to a finite number if and only if \( -np + (1+n) + p > 0 \), i.e. if and only if \( p < (1+n)/(n-1) = q \).

As far as \( \nabla \Phi \) is concerned, similar reasoning reveals that

\[
\int \int_{B(s,y;2R)} |\nabla_{t\tilde{\tau}} \Phi (t,x,s,y)|^p \, dx \, dt \leq \sum_{k=0}^{\infty} \int \int_{A(s,y;2^k R,2^{k+1} R)} |\nabla_{t\tilde{\tau}} \Phi (t,x,s,y)|^p \, dx \, dt \leq CR^{-np + (1+n) + p} \sum_{k=0}^{\infty} 2^{-k(-np + (1+n))} \leq CR^{-np + (1+n)},
\]

where \( C = C(p) \) is an appropriate positive constant, if and only if \( -np + (1+n) > 0 \), i.e. if and only if \( p < (1+n)/n \). We have thus arrived at the following proposition.
Proposition 3.1.13. Let $\Gamma^p$ be as in (3.24). Then for every $1 \leq p < (1+n)/(n-1)$, there exists a positive constant $C = C(p)$ such that

$$\|\Gamma^p(\cdot, \cdot; s, y)\|_{L^p(B(x,y;R))} \leq CR^{n+\frac{1}{pq}+1},$$

for all $R > 0$ and for all $p > 0$. Moreover, for every $1 \leq p < (1+n)/n$, there exists a positive constant $C = C(p)$ such that

$$\|v\Gamma^p(\cdot, \cdot; s, y)\|_{L^p(B(x,y;R);\mathbb{R}^{1+n})} \leq CR^{n+\frac{1}{pq}},$$

for all $R > 0$ and for all $p > 0$.

3.1.2 Proof of Theorem 3.1.1 and Corollary 3.1.2

Finally, we consider what happens to $\Gamma$ and $v\Gamma$ in the limit, as $\rho$ tends to zero, see [31, Section 3.5]. The resulting limiting function $\Gamma$ is the much sought-after fundamental solution. This will prove Theorem 3.1.1, which is the main result of this section.

The proof relies heavily on tools and concepts from Functional Analysis, such as duality and weak convergence. For details we refer, for example, to [12, Chapter 3]. We also use the elementary fact that $C_c^\infty(\mathbb{R}^{1+n})$ is a dense subspace of $L^p(\mathbb{R}^{1+n})$ and of the Sobolev space $W^{1,p}(\mathbb{R}^{1+n})$, for $k \geq 0$ and $1 \leq p < \infty$. Recall also that the dual of $L^1(B(s,y;R))$ is precisely $L^\infty(B(s,y;R))$.

Proof of Theorem 3.1.1: Fix any $1 < q < (1+n)/n$. Then, by Proposition 3.1.13, we have that for every $R > 0$, there exists a positive constant $C = C(R)$, such that

$$\|\Gamma^p(\cdot, \cdot; s, y)\|_{W^{1,q}(B(x,y;R))} \leq C(R),$$

for all $p > 0$. We note that the constant depends only on $R$ because $q$ is now fixed. Since this bound is uniform with respect to $p$ and since $W^{1,q}(B(s,y;R))$ is a reflexive Banach space, by a diagonalization argument and [12, Theorem 3.18] (see also [41, Theorem 10.44]), we obtain a sequence $(\rho_k)_{k=1}^\infty$ of positive numbers and a function $\Gamma(\cdot, \cdot; s, y) \in W^{1,q}_{\text{loc}}(\mathbb{R}^{1+n})$, such that $\rho_k \to 0$ as $k \to \infty$ and

$$\Gamma^p(\cdot, \cdot; s, y) \rightharpoonup \Gamma(\cdot, \cdot; s, y),$$

in $W^{1,q}(B(s,y;R))$, where the symbol “$\rightharpoonup$” indicates weak convergence. Let $\phi \in C_c^\infty(\mathbb{R}^{1+n})$ with $\text{supp} \phi \subset B$, where $B$ is some ball of fixed radius in $\mathbb{R}^{1+n}$. We compute

$$\left| \int_{\mathbb{R}^{1+n}} A(x) [\nabla_t \Gamma^p(t,x,s,y) - \nabla_t \Gamma(t,x,s,y)] \cdot \nabla t,\phi(t,x) \, dt \, dx \right| \leq \left| \int_{\mathbb{R}^{1+n}} A(x) [\nabla_t \Gamma^p(t,x,s,y) - \nabla_t \Gamma(t,x,s,y)] \cdot \nabla t,\phi(t,x) \, dt \, dx \right| \leq \left| \int_{\mathbb{R}^{1+n}} [\nabla_t \Gamma^p(t,x,s,y) - \nabla_t \Gamma(t,x,s,y)] \cdot A^T(x) \nabla t,\phi(t,x) \, dt \, dx \right|,$$
where the latter quantity tends to zero as $k \to \infty$, essentially by the very definition of weak convergence. Thus
\[ \int_{\mathbb{R}^n} A(x) \nabla \phi \cdot \nabla \phi \, dx \to \int_{\mathbb{R}^n} A(x) \nabla \phi \cdot \nabla \phi \, dx, \]
as $k \to \infty$, so, by (3.23), we deduce that
\[ \int_{\mathbb{R}^n} A(x) \nabla \phi \cdot \nabla \phi \, dx = \phi(s,y), \]
which proves (3.6).

Furthermore, let $\phi \in L^\infty(\mathbb{R}^{1+n}, \mathbb{R}^{1+n})$ such that supp $\phi \subset A(s,y; R; 2R)$. By Proposition 3.1.12 and the definition of weak convergence, it is seen that
\[ \left| \int_{A(s,y; R; 2R)} \nabla \phi \cdot \nabla \phi \, dx \right| \leq CR^{1-n}, \]
from which (3.9) follows by duality. Similarly, let $\phi \in L^\infty(\mathbb{R}^{1+n})$ such that supp $\phi \subset A(s,y; R; 2R)$. Then Proposition 3.1.12 and weak convergence yield
\[ \left| \int_{A(s,y; R; 2R)} \phi \, dx \right| \leq CR^{1-n}, \]
from which (3.12) follows via duality.

Now, let $\phi \in L^\infty(\mathbb{R}^{1+n})$ such that supp $\phi \subset B(s,y; R)$. Due to weak convergence in $W^{1,q}(B(s,y; R))$ we have that
\[ \int_{\mathbb{R}^{1+n}} \Gamma(t,x,s,y) \phi(t,x) \, dx \to \int_{\mathbb{R}^{1+n}} \Gamma(t,x,s,y) \phi(t,x) \, dx, \quad k \to \infty. \]
By Proposition 3.1.13 and Hölder’s inequality, we see that for every $1 \leq p < (1+n)/(n-1)$, there exists a positive constant $C = C(p)$, such that
\[ \left| \int_{\mathbb{R}^{1+n}} \Gamma(t,x,s,y) \phi(t,x) \, dx \right| \leq CR^{1-n+\frac{1+n}{p}} \| \phi \|_{L^p(B(s,y; R))}, \]
for all $p_\ell > 0$. As usual, $p'$ denotes the conjugate exponent of $p$. It follows that, for every $1 \leq p < (1+n)/(n-1)$, there exists a positive constant $C = C(p)$, such that
\[ \| \Gamma(t, \cdot; s,y) \|_{L^p(B(s,y; R))} \leq CR^{1-n+\frac{1+n}{p}}, \]
which proves (3.10).
3.2 Estimates on the boundary

Next we mimic the proof of Proposition 3.1.6, working with the fundamental solution away from the pole to show the pointwise estimates (3.8) for $\Gamma$. From (3.6) it follows that $\Gamma(\cdot, \cdot; s, y)$ satisfies $L \phi = 0$ in the weak sense in $B(t, x; R)$, where $R := 2 \{ (t, x) - (s, y) \}/3$. From (3.9) (or from (3.6) again, using suitable test functions supported in $B(t, x; R)$ and duality) and from (3.12), it is deduced that $\Gamma$ is in $W^{1,2}(B(t, x; R))$. Thus, (3.14) and (3.10) yield

$$|\Gamma(t, x; s, y)| \leq CR^{-n-1} \|\Gamma(\cdot, \cdot; s, y)\|_{L^1(B(t, x; R))} \leq CR^{-n-1} \|\Gamma(\cdot, \cdot; s, y)\|_{L^1(B(s, y; 3R))} \leq CR^{-n-1} R^2 = CR^{1-n} = C |(t, x) - (s, y)|^{1-n}. $$

Finally, (3.7) is established using the Lax-Milgram lemma and weak convergence as in [31, Section 3.7], from which the uniqueness follows. This can also be found in [31, Section 3.7].

