

DISPERSION FORCES BETWEEN REAL METAL PLATES AT FINITE TEMPERATURE AND THE NERNST HEAT THEOREM

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We derive the zero-and-finite-temperature dispersion-forces in terms of changes in the zero-point energy of the electromagnetic normal modes of the system. At finite temperature the Helmholtz free energy of interaction determines the force. We express this energy in two complementary ways; in the first as a discrete summation over complex frequencies, Matsubara frequencies; in the second as an integral along the real frequency axis. We use these expressions to get the entropy of interaction and demonstrate that this entropy obeys the Nernst heat theorem, one version of the third law of thermodynamics.

1 Introduction

In 1948 Casimir predicted the attraction between two parallel, perfectly conducting, surfaces at zero temperature.¹ This Casimir force has been measured between metal surfaces.^{2,3,4,5} Recently, a number of studies have focused on the combined effects of finite conductivity and finite temperature.^{6,7,8,9} As we will discuss below, one way to calculate the Casimir interaction at finite temperature leads to a frequency summation. In this summation the zero frequency term is of special interest since it dominates at high temperatures and also at large separations for any finite temperature. We used⁶ this frequency summation together with the Drude dielectric function to describe the Casimir interaction between real metal surfaces. This gave a high temperature asymptote that is half the value of that for perfect conductors. A number of different groups have later come to different results using different approaches.^{10,11,12,13} A recent objection^{12,13,14} to our approach is that its results violate the Nernst heat theorem. We demonstrate in this work that it is not so.

2 Dispersion forces in general

The dispersion forces between objects are due to the change in energy of the electromagnetic normal modes of the system with separation between the objects. The normal modes are so called massless bosons. The total energy for massless bosons is:

$$E = \sum_i \hbar\omega_i \left[n(\omega_i) + \frac{1}{2} \right] ; n(\omega) = 1/[e^{\beta\hbar\omega} - 1]. \quad (1)$$

At zero temperature the occupation numbers, $n(\omega_i)$, vanish and only the zero-point energy contributes:

$$F = -\frac{dV(r)}{dr} = -\frac{dE(r)}{dr} = -\frac{d}{dr} \sum_i \frac{1}{2} \hbar \omega_i(r). \quad (2)$$

At non-zero temperature the change in internal energy also has contributions from heat transfer. These do not contribute to the force. It is Helmholtz' free energy, \mathcal{F} , that governs the force:

$$F = -\frac{dV(r)}{dr} = -\frac{d\mathcal{F}(r)}{dr} = -\frac{d}{dr} \sum_i \frac{1}{\beta} \ln [2 \sinh \frac{1}{2} \beta \hbar \omega_i(r)]. \quad (3)$$

Sometimes it is easy to find the energy of the modes and to calculate the force. It is not always so; the modes may form a continuum; when retardation is included also modes from the surrounding vacuum contribute and it is difficult to solve the equations leading to the modes. Then it is better to use another approach, based on an extension of the so-called argument principle. Let us introduce two functions of the complex variable z . Function $f(z)$ has poles and zeros inside a given closed contour in the complex frequency plane, but no poles or zeros on the contour itself; function $\varphi(z)$ is analytic inside and on the contour. Then the following relation holds:

$$\frac{1}{2\pi i} \oint dz \varphi(z) \frac{d}{dz} \ln f(z) = \sum \varphi(z_o) - \sum \varphi(z_\infty), \quad (4)$$

where the first (second) summation runs over the zeros (poles) of function $f(z)$. If we now choose our functions to be

$$f(z) = A(z) ; \varphi(z) = \begin{cases} \hbar z/2 ; & T = 0 \\ \ln (2 \sinh \frac{1}{2} \beta \hbar z) / \beta ; & T \neq 0 \end{cases}, \quad (5)$$

where $A(z) = 0$ is the condition for having normal modes, we obtain the interaction energy.

Now, depending on how one chooses the contour ¹⁵ one ends up with different expressions. Let us choose as reference system the geometry when the objects are at infinite separation. One may then have for zero temperature and at the separation d :

$$\Delta E(d) = \hbar \int_0^\infty \frac{d\omega}{2\pi} \ln \left[\frac{A(d, i\omega)}{A(\infty, i\omega)} \right], \quad (6)$$

i.e., one obtains the results as an integration along the imaginary frequency axis. To arrive at this result we have used the contour in Fig. 1a. For finite

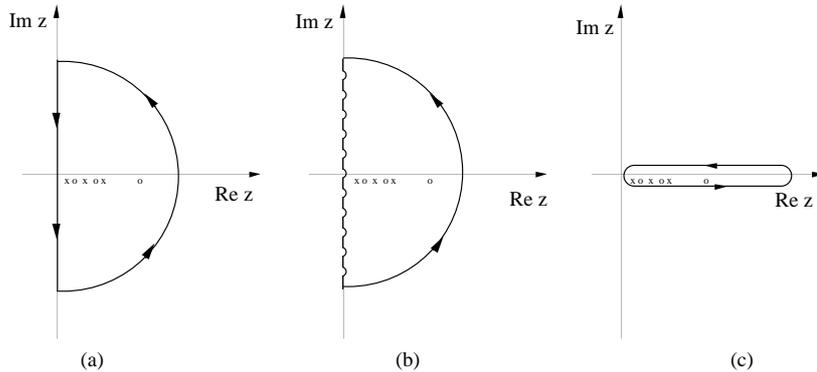


Figure 1. Contours used in the derivation of Eqs. (6)-(8). The crosses (circles) are the positions of the poles (zeros) of function $f(z)$.

temperature one arrives at

$$\Delta \mathcal{F}(d) = \frac{1}{\beta} \sum'_{\omega_n} \ln \left[\frac{A(d, i\omega_n)}{A(\infty, i\omega_n)} \right]; \quad \omega_n = \frac{2\pi n}{\hbar\beta}; \quad n = 0, 1, 2, \dots, \quad (7)$$

i.e., one obtains the result as a discrete summation along the imaginary frequency axis. To arrive at this result we have used the contour in Fig. 1b. The contour contains small semicircles around each $i\omega_n$. The prime on the summation sign indicates that the $n = 0$ term should be reduced by a factor of two. By instead choosing the contour in Fig. 1c one gets the result as an integration along the real frequency axis:

$$\Delta \mathcal{F}(d) = 2\hbar \text{Im} \int_0^{\infty} \frac{d\omega}{2\pi} \left[n(\omega, T) + \frac{1}{2} \right] \ln \left[\frac{A(d, \omega)}{A(\infty, \omega)} \right]. \quad (8)$$

3 Dispersion forces between metal plates

To find the electromagnetic normal modes for a system consisting of two metal plates we solve Maxwell's equations in the different regions and match the solutions at the boundaries. We use the *standard idealizations*: the dielectric properties are valid all the way up to the sharp interfaces neglecting the fact that the properties are macroscopic averages; \mathbf{k} -conservation along the surface (no roughness); standard boundary conditions for the \mathbf{E} -, \mathbf{D} -, \mathbf{B} - and \mathbf{H} -fields; assume no spatial dispersion, i.e., the dielectric function has no momentum dependence.

There will be two types of mode. Transverse magnetic (TM) and Transverse electric (TE). If the metal plates are thick enough one may consider two metal half spaces. Then the two conditions for normal modes are

$$A(d, \mathbf{k}, \omega) = \left[\varepsilon(\omega) \sqrt{k^2 - (\omega/c)^2} + \sqrt{k^2 - \varepsilon(\omega) (\omega/c)^2} \right]^2 - e^{-2\sqrt{k^2 - (\omega/c)^2}d} \left[\varepsilon(\omega) \sqrt{k^2 - (\omega/c)^2} - \sqrt{k^2 - \varepsilon(\omega) (\omega/c)^2} \right]^2 \quad (9)$$

and

$$A(d, \mathbf{k}, \omega) = \left[\sqrt{k^2 - \varepsilon(\omega) (\omega/c)^2} + \sqrt{k^2 - (\omega/c)^2} \right]^2 - e^{-2\sqrt{k^2 - (\omega/c)^2}d} \left[\sqrt{k^2 - \varepsilon(\omega) (\omega/c)^2} - \sqrt{k^2 - (\omega/c)^2} \right]^2 \quad (10)$$

for *TM* and *TE* modes, respectively. Thus the modes are characterized by the two-dimensional wave vector \mathbf{k} .

Then from Eqs. (7) and (8) we get two quite different but equivalent expressions for the interaction potential between two half spaces:

$$\mathcal{F}(d) = \frac{1}{\beta} \sum_n' \int \frac{d^2k}{(2\pi)^2} \{ \ln [G^{TM}(\mathbf{k}, i\omega_n)] + \ln [G^{TE}(\mathbf{k}, i\omega_n)] \} \quad (11)$$

and

$$\mathcal{F}(d) = 2\hbar \text{Im} \int \frac{d^2k}{(2\pi)^2} \int_0^\infty \frac{d\omega}{2\pi} \left[n(\omega, T) + \frac{1}{2} \right] \times \{ \ln [G^{TM}(\mathbf{k}, \omega)] + \ln [G^{TE}(\mathbf{k}, \omega)] \}. \quad (12)$$

The G functions are defined as

$$G^{TE}(\mathbf{k}, z) = 1 - \left(\frac{\gamma_1 - \gamma_0}{\gamma_1 + \gamma_0} \right)^2 e^{-2\gamma_0 d}; \quad G^{TM}(\mathbf{k}, z) = 1 - \left(\frac{\epsilon_0 \gamma_1 - \gamma_0 \epsilon_1}{\epsilon_0 \gamma_1 + \gamma_0 \epsilon_1} \right)^2 e^{-2\gamma_0 d}, \quad (13)$$

where the subscripts 0 and 1 represent the medium between the half-spaces and the half-spaces themselves, respectively and $\gamma_i(z) = \sqrt{k^2 - \epsilon_i(z) z^2/c^2}$.