**Proof of Corollary 3.1.2:** Using the pointwise estimate (3.8) from Theorem 3.1.1(iii) and integrating introducing polar coordinates yields (iii); alternatively one could use a suitable cut-off function, Proposition 3.1.12 for $p = 2$ and duality, as mentioned in the proof above.

Furthermore, let $\phi \in L^{\infty}(\mathbb{R}^{1+n}; \mathbb{R}^{1+n})$ such that $\text{supp} \phi \subset B(s, y; R)$. Proposition 3.1.13 and weak convergence shows that, for every $1 \leq p < (1+n)/n$, there exists a positive constant $C = C(p)$ such that

$$\left| \int_{\mathbb{R}^{1+n}} V_t \Gamma^\phi(t, x; s, y) \cdot \phi(t, x) \, dt \, dx \right| \leq CR^{-n+1/n} \|\phi\|_{L^p(B(t, x; R))},$$

for all $\rho_k > 0$. Accordingly, for every $1 \leq p < (1+n)/n$, there exists a positive constant $C = C(p)$ such that

$$\|\nabla \Gamma(\cdot, \cdot; s, y)\|_{L^p(B(s, y; R); \mathbb{R}^{1+n})} \leq CR^{-n+1/n},$$

which proves (ii).

Item (i), i.e. (3.10), has already been established in the course of the previous proof.

\[\square\]

### 3.2 Estimates on the boundary

As already illustrated in Chapter 1, in order to speak of a (principal value) double layer potential operator, it is necessary for the fundamental solution and its gradient to possess a well-defined trace on the boundary.

Since the fundamental solution of the differential operator $L$ defined in (3.4) is continuous away from the diagonal, as seen in the previous section, there is no problem in making sense of what $\Gamma(0, x; s, y)$, with $(s, y) \in \mathbb{R}^{1+n}$ fixed, stands for; it is simply the value of the function $\Gamma(\cdot, \cdot; s, y) : \mathbb{R}^{1+n} \setminus \{(s, y)\} \to \mathbb{R}$ at the point $(0, x) \in \mathbb{R}^{1+n}$.

Unfortunately, the situation is different for the gradient of the fundamental solution. On one hand, we cannot simply treat it as a locally square integrable function in $\mathbb{R}^{1+n}$,
because functions in $L^2_{\text{loc}}(\mathbb{R}^{1+n})$ do not have a trace on $\mathbb{R}^n$. On the other, viewing $\Gamma$ as a function in $W^{1,p}_{\text{loc}}(\mathbb{R}^{1+n})$ does not help much either. Since $1 < p < (1 + n)/n < 2$, the standard theory for traces of functions in Sobolev spaces would give a trace in $W^{1-1/p,p}(\mathbb{R}^n)$ (see, for example, [12, Chapter 9], [41, Chapter 15]) and this does not provide information about the $L^2$ behaviour of the function $\nabla x, \Gamma(0, \cdot; y) : \mathbb{R}^n \to \mathbb{R}$.

In this section, we prove estimates satisfied by the gradient of the fundamental solution on $\mathbb{R}^n$. These will be used in the next chapter, in order to derive a useful representation formula for (a particular subclass of) the solutions of equation (3.4). As will be apparent from the proofs, that use properties of the operator $L$ such as uniform ellipticity, real-valuedness and t-independence of the coefficients that are shared by the transposed operator $L^t$, the transposed fundamental solution will satisfy the same estimates. Note that this holds in spite of the fact that the coefficients are not necessarily symmetric; in other words, $\Gamma = \Gamma^T$ is not assumed.

Estimates for the normal derivative of $\Gamma$ are given first. Since the coefficient matrix $A$ is assumed $t$-independent, it is natural to expect some more regularity along this direction. The next proposition is taken from [2, Lemma 2.2].

**Proposition 3.2.1.** Let $\Gamma$ be the fundamental solution to (3.4). Then there exists a positive constant $C$, such that

$$|\partial_t \Gamma(t,x;0,y)| \leq C |(t,x-y)|^{-n},$$

(3.40)

for all $t \in \mathbb{R}$ and for all $x, y \in \mathbb{R}^n$.

**Proof:** Consider the function $\partial_t \Gamma(\cdot, \cdot; 0, y)$ on the ball $B(t,x;R/17) \subset \mathbb{R}^{1+n}$, where $R := |(t,x) - (0,y)|$. Since $(0,y) \notin B(t,x;R/17)$, $\Gamma(\cdot, \cdot; 0, y)$ and therefore $\partial_t \Gamma(\cdot, \cdot; 0, y)$ (see Section 2.3) is a solution of $L_{x,y} U = 0$, by Theorem 3.1.1. Applying (3.14) with $p = 2$ for the latter solution we see that there exists a positive constant $C$ such that

$$|\partial_t \Gamma(t,x;0,y)| \leq CR^{-(1-n)/ 2} \|\nabla\Gamma(\cdot, \cdot; 0, y)\|_{L^2(B(t,x;2R/17))}$$

$$\leq CR^{-(1-n)/ 2} \|\nabla \nabla^{T} \Gamma(\cdot, \cdot; 0, y)\|_{L^2(B(t,x;2R/17);\mathbb{R}^{1+n})}.$$

By Caccioppoli’s inequality (see Lemma 3.1.10) and (3.12)

$$|\partial_t \Gamma(t,x;0,y)| \leq CR^{-(1-n)/ 2} R^{-1} \|\Gamma(\cdot, \cdot; 0, y)\|_{L^2(B(t,x;4R/17))}$$

$$\leq CR^{-(1-n)/ 2} R^{-1} R^{1/2} = CR^{-n}.$$ 

Note that using (3.9) instead of (3.12) we could have skipped the last step in the previous proof. Moreover, by iteration, one gets estimates for higher order vertical derivatives of the fundamental solution; for example

$$|\partial_t^2 \Gamma(t,x;0,y)| \leq C |(t,x-y)|^{-n-1},$$

(3.41)

for all $t \in \mathbb{R}$ and for all $x, y \in \mathbb{R}^n$.

Now, for $t > 0$, let $A_k := A(x; 2^k t, 2^{k+1} t) \subset \mathbb{R}^n$ denote the annulus consisting of all $y \in \mathbb{R}^n$ such that $2^k t < |y-x| < 2^{k+1} t$, for $k = 1, 2, \ldots$; $B(x,R)$ stands for the ball in $\mathbb{R}^n$ centred at $x$, as usual. The following proposition is essentially a restatement of [2, Lemma 2.5].
**Proposition 3.2.2.** Let $\Gamma$ be the fundamental solution to $L = -\text{div}AV$ as in (3.4). Then there exists a positive constant $C$, such that

$$
\int_{\mathbb{R}^{1+n}} |\nabla_y \Gamma(t,x;0,y)|^2 \, dy \leq C(2^k t)^{-n},
$$

(3.42)

for all $(t,x) \in \mathbb{R}^{1+n}$; here $A_k = A_k(x;2^k t,2^k t+1)$. Moreover, there exists a positive constant $C$, such that

$$
\int_{\mathbb{R}^{1+n}} |\nabla_y \Gamma(t,x;0,y)|^2 \, dy \leq C t^{-n},
$$

(3.43)

for all $(t,x) \in \mathbb{R}^{1+n}$.

**Proof:** Consider a cut-off function $\phi_k \in C_c^\infty(\mathbb{R}^n)$ such that supp $\phi_k \subset A(x;3(2^k t),3(2^k t))$, $\phi_k \equiv 1$ on $A_k = A_k(x;2^k t,2^{k+1} t)$ and $|\nabla_y \phi_k(y)| \leq C/|y|$, for all $y \in \mathbb{R}^n$. The adjoint fundamental solution $\Gamma^T(\cdot,\cdot;0,x)$ has pole at $(t,x) \in \mathbb{R}^{1+n}$.