For a real metal the dielectric function can be modeled with the Drude dielectric function, $\epsilon(\omega) = 1 - \omega_p^2/[\omega(\omega + i\eta)]$ and $\epsilon(i\omega) = 1 + \omega_p^2/[\omega(\omega + \eta)]$, where ω_p is the plasma frequency and η is the result of dissipation. The parameter η is a function of both frequency and temperature. The frequency dependence is negligible for frequencies below the plasma frequency. The contributions to the free energy and entropy decrease with frequency and have long ago faded away before the frequency dependence of η becomes important. Thus we may use the static value. This is related to the resistivity, ρ , through the relation $\eta = \omega_p^2 \rho / 4\pi$. Now, ρ and hence η is temperature dependent. The resistivity approaches a constant value as the temperature goes towards zero. This constant is caused by elastic scattering due to defects or impurities. For

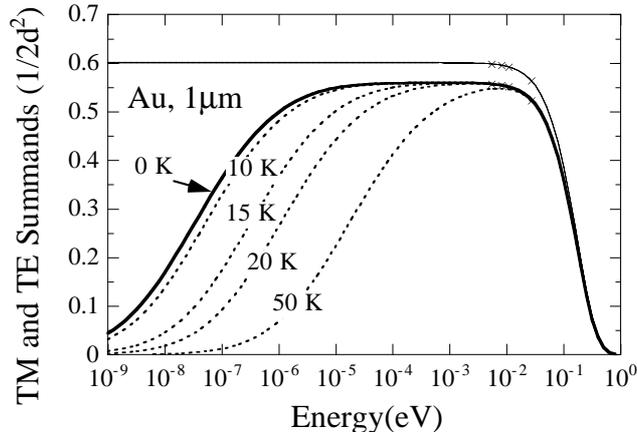


Figure 2. The summands of Eq. (11) for two gold half-spaces. See the text for more details.

higher temperatures inelastic scattering also contributes. This contribution is mainly due to phonons. There have been arguments in the literature^{12,13,14} in favor of the importance of this temperature dependence for the free energy and entropy, but it turns out that also the temperature dependence can be neglected. In Fig. 2 we show the integrands or summands of Eq. (11) for the TE and TM contributions, after the momentum integration has been performed. The figure is for gold half-spaces, $1 \mu\text{m}$ apart. We have chosen to discuss the separation $1 \mu\text{m}$ since this is the separation where a comparison between the experimental and theoretical temperature dependence of the Casimir effect is most likely to be feasible.^{2,6} Integrands or summands for different separations can be found elsewhere.¹⁵ The temperature dependence comes entirely from the temperature dependence of η .¹⁷ The thin solid curve is for the TM contribution. Here no temperature dependence at all is visible. The thick solid curve is the TE contribution at zero temperature. The dashed curves are the TE contributions at different temperatures as indicated in the figure. We see that the temperature dependence occurs for small frequencies, or energies, and extend to higher and higher frequencies with increasing temperature. Each curve carries a cross which indicates the first non-zero frequency value in the summation. The leftmost cross is for 10 K, next is for 15 K and so on. We note that the cross is always appearing at a frequency where the temperature dependence has vanished. Thus we can neglect the temperature dependence and just use the constant value for η , due to the elastic scattering. Bezerra et al.¹² neglected this very crucial constant and found a violation of the Nernst heat theorem.

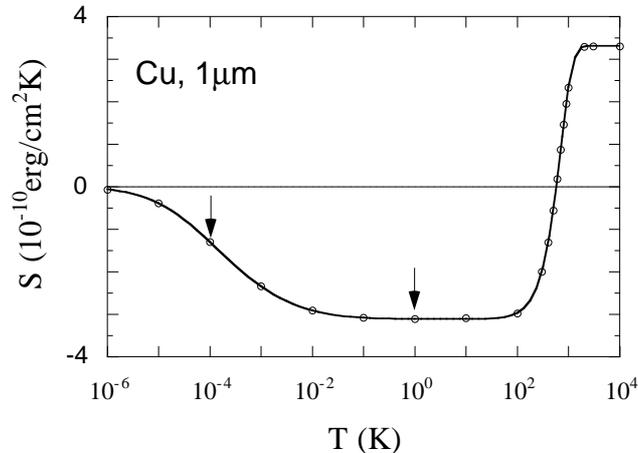


Figure 3. The entropy as a function of temperature for two copper half-spaces. The solid curve is from using Eq. (12) and the circles from using Eq. (14). Note that the results represent the entropy value at $1\mu m$ separation relative the value at infinite separation.