Since $A$ is strictly pointwise accretive, $A^T$ and $A^T_{\|\|}$ are also strictly pointwise accretive; consequently

$$
\int_{\mathbb{R}^n} |\nabla_y \Gamma(t,x;0,y)|^2 \phi_k^2(y) \, dy \leq C \text{Re} \int_{\mathbb{R}^n} A^T_{\|\|}(y) \nabla_y^2 \Gamma(t,x;0,y) \cdot \nabla_y \Gamma(t,x;0,y) \phi_k^2(y) \, dy,
$$

for an appropriate positive constant $C$. Setting $L^T_{\|\|} := -\text{div} A^T_{\|\|} \nabla$, the last inequality can be rewritten as

$$
\int_{\mathbb{R}^n} |\nabla_y \Gamma(t,x;0,y)|^2 \phi_k^2(y) \, dy \leq C \text{Re} \int_{\mathbb{R}^n} L^T_{\|\|} \Gamma(t,x;0,y) \cdot \Gamma(t,x;0,y) \phi_k^2(y) \, dy
$$

$$
- C \text{Re} \int_{\mathbb{R}^n} A^T_{\|\|}(y) \nabla_y^2 \Gamma(t,x;0,y) \cdot \Gamma(t,x;0,y) \nabla_y \phi_k^2(y) \, dy.
$$

Using integration-by-parts and that

$$
L^T_{\|\|} = -\partial_s A^T_{\|\|}(y) \partial_s - \partial_s A^T_{\|\|}(y) \nabla y \cdot A^T_{\|\|}(y) \partial_s + L^T_{\|\|},
$$
together with the fact that the matrix $A$ is $s$-independent, we see that
\[
\int_{\mathbb{R}^n} \left| \nabla \Gamma(t, x; 0, y) \right|^2 \phi_k^2(y) \, dy \leq C \text{Re} \int_{\mathbb{R}^n} L^T_{0,y} \Gamma(t, x; 0, y) \Gamma(t, x; 0, y) \phi_k^2(y) \, dy 
+ C \text{Re} \int_{\mathbb{R}^n} A_{1\perp}^T(y) \partial_t^2 \Gamma(t, x; 0, y) \Gamma(t, x; 0, y) \phi_k^2(y) \, dy 
+ C \text{Re} \int_{\mathbb{R}^n} A_{1\perp}^T(y) \partial_t \Gamma(t, x; 0, y) \cdot \nabla \Gamma(t, x; 0, y) \phi_k^2(y) \, dy 
- C \text{Re} \int_{\mathbb{R}^n} A_{1\perp}^T(y) \partial_t \Gamma(t, x; 0, y) \cdot \Gamma(t, x; 0, y) \cdot 2 \phi_k(y) \nabla \phi_k(y) \, dy 
- C \text{Re} \int_{\mathbb{R}^n} A_{1\parallel}^T(y) \partial_t \Gamma(t, x; 0, y) \cdot \Gamma(t, x; 0, y) \cdot 2 \phi_k(y) \nabla \phi_k(y) \, dy
\]
\[=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \tag{3.44} \]

Since away from $(t, x)$ the function $u(t, \cdot, \cdot) := \Gamma^T(\cdot, \cdot, \cdot; t, x) = \Gamma(t, x; \cdot, \cdot, \cdot)$ is a solution of $L^T_{x,y} u(s, y) = 0$, it follows that $I_1 = 0$.

By the triangle inequality, the definition of $\phi_k$, the $L^\infty$-boundedness of $A$, Proposition 3.2.1 via (3.41), Theorem 3.1.1(iii) and after introducing polar coordinates, we see that
\[
|I_2| \leq C (2^k t)^{-\eta}.
\]

Along similar lines, by the triangle inequality, the absorption inequality (see the proof of Proposition 3.1.9), the definition of $\phi_k$, the $L^\infty$-boundedness of $A$, Proposition 3.2.1 and after introducing polar coordinates, we see that
\[
|I_4| \leq \frac{C}{\varepsilon} (2^k t)^{-\eta} + C \varepsilon \int_{\mathbb{R}^n} \left| \nabla \Gamma(t, x; 0, y) \right|^2 \phi_k^2(y) \, dy,
\]
for $\varepsilon > 0$ to be chosen later.

Following the same train of thought, due to the triangle inequality, the definition of $\phi_k$ (and the estimate for its derivative), Proposition 3.2.1, the $L^\infty$-boundedness of $A$, Theorem 3.1.1(iii) and after introducing polar coordinates once again, we obtain
\[
|I_5| \leq C (2^k t)^{-\eta}.
\]

As for $I_6$, applying the absorption inequality and making use of the triangle inequality, the definition of $\phi_k$ (and the estimate for its derivative), the $L^\infty$-boundedness of $A$, Theorem 3.1.1(iii) and after introducing polar coordinates, we have that
\[
|I_6| \leq \frac{C}{\varepsilon} (2^k t)^{-\eta} + C \varepsilon \int_{\mathbb{R}^n} \left| \nabla \Gamma(t, x; 0, y) \right|^2 \phi_k^2(y) \, dy.
\]
for \( \varepsilon > 0 \) to be chosen later. Note that this \( \varepsilon \) may as well differ from the previous one.

It remains to estimate \( I_3 \), which is the most complicated integral, due to the presence of the second derivative term, namely \( \partial_t \nabla^3 \Gamma(t,x,0,y) \). The absorption inequality together with the definition of \( \varphi_\delta \), the \( L^\infty \)-boundedness of \( A \) and the introduction of polar coordinates yield

\[
|I_3| \leq C \int_{\mathbb{R}^n} \left| \partial_t \nabla^3 \Gamma(t,x,0,y) \right|^2 \varphi_\delta^2(y) \, dy + C \int_{\mathbb{R}^n} \left| \Gamma(t,x,0,y) \right|^2 \varphi_\delta^2(y) \, dy
\]

for \( \varepsilon > 0 \) to be chosen later and an appropriate positive constant \( C \); here \( \mathcal{A}_K \) stands for the annulus \( \mathcal{A}_K = A(x;3(2^{k-2}t),3(2^k t)) \). We proceed to estimate the remaining integral in the right-hand-side of (3.45) using duality:

\[
\left( \int_{\mathcal{A}_K} \left| \partial_t \nabla^3 \Gamma(t,x,0,y) \right|^2 \, dy \right)^{\frac{1}{2}} = \sup_{h} \left| \int_{\mathcal{A}_K} \partial_t \nabla^3 \Gamma(t,x,0,y) \cdot h(y) \, dy \right|,
\]

where the supremum is taken over all \( h \in L^2(\mathcal{A};\mathbb{R}^{1+n}) \) with \( L^2 \)-norm less or equal to one. Now, since \( A \) is \( s \)-independent, it follows that we can replace \( \partial_t \) with \( \partial_t \); thus

\[
\sup_{h} \left| \int_{\mathcal{A}_K} \partial_t \nabla^3 \Gamma(t,x,0,y) \cdot h(y) \, dy \right| = \sup_{h} \left| \int_{\mathcal{A}_K} \nabla^3 \Gamma(t,x,0,y) \cdot h(y) \, dy \right|.
\]

For brevity, set \( u(t,x) := \int_{\mathcal{A}_K} \nabla^3 \Gamma(t,x,0,y) \cdot h(y) \, dy \). By the linearity of the inner product, \( u(\cdot, \cdot) \), hence \( \partial_t u(\cdot, \cdot) \) as well, is a solution to \( L_{t,x}u(t,x) = 0 \) away from \( (0,y) \). By (3.14) for \( p = 2 \) and Lemma 3.1.10 it follows that

\[
|\partial_t u(t,x)| \leq C(2^k t)^{\frac{1-n}{4}} \left( \int_{B(t,x;1/2^k)} \left| \partial_t u(t',x') \right|^2 \, dt' \, dx' \right)^{\frac{1}{2}}
\]

\[
\leq C(2^k t)^{\frac{1-n}{4}} (2^{2k})^{1-1} \left( \int_{B(t,x;1/4)} \left| u(t',x') \right|^2 \, dt' \, dx' \right)^{\frac{1}{2}}
\]

\[
\leq C(2^k t)^{-1} \sup_{(t',x')} |u(t',x')| = C(2^j t)^{-1} \sup_{(t',x')} \left| \int_{\mathcal{A}_K} \nabla^3 \Gamma(t',x';0,y) \cdot h(y) \, dy \right|,
\]

where the supremum is taken over all \( (t',x') \in B(t,x;1/4) \). Since the distance \( |x-y| \) is comparable to the distance \( |(t,x) - (0,y)| \), by the Cauchy-Schwartz inequality, (3.46)
and the fact that \( \|h\|_{L^2(\mathbb{R}_{\mathbb{A}_k};\mathbb{R}^{1+n})} \leq 1 \), it follows that

\[
\int_{\mathbb{A}_k} \partial_s V(t,x;0,y) \left| \nabla y \right|_{\Gamma(t,x;0,y)}^2 \, dy \leq C \sup_{(t',x') \in \mathbb{A}_k} \left| \nabla y \right|_{\Gamma(t',x';0,y)}^2 \, dy \leq C (2^k t)^{-n}.
\]

To finish the proof of (3.42), define

\[
\tilde{C} := \sup_{(t,x)} \left\{ \left( 2^k t \right)^n \int_{\mathbb{A}_k} \left| \nabla y \right|_{\Gamma(t,x;0,y)}^2 \, dy : (t,x) \in \mathbb{R}_{1+n}^1, k \in \mathbb{Z}^+ \right\}.
\] (3.46)

The finiteness of \( \tilde{C} \) is guaranteed whenever \( A \in C^\infty(\mathbb{R}^n; M_{1+n}(\mathbb{R})) \); in order to remove this assumption one works as in the proof of [2, Lemma 2.5].

Collecting all estimates above, we have proved that

\[
\tilde{C} \leq \varepsilon \tilde{C} + \frac{1}{\varepsilon} C.
\]

Choosing \( \varepsilon > 0 \) small enough, this implies \( \tilde{C} \leq C \), provided \( \tilde{C} < \infty \).

To prove (3.43) we argue similarly, making use of a cut-off function \( \phi \in C^\infty(\mathbb{R}^n) \) which is identically 1 on \( B(x;2t) \), identically zero outside \( B(x;3t) \) and whose derivative decays like \( 1/|x| \) and assuming that \( \sup_{(t,x) \in \mathbb{R}_{1+n}^1} \int_{B(x,2t)} \left| \nabla y \right|_{\Gamma(t,x;0,y)}^2 \, dy < \infty \), which is true for smooth coefficients. \( \Box \)

The above proof contains some novelties as compared to [2], in the estimate of (3.45).
4.1 Green’s formula

The goal of this section is to obtain Proposition 4.1.1, which gives a representation formula for the particular subclass of the solutions of (3.4) in the upper half-space, that are obtained via functional calculus as in Chapter 2. In the next section, we shall employ this representation for \( u \) in order to establish the boundedness of the double layer potential operator for real coefficients, without assuming that the latter are symmetric.

For convenience, we start off by recalling the estimates satisfied by the solutions constructed in Section 2.3 and by the fundamental solution, constructed in Chapter 3.

(S1) By Theorem 2.3.14, Proposition 2.3.16 and (2.48), the following estimates are valid for solutions \( u \in W^{1,2}_{bc}(\mathbb{R}^{1+n}) \), of equation (3.4) in \( \mathbb{R}^{1+n} \), that satisfy the square function estimate (2.44), vanish at infinity (i.e. \( \lim_{s \to \infty} \| u(s, \cdot) \|_{L^2(\mathbb{R}^n)} = 0 \)) and have \( L^2 \)-trace \( u(0, \cdot) = u|_{\mathbb{R}^n} = u_0(\cdot) \in L^2(\mathbb{R}^n) \) (i.e. \( \lim_{s \to 0} \| u(s, \cdot) - u_0(\cdot) \|_{L^2(\mathbb{R}^n)} = 0 \)):

\[
\sup_{s > 0} \| u(s, \cdot) \|_{L^2(\mathbb{R}^n)} \leq C \| u(0, \cdot) \|_{L^2(\mathbb{R}^n)} < \infty,
\]

and

\[
\| \nabla_y u(s, \cdot) \|_{L^2(\mathbb{R}^n; \mathbb{R}^{1+n})} \leq C \min \left\{ \frac{1}{s}, 1 \right\} \| v_0 \|_{L^2(\mathbb{R}^n; \mathbb{R}^{1+n})} + \| Dv_0 \|_{L^2(\mathbb{R}^n; \mathbb{R}^{1+n})} \leq C \min \left\{ \frac{1}{s}, 1 \right\} < \infty,
\]

where \( C \) is an appropriate positive constant and \( v_0 \in L^2(\mathbb{R}^n; \mathbb{R}^{1+n}) \) is a vector field with normal component \( v_0, \perp = -u_0 \) as in Subsection 2.3.1; \( \nabla_y u(0, \cdot) = (\nabla_y u)|_{\mathbb{R}^n} (\cdot) \) stands for the \( L^2 \)-trace of \( \nabla_y u : \mathbb{R}^{1+n} \to \mathbb{R}^{1+n} \).

(FS1) By Theorem 3.1.1(iii, iv), the following are valid for the fundamental solution \( \Gamma^T \),
where $L_+^T \Gamma^T(s,y;\ell,x) = \delta(s,y)$ for $(s,y) \in \mathbb{R}_e^{1+n}$ and with pole $(t,x)$ in $\mathbb{R}_e^{1+n}$:

$$|\Gamma^T(s,y;\ell,x)| \leq \frac{C}{|(s,y) - (t,x)|^{n+T}}, \quad (s,y) \neq (t,x)$$

$$\int_{A_\ell(t,x;\mathbb{R};2R)} |\nabla_{s,y} \Gamma^T(s,y;\ell,x)|^2 \, ds \, dy \leq C \frac{1}{R^{n-\ell}}, \quad R > 0,$$

for an appropriate positive constant $C$, where $A_\ell(t,x;\mathbb{R};R) := A(t,x;\mathbb{R};2R) \cap \mathbb{R}_e^{1+n}$, the part of the annulus centred at $(t,x)$ that lies in the upper half-space. Observe that the same estimate holds over annuli centred at $(0,0)$.

(FS2) By Proposition 3.2.2, the following holds for the trace of $\Gamma^T$ on $\mathbb{R}^n$:

$$\int_{A(0,\mathbb{R};2R)} |\nabla_{s,y} \Gamma^T(0,y;\ell,x)|^2 \, ds \, dy \leq CR^{-n},$$

for all $R > 0$ such that $(t,x) \in B_\ell(0,0;R) := B(0,0;R) \cap \mathbb{R}_e^{1+n}$ and for an appropriate positive constant $C$.

**Proposition 4.1.1** (Green’s formula). Let $\Gamma^T$ be the fundamental solution of the transpose operator $L^T$ and let $u \in W^{1,2}_{\text{loc}}(\mathbb{R}_e^{1+n})$ be a solution of (3.4) in $\mathbb{R}_e^{1+n}$, vanishing at infinity and satisfying quadratic estimates. Then for every $(t,x) \in \mathbb{R}_e^{1+n}$

$$u(t,x) = \int_{\mathbb{R}^n} \partial_{\lambda,\tau} \Gamma^T(0,y;\ell,x)u(0,y) \, dy - \int_{\mathbb{R}^n} \Gamma^T(0,y;\ell,x)\partial_{\lambda,\tau} u(0,y) \, dy,$$

(4.1)

where $\partial_{\lambda,\tau} \Gamma^T(0,y;\ell,x) = (A^T(y)\nabla_{s,y} \Gamma^T(0,y;\ell,x))_{\perp}$ and $\partial_{\lambda,\tau} u(0,y) = (A(y)\nabla_{s,y} u(0,y))_{\perp}$.

In order to understand this formula, consider the following two vector fields on $\mathbb{R}_e^{1+n}$

$$F : \mathbb{R}_e^{1+n} \longrightarrow \mathbb{R}_e^{1+n} ; \quad (s,y) \longmapsto u(s,y)A^T(y)\nabla_{s,y} \Gamma^T(s,y;\ell,x),$$

(4.2)

and

$$G : \mathbb{R}_e^{1+n} \longrightarrow \mathbb{R}_e^{1+n} ; \quad (s,y) \longmapsto \Gamma^T(s,y;\ell,x)A(y)\nabla_{s,y} u(s,y).$$

(4.3)

Assuming that the Gauß/Divergence theorem (see, for example, [33, Section 3.1]) can be applied for $F$ and $G$ in $\mathbb{R}_e^{1+n}$, (4.1) is basically Green’s second identity (see, for example, [40, Theorem 6.3]). To see this, apply the divergence theorem to both of the vector fields and subtract. Then use that $\nabla u \cdot A^T \Gamma^T = \nabla u \cdot \nabla^T \Gamma^T$, by the definition of the adjoint; that $-\text{div}A^T \nabla^T = \delta$, by the definition of $\Gamma^T$ and, finally, that $\text{div} \nabla u = 0$ in $\mathbb{R}_e^{1+n}$, since $u$ is a solution; $\partial \mathbb{R}_e^{1+n} = \mathbb{R}^n$. In fact, this is exactly what we did in Section 1.1, where we formally obtained an expression for the double layer potential operator for the Laplace equation. The representation formula (4.1) can therefore be viewed as a generalisation of Green’s formula for harmonic functions, hence the term “Green’s formula”.