4 The entropy and the Nernst heat theorem

The Nernst heat theorem¹⁶ states that the entropy change in a reaction between pure substances approaches zero as the temperature goes to zero. This would in the present problem correspond to that the entropy change, when one brings the two metal half-spaces from a finite separation to infinite separation, should approach zero as the temperature goes towards zero. The entropy is obtained from the Helmholtz free energy, $S = -d\mathcal{F}/dT$. We use both versions of the free energy, Eq. (11) and Eq. (12), in the calculation of the entropy. We demonstrate that the entropy change goes to zero as the temperature goes to zero, i.e. the Nernst heat theorem is not violated.

We have calculated the entropy (per unit area) in the case of two copper half-spaces and only used the low-temperature constant for η . It equals¹⁸ $4.04 \times 10^{-10} \Omega cm$. The result from using Eq. (12) is shown as the solid curve in Fig. 3. The circles are from using Eq. (11). However a straight forward use of Eq. (11) would not be feasible at the low-temperature end of the figure. One would need of the order of 10^9 points in the summation. We have made a move that makes us get away with using just 10 points in the summation, a dramatic reduction in computer time. We just sum over the first 10 points and then replace the rest of the summation by an integration. When we then take the derivative of this integral with respect to temperature the only temperature dependence of the integral is in the lower integration limit. Hence the integration picks out the integrand at the lower integration limit and the entropy is obtained as

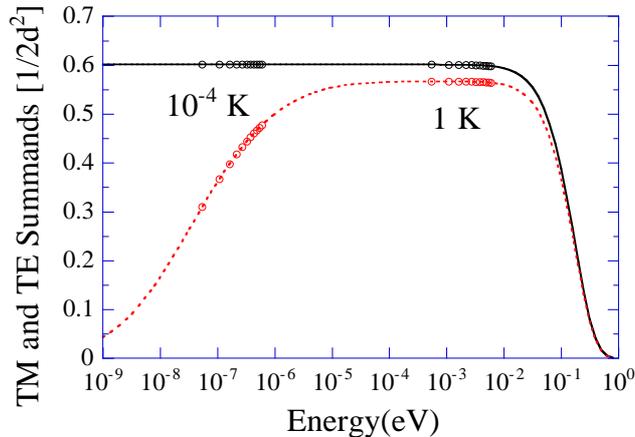


Figure 4. The circles indicate the values of the summands of Eq. (14) contributing to the entropy values at the arrows in Fig. 3. The clusters of circles to the left are for $T = 10^{-4}K$ and the clusters to the right for $T = 1K$. The solid and dashed curves are for the TM and TE contributions, respectively.

$$S = k_B \left\{ \frac{1}{2}I(0) + \sum_{n=1}^{n_{\max}} \frac{d}{dT} [T \cdot I(\omega_n)] - (n_{\max} + \frac{1}{2})I(\omega_{n_{\max} + \frac{1}{2}}) \right\}, \quad (14)$$

where I is the summand. We find that both derivations, Eqs. (11) and (12), give the same result, a result in accordance with the Nernst heat theorem. The entropy difference vanishes at zero temperature. We have to go to very low temperatures before this decrease begins though. We further realize that a mere manipulation of the zero frequency term will instead have the effect that the Nernst heat theorem will no longer be obeyed. In Fig. 4. we show the summands at the temperatures indicated by the arrows in Fig. 3. For $T = 1K$ the contributing terms in the summation are on the flat part of the summand and for $T = 10^{-4}K$ they are on the left sloping part of the summand. They have to reach this part before the entropy starts to decrease in size.

Summary and discussions

We have presented results for the entropy between two real metal half spaces. The results were based on two complementary expressions for the Helmholtz free energy; one obtained as a discrete summation along the imaginary frequency axis; one as an integration along the real frequency axis. Both approaches gave the same result and we could demonstrate that the Nernst heat theorem was not violated.

Dissipation has two main effects on the Casimir interaction, viz., a reduction of the high-temperature Casimir-asymptote with a factor of two and the appearance of a region of negative entropy; see Fig. 3. These effects are due to the decrease of the TE summands when the frequency goes towards zero; see Figs. 2 and 4. The reason is that the dielectric function of the metal no longer goes towards infinity fast enough when frequency approaches zero.

Acknowledgments

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