However, the aforementioned arguments do not constitute a rigorous proof. First of all, $\mathbb{R}_e^{1+n}$ is an unbounded domain. Secondly, it is not clear whether $uA^T \Gamma^T$ and $\Gamma^T A \nabla u$ are integrable in $\mathbb{R}_e^{1+n}$. In order to overcome these obstacles, we consider the truncations $F\phi_R$ and $G\phi_R$ of the vector fields, where $\phi_R \in C_0^\infty(\mathbb{R}_e^{1+n})$ is a test function such that
4.1 Green’s formula

(i) \( \varphi_R \equiv 1 \) on \( B(0,0;R) \).

(ii) \( \varphi_R \equiv 0 \) on \( \mathbb{R}^{1+n} \setminus B(0,0;2R) \).

(iii) \( |\nabla_x \varphi_R(t,x)| \leq C / |(t,x)| \), for some positive constant \( C \) and for all \( (t,x) \in \mathbb{R}^{1+n} \).

Note that we can assume that the radius \( R \) is large enough, so that the pole \( (t,x) \) of the fundamental solution \( \Gamma^T(\cdot,t;\cdot) \) is in \( B(0,0;R) \cap \mathbb{R}^{1+n}_+ \).

We begin with a series of lemmata that will be used to establish Proposition 4.1.1. The first two deal with the behaviour of \( \nabla \varphi \) and \( \nabla \varphi \cdot \mathbf{F} \) in \( \mathbb{R}^{1+n}_+ \) where \( \varphi \) is the fundamental solution of \( A \cdot \nabla \varphi + \mathbf{F} \cdot \nabla \varphi = 0 \). Then, by the triangle inequality and the Cauchy-Schwarz inequality, it follows that

\[
|I|^2 \leq \int_{A_+(R)} \left| \nabla_x \varphi_R(s,y) \cdot A^T(y) \nabla_x \Gamma^T(s,y;\cdot,\cdot) \right|^2 \, dsdy \leq \int_{A_+(R)} |\nabla_x \varphi_R(s,y)|^2 \, dsdy = I_1 I_2.
\]

Hence, it suffices to show that the product \( I_1 I_2 \) tends to zero, as \( R \) tends to infinity.

We estimate the integral \( I_1 \) first. By the \( L^\infty \)-boundedness of \( A \) (hence also of \( A^T \)), the bound for the gradient of \( \varphi \), (FS1) and after introducing polar coordinates, it follows that

\[
I_1 \leq C \| \nabla \varphi_R \|_{L^\infty(A_+(R) \cap \mathbb{R}^{1+n})}^2 \int_{A_+(R)} \left| \nabla_x \Gamma^T(s,y;\cdot,\cdot) \right|^2 \, dsdy \leq C \frac{1}{R^2} \frac{1}{R^{n+1}} = C \frac{1}{R^{1+n}}.
\]

To estimate the integral \( I_2 \), we use (S1). Fubini’s theorem and the introduction of polar coordinates yield

\[
I_2 \leq C \int_0^{2R} \frac{2}{0} |u(0,\cdot)|^2_{L^2(\mathbb{R}^n)} \, ds = CR.
\]

Proof: Since \( \varphi_R \) is constant on \( B(0,0;R) \) and on \( \mathbb{R}^{1+n} \setminus B(0,0;2R) \), it is clear that

\[
\int_{A_+(R)} \left| \nabla_x \varphi_R(s,y) \cdot A^T(y) \nabla_x \Gamma^T(s,y;\cdot,\cdot) \right|^2 \, dsdy = \int_{A_+(R)} |\nabla_x \varphi_R(s,y)|^2 \, dsdy = I_1.
\]

\[
\int_{A_+(R)} |\nabla_x \varphi_R(s,y) \cdot A^T(y) \nabla_x \Gamma^T(s,y;\cdot,\cdot) \, dsdy = I,
\]

where \( A_+(R) := A(0,0;R,2R) \cap \mathbb{R}^{1+n}_+ \). Then, by the triangle inequality and the Cauchy-Schwarz inequality, it follows that

\[
|I|^2 \leq \int_{A_+(R)} \left| \nabla_x \varphi_R(s,y) \cdot A^T(y) \nabla_x \Gamma^T(s,y;\cdot,\cdot) \right|^2 \, dsdy \leq \int_{A_+(R)} |\nabla_x \varphi_R(s,y)|^2 \, dsdy = I_1 I_2.
\]
Thus, combining the estimates for $I_1$ and $I_2$, we see that

$$|I|^2 \leq C \frac{1}{R^{1+n}} R = C \frac{1}{R^{n-1}} \to 0, \quad R \to \infty. \quad \Box$$

**Lemma 4.1.3.** Let $\Gamma^T$ be the fundamental solution of the transpose operator $L^T$, $u$ be a solution of (3.4) in $\mathbb{R}^{1+n}_+$ and $\phi \in C_c(\mathbb{R}^{1+n})$ be a cut-off function as above. Then

$$\iint_{\mathbb{R}^{1+n}_+} \nabla_{x,y} \phi_R(s,y) \cdot (\Gamma^T(s,y,t,x)A(y)\nabla_{s,x} u(s,y)) \, dsdy \to 0, \quad (4.5)$$
as $R \to \infty$.

**Proof:** Clearly

$$\iint_{\mathbb{R}^{1+n}_+} \nabla_{x,y} \phi(s,y) \cdot (\Gamma^T(s,y,t,x)A(y)\nabla_{s,x} u(s,y)) \, dsdy =$$

$$\iint_{A_+(R)} \nabla_{x,y} \phi_R(s,y) \cdot (\Gamma^T(s,y,t,x)A(y)\nabla_{s,x} u(s,y)) \, dsdy =: I,$$

just like in the proof of the previous lemma. By the triangle inequality, the Cauchy-Schwartz inequality and the $L^\infty$-boundedness of the coefficient matrix $A$, it follows that

$$|I|^2 \leq \iint_{A_+(R)} |A(y)\nabla_{s,x} u(s,y)|^2 \, dsdy \iint_{A_+(R)} |\nabla_{s,x} \phi_R(s,y)\Gamma^T(s,y,t,x)|^2 \, dsdy$$

$$\leq C \iint_{A_+(R)} |\nabla_{s,x} u(s,y)|^2 \, dsdy \iint_{A_+(R)} |\nabla_{s,x} \phi_R(s,y)\Gamma^T(s,y,t,x)|^2 \, dsdy.$$

Hence, as earlier, it suffices to show that the product $I_1 I_2$ tends to zero, as $R$ tends to infinity.

First we treat $I_2$ by introducing polar coordinates and using the pointwise estimates for $\Gamma^T$ from (FS1), as well as those for $\nabla \phi_R$. It follows that

$$I_2 \leq C \int_{1/R}^{2R} \frac{1}{r^{n-1}} 1 \, dr = C \int_{1/R}^{2R} \frac{1}{r^n} \, dr = C \frac{1}{R^{n-1}}.$$

We estimate $I_1$ by considering the cases $s \leq 1$ and $s > 1$ for $(s,y) \in A_+(R)$ separately and invoking (S1). It follows that

$$I_1 \leq C \int_0^1 ds + C \int_{1}^{2R} \frac{1}{s^2} \, ds \leq C \left( 1 + \frac{1}{R} \right).$$
Hence, combining the estimates for $I_1$ and $I_2$ we see that
\[ |I|^2 \leq C \left( \frac{1}{R^n-1} + \frac{1}{R^n} \right) \longrightarrow 0, \quad R \longrightarrow \infty. \]

The next two lemmata deal with the behaviour of the traces of the truncated vector fields $\phi_R F$ and $\phi_R G$ on $\mathbb{R}^n$ (the boundary of the upper half-space) as $R$ tends to infinity.

**Lemma 4.1.4.** Let $\Gamma^T$ be the fundamental solution of the transpose operator $L^T$, $u$ be a solution of (3.4) in $\mathbb{R}^{1+n}^+$ and $\phi \in C_c^\infty(\mathbb{R}^{1+n})$ be a cut-off function as before. Then
\[ \int_{\mathbb{R}^n} (1 - \phi_R(0,y)) u(0,y) \left( A^T(y) \nabla_{x,y} \Gamma^T(0,y;t,x) \right)_\perp dy \longrightarrow 0, \quad \text{as } R \to \infty. \] (4.6)

**Proof:** Clearly, from the definition of $\phi_R$ and by the $L^\infty$-boundedness of the coefficient matrix, we have
\[ \left| \int_{\mathbb{R}^n} (1 - \phi_R(0,y)) u(0,y) \left( A^T(y) \nabla_{x,y} \Gamma^T(0,y;t,x) \right)_\perp dy \right| \leq C \int_{\mathbb{R}^n \setminus B(0;R)} |u(0,y)| \left| \nabla_{x,y} \Gamma^T(0,y;t,x) \right| \perp dy =: I. \]

Partition $\mathbb{R}^n \setminus B(0;R)$ into annuli $A_k(R) := A_k(0;2^k R, 2^{k+1} R)$, for $k = 0, 1, 2, \ldots$, and notice that $I \leq C \sum_{k=0}^\infty I_k$, where
\[ I_k := \int_{A_k(R)} |u(0,y)| \left( \nabla_{x,y} \Gamma^T(0,y;t,x) \right)_\perp dy. \]

By the Cauchy-Schwartz inequality, (S1) and (FS2), it follows, via the introduction of polar coordinates, that
\[ I_k \leq \left( \int_{A_k(R)} |u(0,y)|^2 dy \right)^{\frac{1}{2}} \left( \int_{A_k(R)} \left| \nabla_{x,y} \Gamma^T(0,y;t,x) \right|_\perp^2 dy \right)^{\frac{1}{2}} \]
\[ \leq \left( \int_{\mathbb{R}^n} |u(0,y)|^2 dy \right)^{\frac{1}{2}} \left( \int_{A_k(R)} \left| \nabla_{x,y} \Gamma^T(0,y;t,x) \right|_\perp^2 dy \right)^{\frac{1}{2}} \]
\[ \leq C \frac{1}{2^k R^{n/2}} = C \frac{1}{R^{n/2} \left( 2^{n/2} \right)^k}. \]

Consequently
\[ I \leq C \frac{1}{R^n} \sum_{k=0}^\infty \frac{1}{\left( 2^{n/2} \right)^k} \leq C \frac{1}{R^n} \longrightarrow 0, \quad R \longrightarrow \infty. \]
Lemma 4.1.5. Let $\Gamma^T$ be the fundamental solution of the transpose operator $L^T$, $u$ be a solution of (3.4) in $\mathbb{R}^{1+n}_+$ and $\phi \in C^\infty_c(\mathbb{R}^{1+n})$ be a cut-off function as before. Then

$$\int_{\mathbb{R}^n} (1 - \phi_R(0,y)) \Gamma^T(0,y;t,x) (A(y)\nabla_{x,y} u(0,y))_\perp \, dy \longrightarrow 0,$$

as $R \to \infty$.

Proof: In light of the definition of $\phi_R$ and the $L^\infty$-boundedness of the coefficients we clearly get

$$\left| \int_{\mathbb{R}^n} (1 - \phi_R(0,y)) \Gamma^T(0,y;t,x) (A(y)\nabla_{x,y} u(0,y))_\perp \, dy \right| \leq C \int_{\mathbb{R}^n \setminus B(0;R)} \left| \Gamma^T(0,y;t,x) (\nabla_{x,y} u(0,y))_\perp \right| \, dy =: I.$$

By the Cauchy-Schwartz inequality, (FS1), (S1) and since $\int_{\mathbb{R}^n} |\nabla_{x,y} u(s,y)|^2 \, dy < \infty$ so that $\int_{\mathbb{R}^n \setminus B(0;R)} |\nabla_{x,y} u(s,y)|^2 \, dy \to 0$ as $R \to \infty$, it follows, via the introduction of polar coordinates, that

$$I \leq C \left( \int_{\mathbb{R}^n \setminus B(0;R)} \left| \Gamma^T(0,y;t,x) \right|^2 \, dy \right)^{1/2} \left( \int_{\mathbb{R}^n \setminus B(0;R)} \left| \nabla_{x,y} u(0,y) \right|^2 \, dy \right)^{1/2} \leq C \left( \int_{\mathbb{R}^n \setminus B(0;R)} \left| \nabla_{x,y} u(0,y) \right|^2 \, dy \right)^{1/2} \to 0, \quad R \longrightarrow \infty.$$

Before we turn to the proof of Proposition 4.1.1, we mention that the vector fields $F \phi_R$ and $G \phi_R$ are locally integrable and in fact belong to $W^{1,1}(B_+ (0,0,2R))$. This follows from Theorem 3.1.1(iii, iv) and from Theorem 3.1.3. Therefore, the Gauss theorem/Green’s formula for bounded domains and weak derivatives holds; see [28, Theorem 1.5.3.1] and [33, Section 3.1].

Note that the following proof is really similar to the informal argument given in the beginning of this section; the extra terms that appear due to the product rule for weak derivatives disappear when $R$ tends to infinity.
Proof of Proposition 4.1.1: On one hand

\[
\iint_{\mathbb{R}^{1+n}} \text{div}_{s,y} (\phi_R(s,y)F(s,y)) \, ds \, dy = \iint_{B_{\epsilon}(0,0;2R)} \text{div}_{s,y} (\phi_R(s,y)F(s,y)) \, ds \, dy
\]

\[
= - \int_{B(0,2R)} (\phi_R(0,y)F(0,y))_y \, dy
\]

\[
= - \int_{\mathbb{R}^n} (\phi_R(0,y)F(0,y))_y \, dy,
\]

(4.8)

and similarly

\[
\iint_{\mathbb{R}^{1+n}} \text{div}_{s,y} (\phi_R(s,y)G(s,y)) \, ds \, dy = - \int_{\mathbb{R}^n} (\phi_R(0,y)G(0,y))_y \, dy.
\]

(4.9)

Observe that

\[
- \int_{\mathbb{R}^n} (\phi_R(0,y)F(0,y))_y \, dy \rightarrow - \int_{\mathbb{R}^n} F_y(0,y) \, dy, \quad R \rightarrow \infty,
\]

and similarly for the right-hand side of (4.9), as follows from Lemmata 4.1.4 and 4.1.5 respectively.

On the other hand

\[
\iint_{\mathbb{R}^{1+n}} \text{div}_{s,y} (\phi_R(s,y)F(s,y)) \, ds \, dy = \iint_{\mathbb{R}^{1+n}} (\nabla_{s,y}\phi_R(s,y) \cdot F(s,y) + \phi_R(s,y) \cdot \text{div}_{s,y}F(s,y)) \, ds \, dy
\]

\[
= \iint_{\mathbb{R}^{1+n}} \phi_R(s,y)\nabla_{s,y}u(s,y) \cdot A_T(y) \nabla_{s,y}\Gamma_T(s,y;t,x) \, ds \, dy
\]

\[
+ \iint_{\mathbb{R}^{1+n}} \phi_R(s,y)A_T(y) \nabla_{s,y}\Gamma_T(s,y;t,x) \, ds \, dy
\]

\[
+ \iint_{\mathbb{R}^{1+n}} \nabla_{s,y}\phi_R(s,y) \cdot F(s,y) \, ds \, dy,
\]

(4.10)
and similarly
\[
\iint_{\mathbb{R}^{1+n}_+} \text{div}_x (\Phi_R(s,y)G(s,y)) \, dsdy = \int \int_{\mathbb{R}^{1+n}_+} \Phi_R(s,y) \nabla_s \Gamma^T(s,y;t,x) : A(y) \nabla_s u(s,y) \, dsdy
\]
\[
+ \int \int_{\mathbb{R}^{1+n}_+} \Phi_R(s,y) \Gamma^T(s,y;t,x) \text{div}_s A(y) \nabla_s u(s,y) \, dsdy
\]
\[
+ \int \int_{\mathbb{R}^{1+n}_+} \nabla_s \Phi_R(s,y) \cdot G(s,y) \, dsdy.
\] (4.11)

Combine (4.8) with (4.10) and (4.9) with (4.11) and subtract. The terms \(I_{F,1}\) and \(I_{G,1}\) cancel each other out; \(I_{F,2} = -u(t,x)\), \(I_{G,2} = 0\). Taking the limit \(R \to \infty\) and invoking Lemmata 4.1.2, 4.1.3 in order to deal with \(I_{F,3}\) and \(I_{G,3}\), together with the aforementioned observation yields the result.

\[\square\]

### 4.2 The main result

We are finally, after all the important preliminaries in the previous sections and chapters, in position to prove the \(L^2\)-boundedness of the double layer potential operator for real, possibly non-symmetric, coefficients; see Corollary 4.2.2. This follows from Theorem 4.2.1, which is the main result of the thesis.

We start by recalling that since the operator \(BD\) satisfies quadratic estimates we have the following topological splitting

\[
L^2(\mathbb{R}^n,\mathbb{R}^{1+n}) = E_{BD}^+ L^2(\mathbb{R}^n,\mathbb{R}^{1+n}) \oplus E_{BD}^- L^2(\mathbb{R}^n,\mathbb{R}^{1+n}) \oplus \mathcal{N}(BD).
\]

Moreover, recall that \(D(D)\) is dense in \(E_{BD}^+ L^2(\mathbb{R}^n,\mathbb{R}^{1+n}) \oplus E_{BD}^- L^2(\mathbb{R}^n,\mathbb{R}^{1+n})\). Note that \(E_{BD}^+ L^2(\mathbb{R}^n,\mathbb{R}^{1+n}) = E_{BD}^+ \mathcal{H} = E_{BD}^+ \mathcal{H}_{\text{cut}},\) where now \(\mathcal{H} = L^2(\mathbb{R}^n,\mathbb{R}^{1+n})\). See Sections 2.2 and 2.3.

Now, let \(v_0^+ \in D(D) \cap E_{BD}^+ L^2(\mathbb{R}^n,\mathbb{R}^{1+n})\). Then, \(E_{BD}^+ v_0^+ = v_0^+\), \(e^{-i|BD|} E_{BD}^+ = e^{-i|BD|} E_{BD}^+\) and the function \(u = -v_\perp\) where

\[
v_\perp : \mathbb{R}^{1+n}_+ \longrightarrow \mathbb{R}; \ (t,x) \longmapsto (e^{-i|BD|} v_0^+(x))_\perp,
\]

has been seen to satisfy equation (3.4) in \(\mathbb{R}^{1+n}_+\) and to have boundary trace \(-v_0^+;\) see Subsection 2.3.1. Applying Green’s formula from Proposition 4.1.1 in the previous section, we see that for all \((t,x) \in \mathbb{R}^{1+n}_+\)

\[
- (e^{-i|BD|} v_0^+(x))_\perp = - \int_{\mathbb{R}^n} \Gamma^T(0,y;t,x) \text{div}_y v_0^{\|} \, dy - \int_{\mathbb{R}^n} \partial_{\nu,x} \Gamma^T(0,y;t,x) v_0^{\perp} \, dy.
\]
If \( v_0 \in D(D) \cap E_{BD}^{-}\mathcal{L}^2(\mathbb{R}^n; \mathbb{R}^{1+n}) \), then \( E_{BD}^{-}v_0 = v_0 \) and the vector field \( v(t,x) = e^{-BDt}E_{BD}^{-}v_0(x) = e^{-BDt}E_{BD}^{-}v_0(0) \) satisfies \( \partial_t v(t,x) + BDv(t,x) = 0 \) in \( \mathbb{R}^{1+n} \). The function \( -v_\perp \) is a solution of (3.4) in \( \mathbb{R}^{1+n} \) in the case. The pole of the fundamental solution \( \Gamma^T \) laid in the upper half-space when we proved (4.1). Accordingly, repeating the same process with the pole still in the upper half-space but for vector fields in \( \mathbb{R}^{1+n} \) yields

\[
0 = \int_{\mathbb{R}^n} \Gamma^T(0,y;t,x) \text{div} v_\perp(y) \, dy + \int_{\mathbb{R}^n} \partial_{x,y}^{\perp} \Gamma^T(0,y;t,x) v_\perp(y) \, dy.
\]

Adding the previous two equations together and recalling that for \( v_0 \in D(D) \cap N(BD) = D(D) \cap E_{BD}^{-}\mathcal{L}^2(\mathbb{R}^n; \mathbb{R}^{1+n}) \) both \( \text{div} v_\parallel \) and \( v_\parallel \) are identically zero, we arrive at the formula

\[
(e^{-BDt}E_{BD}^{+}v_0(x))_\perp = \int_{\mathbb{R}^n} \partial_{x,y}^{\perp} \Gamma^T(0,y;t,x) v_\perp(y) \, dy - \int_{\mathbb{R}^n} \nabla_{y}^{\perp} \Gamma^T(0,y;t,x) \cdot v_\parallel(y) \, dy \tag{4.12}
\]

where \((t,x) \in \mathbb{R}^{1+n}\), which is valid for all \( v_0 \in D(D) \cap L^2(\mathbb{R}^n; \mathbb{R}^{1+n}) = D(D) \). Naturally, for \((t,x) \in \mathbb{R}^{1+n}\), the analogous formula holds. Here \( \nabla_{y} \Gamma^T(0,y;t,x) = \nabla_{y} \Gamma^T(0,y;t,x) \) and \( \partial_{x,y}^{\perp} \Gamma^T(0,y;t,x) = (A(x) \nabla \Gamma^T(0,y;t,x))_\perp \).

The following theorem is the main result of this thesis. It proof relies on the fact that the operator \( BD \) satisfies quadratic estimates (by Theorem 2.3.9); therefore it has a bounded \( \mathcal{H}^m \) functional calculus and a well-defined holomorphic semigroup on the appropriate spectral subspaces; see Theorem 2.2.3, Corollary 2.2.9, Proposition 2.2.13 and Proposition 2.2.14.

**Theorem 4.2.1.** Consider the family of operators of operators \( \Theta_t^+ \) where for \( t > 0 \)

\[
\Theta_t^+ : L^2(\mathbb{R}^n; \mathbb{R}^{1+n}) \longrightarrow L^2(\mathbb{R}^n) : f(\cdot) \longmapsto \Theta_t^+ f(\cdot) := (e^{-BDt}E_{BD}^{+}f(\cdot))_\perp.
\]

Then there exists a positive constant \( C \), such that for all \( t > 0 \)

\[
||\Theta_t^+ f||_{L^2(\mathbb{R}^n)} \leq C ||f||_{L^2(\mathbb{R}^n; \mathbb{R}^{1+n})}, \tag{4.13}
\]

for all \( f \in L^2(\mathbb{R}^n; \mathbb{R}^{1+n}) \), in other words, \( (\Theta_t^+)_{t>0} \) is uniformly bounded in \( t \). Moreover, the operators \( \Theta_t^+ \) converge strongly as \( t \to 0 \) to a bounded operator \( \Theta^+ \), where

\[
\Theta^+ : L^2(\mathbb{R}^n; \mathbb{R}^{1+n}) \longrightarrow L^2(\mathbb{R}^n) : f \longmapsto \Theta^+ f := \lim_{t \to 0} \Theta_t^+ f = (E_{BD}f)_\perp. \tag{4.14}
\]

In addition, the operators \( (\Theta_t^+)_{t>0} \) converge strongly to the zero operator as \( t \to \infty \). Furthermore, for \( f \in D(D) \) the following formula holds

\[
\Theta_t^+ f(x) = \int_{\mathbb{R}^n} \nabla_{y}^{\perp} \Gamma^T(0,y;t,x) \cdot f(y) \, dy - \int_{\mathbb{R}^n} \partial_{x,y}^{\perp} \Gamma^T(0,y;t,x) f(y) \, dy, \tag{4.15}
\]

for all \( x \in \mathbb{R}^n \) and \( t > 0 \).
Note that since \((\Theta^+_t)_{t \geq 0}\) is uniformly bounded, \(\mathcal{D}(D)\) is dense in \(L^2(\mathbb{R}^n; \mathbb{R}^{1+n})\) and \(\Theta^+_t f\) converges as \(t \to 0\), and as \(t \to \infty\) for all \(f \in \mathcal{D}(D)\), by Lemma 2.2.4 it follows that \(\Theta^+_t f\) converges as \(t \to 0, \infty\) for all \(f \in L^2(\mathbb{R}^n; \mathbb{R}^{1+\alpha})\). Therefore, the limit of the right-hand-side of (4.15) as \(t \to 0\) exists for all \(f \in L^2(\mathbb{R}^n; \mathbb{R}^{1+\alpha})\) and

\[
\lim_{t \to 0} \left( \int_{\mathbb{R}^n} \nabla \Gamma^T(0,y;t,x) \cdot f(y) \, dy - \int_{\mathbb{R}^n} \partial_{x^T} \Gamma^T(0,y;t,x) f(y) \, dy \right) = (E^+_\mathcal{B}D f(x))_{\perp} \in \mathbb{R},
\]

for all \(f \in L^2(\mathbb{R}^n; \mathbb{R}^{1+\alpha})\), \(x \in \mathbb{R}^n\). For the same reasons

\[
\lim_{t \to -\infty} \left( \int_{\mathbb{R}^n} \nabla \Gamma^T(0,y;t,x) \cdot f(y) \, dy - \int_{\mathbb{R}^n} \partial_{x^T} \Gamma^T(0,y;t,x) f(y) \, dy \right) = 0,
\]

for all \(f \in L^2(\mathbb{R}^n; \mathbb{R}^{1+\alpha})\), \(x \in \mathbb{R}^n\).

**Proof:** By Proposition 2.2.14 the family of operators \((e^{-itBDE^+_\mathcal{B}D})_{t \geq 0}\) is uniformly bounded on \(L^2(\mathbb{R}^n; \mathbb{R}^{1+\alpha})\). A fortiori, the family of operators involving only its “normal part”, i.e. \((\Theta^+_t)_{t \geq 0}\) is also uniformly bounded. The strong limits of \((\Theta^+_t)_{t \geq 0}\) follow from the \(L^2\)-limits of \(e^{-itBDE^+_\mathcal{B}D}\), namely \(\lim_{t \to 0} \|e^{-itBDE^+_\mathcal{B}D} \|_{L^2(\mathbb{R}^n; \mathbb{R}^{1+\alpha})} = 0\) and \(\lim_{t \to \infty} \|e^{-itBDE^+_\mathcal{B}D} \|_{L^2(\mathbb{R}^n; \mathbb{R}^{1+\alpha})} = 0\); see also Proposition 2.3.13. Since the spectral projection \(E^+_\mathcal{B}D : L^2(\mathbb{R}^n; \mathbb{R}^{1+\alpha}) \to L^2(\mathbb{R}^n; \mathbb{R}^{1+\alpha})\) is bounded, it is clear that the operator \(\Theta^+_t = \text{pr}_\perp \circ E^+_\mathcal{B}D\), where \(\text{pr}_\perp(f) = f_{\perp}, f \in L^2(\mathbb{R}^n; \mathbb{R}^{1+\alpha})\) is also bounded. Formula (4.15) is simply a restatement of (4.12). \(\square\)

The next two corollaries follow immediately from Theorem 4.2.1, by considering square integrable vector fields with zero parallel components for the first one, and square integrable vector fields with zero normal components for the second one.

**Corollary 4.2.2.** The family of operators \((\mathcal{D}^+_t)_{t \geq 0}\), where

\[
\mathcal{D}^+_t : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) : f \mapsto \mathcal{D}^+_t f := -\Theta^+_t \left[ \begin{array}{c} f \\ 0 \end{array} \right]
\]

is uniformly bounded in \(t\). Moreover, the operators \((\mathcal{D}^+_t)_{t \geq 0}\) converge strongly as \(t \to 0\) to a bounded operator \(\mathcal{D}^+ := \lim_{t \to 0} \mathcal{D}^+_t\) on \(L^2(\mathbb{R}^n)\) and \(\lim_{t \to \infty} \mathcal{D}^+_t = 0\). In addition, for \(f \in W^{1,2}(\mathbb{R}^n)\) the following formula holds

\[
\mathcal{D}^+_t f(x) = \int_{\mathbb{R}^n} \partial_{x^T} \Gamma^T(0,y;t,x) f(y) \, dy, \quad x \in \mathbb{R}^n.
\]

**Corollary 4.2.3.** The family of operators \((\mathcal{S}^+_t)_{t \geq 0}\), where

\[
\mathcal{S}^+_t : L^2(\mathbb{R}^n; \mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) : f \mapsto \mathcal{S}^+_t f := \Theta^+_t \left[ \begin{array}{c} 0 \\ f \end{array} \right]
\]
is uniformly bounded in \( t \). Moreover, the operators \( \{ \mathcal{R}^+_t \}_{t \geq 0} \) converge strongly as \( t \to 0 \) to a bounded operator \( \mathcal{R}^+_0 := \lim_{t \to 0} \mathcal{R}^+_t \) on \( L^2(\mathbb{R}^n) \) and \( \lim_{t \to 0} \mathcal{R}^+_t = 0 \). In addition, for \( f \in W^{1,2}(\mathbb{R}^n; \mathbb{R}^n) \) the following formula holds

\[
\mathcal{R}^+_t h(x) := \int_{\mathbb{R}^n} \nabla \Gamma^T(0,y,t,x) \cdot h(y) \, dy, \quad x \in \mathbb{R}^n.
\]

It is both interesting and instructive to see what happens in the particular case when the coefficient matrix \( A \) is simply the identity.

---

**Example 4.2.4**

Let \( A(x) = I_{[1,n]} \), for all \( x \in \mathbb{R}^n \). Then the equation \( \text{div}_x \mathcal{A}(x) \nabla u(t,x) = 0 \) in \( \mathbb{R}^{1+n}_+ \) reduces to the classical \( \Delta_t u(t,x) = 0 \) in \( \mathbb{R}^{1+n}_+ \) and \( \Gamma = \Gamma^T \) is the fundamental solution of the Laplacian as in (3.1).

(i) Since

\[
\partial_y \nabla \Gamma^T(0,y,t,x) = \partial_y \Gamma^T(0,y,t,x) = \frac{-t}{\sigma_n \left( t^2 + |x-y|^2 \right)^{\frac{n+2}{2}}} = : P(t,x-y),
\]

is the Poisson kernel for the upper half-space (see, for example, [24, Section 8.7], [27, Example 2.1.13]), the operator \( D^+_t \) is the Poisson integral

\[
D^+_t h(x) = -\int_{\mathbb{R}^n} P(t,x-y) h(y) \, dy = (P_t * f)(x),
\]

where “\( \ast \)” denotes convolution. It is known that \( u(t,\cdot) := -D^+_t h(\cdot) \) is a harmonic function in \( \mathbb{R}^{1+n}_+ \) and that \( u(t,\cdot) \to h(\cdot) \) both in \( L^2(\mathbb{R}^n) \) and almost everywhere, as \( t \to 0^+ \). See, for example, [24, Theorem 8.53], [27, Example 2.1.15].

(ii) Since

\[
\nabla \nabla \Gamma^T(0,y,t,x) = \frac{-1(x-y)t}{\sigma_n \left( t^2 + |x-y|^2 \right)^{\frac{n+2}{2}}},
\]

we see, using vector fields of the form \( h = [0, \ldots, 0, h_j, 0, \ldots, 0]' \), \( 1 \leq j \leq n \) and omitting the minus sign, that the operator \( \mathcal{R}^+_t \) give rise to bounded operators

\[
(\mathcal{R}^+_t)_j : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) ; f \mapsto (\mathcal{R}^+_t)_j f, \quad 1 \leq j \leq n,
\]

where for \( x \in \mathbb{R}^n \)

\[
(\mathcal{R}^+_t)_j f(x) := \frac{1}{\sigma_n} \int_{\mathbb{R}^n} \frac{x_j - y_j}{(t^2 + |x-y|^2)^{\frac{n+2}{2}}} f(y) \, dy.
\]
The $L^2$-limit of $\left( R_j^+ \right)_j$, as $t \to 0^+$, is the $j$th Riesz transform, namely

$$R_j f(x) := \frac{1}{\sigma_n} \p.v. \int \frac{x_j - y_j}{|x - y|^{1+n}} f(y) \, dy.$$ 

For further information on the Riesz transforms, see, for example, [27, Section 4.1.4]. Since the operator $R_j^+$ is bounded by Corollary 4.2.3, we recover the $L^2$-boundedness of the Riesz transforms, see [27, Corollary 4.2.8], which is usually established with the help of the Fourier transform and the classical theory of convolution singular integrals, see [27, Section 4.2]. This generalises a phenomenon we have already seen in one dimension. On one hand the Riesz transform reduces to the Hilbert transform (see, for example, [27, Section 4.1.1]) which, using the Fourier transform for example, is seen to be $L^2$-bounded (see [27, Theorem 4.1.7]). On the other hand the operator $-i\sigma/d\sigma$ satisfies quadratic estimates and has a bounded $\mathcal{H}^\infty$ functional calculus, thus the operator $\text{sgn}(-i\sigma/d\sigma)$ is $L^2$-bounded. It turns out that the latter is precisely the Hilbert transform, see [1, Section J], [46, Example 5.7.22].


